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Towards Loop Quantum Supergravity (LQSG)

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ABSTRACT

Should nature be supersymmetric, then it will be described by Quantum Supergravity at least in some energy regimes. The currently most advanced description of Quantum Supergravity and beyond is Superstring Theory/M-Theory in 10/11 dimensions. String Theory is a top-to-bottom approach to Quantum Supergravity in that it postulates a new object, the string, from which classical Supergravity emerges as a low energy limit. On the other hand, one may try more traditional bottom-to-top routes and apply the techniques of Quantum Field Theory. Loop Quantum Gravity (LQG) is a manifestly background independent and non-perturbative approach to the quantisation of classical General Relativity, however, so far mostly without supersymmetry. The main obstacle to the extension of the techniques of LQG to the quantisation of higher dimensional Supergravity is that LQG rests on a specific connection formulation of General Relativity which exists only in $D+1=4$ dimensions. In this Letter we introduce a new connection formulation of General Relativity which exists in all space–time dimensions. We show that all LQG techniques developed in $D+1=4$ can be transferred to the new variables in all dimensions and describe how they can be generalised to the new types of fields that appear in Supergravity theories as compared to standard matter, specifically Rarita–Schwinger and p-form gauge fields.

String/M Theory (ST) [1,2] and Loop Quantum Gravity (LQG) [3,4] are rather different programmes that aim for a consistent synthesis of the principles of General Relativity and Quantum Theory. String/M Theory is necessarily 10/11-dimensional, necessarily supersymmetric and is perturbatively defined on appropriate background space–times. Loop Quantum Gravity, on the other hand, is to date restricted to 4 space–time dimensions, does not need supersymmetry and by design is background independently and non-perturbatively defined. It is therefore very hard to compare these two approaches.

One possibility to make contact between them consists in the consideration of String/M Theory on space–time manifolds for which the excess dimensions are compactified and in regimes where supersymmetry is broken, so that only an effective 4-dimensional theory of General Relativity and the Standard Model plus quantum corrections survives, which then can be compared to the sector of LQG with small geometry fluctuations around the chosen background. We refer to [5] for the state of the art of String Theory Phenomenology but it transpires that there are many possibilities for doing this and there appears to be no specific model that one can compare LQG to.

Another possibility would be to generalise LQG to higher dimensions including supersymmetric matter in order to compare the methods of the two theories directly in 10/11 dimensions. This idea appears to be easier to implement because in its fundamental dimension, String/M Theory is much simpler to describe and there are much less choices to be made. Concretely, String Theory can be considered as a specific proposal for quantising classical Supergravity [6] in 10/11 dimensions on a background defined by a solution to the classical Supergravity equations of motion. With the possible exception of $N=8$ Supergravity in 4 space–time dimensions [7,8], perturbative approaches seem to fail due to non-renormalisability[1] whence non-perturbative methods appear to be more promising. Thus, the natural question arises, how to apply non-perturbative methods, such as those of LQG, to the quantisation of classical Supergravity in 10/11 dimensions. This is a subject of obvious interest to Supergravity research, but, to the best of our knowledge, the literature on it is rather sparse [9].

We caution the reader that although new steps towards a loop quantisation of higher dimensional Supergravity will be taken

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[1] More precisely, to date no counterterm has been found for $N=8$ SUGRA in 4 dimensions in any loop contribution carried out so far, some of them way beyond 3 loop order. This could mean that the theory is perturbatively finite like QED in 4 dimensions. It does not mean that it is a UV completion of General Relativity because the perturbation series most likely diverges. Therefore strictly speaking, also here non-perturbative methods need to be employed.

in this Letter, there are several issues left to deal with. One of them is to gain a proper understanding of the quantum dynamics. Although well-defined operators corresponding to the classical Hamiltonian and supersymmetry constraints can be constructed, there is at the moment insufficient control over their algebra, in particular off shell. A faithful representation of the super Dirac algebra on the other hand might be considered as one of the benchmarks a quantisation of a Supergravity has to satisfy. Also, it is presently unclear how one could establish a relation to string theory which goes beyond comparing two quantisation schemes for Supergravity. We hope that first hints can be obtained by looking at situations which can be treated by both theories non-perturbatively, like the calculation of black hole entropy.

The results of this Letter can be summarised as follows: We derive a generalisation of Ashtekar variables in all dimensions $D+1 \geqslant 3$ with the properties necessary for a rigorous quantisation. Furthermore, we generalise the construction to standard model matter, Rarita–Schwinger fields,[2] and Abelian p-forms, and thus to the matter fields appearing in many interesting Supergravities. A Hilbert space representation for a suitable point splitting Poisson subalgebra (the holonomy–flux algebra extended to the above matter fields) is provided and densely defined operators corresponding to the constraints of the classical theory can be constructed. The issue of a non-anomalous representation of the constraint algebra has not been investigated so far, however, these problems can, in principle, be circumvented using a Master constraint treatment [10]. Still, having a faithful representation of the classical constraint algebra would be far more satisfactory and we hope to revisit this issue in the future.

When trying to find a generalisation of Ashtekar's variables to higher dimensions, one almost immediately gets stuck:

LQG in 4 dimensions is fundamentally based on a connection formulation of the gravitational degrees of freedom. However, the construction of this connection and the properties it has and which make it so adapted for purposes of quantisation, exist only in 4 space–time dimensions [11–13]. Specifically, in 4 dimensions, General Relativity can be described, in its Hamiltonian form, by an SU(2) Yang–Mills phase space, subject to Gauß, spatial diffeomorphism and Hamiltonian constraints. The key properties of this formulation are:

(I) The connection A_a^j and its canonically conjugate momentum E^a_j (Yang–Mills "electric field") are real-valued, the $*$-relations of the associated Poisson algebra are trivial. Here $a, b, c, \ldots = 1, 2, 3$ are spatial, tensorial indices and $j, k, l, \ldots = 1, 2, 3$ are su(2) indices.

(II) The Poisson algebra is of the usual, simple CCR form

$$\begin{aligned} &\{A_a^j(x), A_b^k(y)\} = \{E^a_j(x), E^b_k(y)\} = 0, \\ &\{E^a_j(x), A_b^k(y)\} = -G\gamma\delta^a_b\delta^k_j\delta(x, y), \end{aligned} \tag{1}$$

where G is Newton's constant and γ is a free, real-valued parameter, called the Immirzi parameter [12,13].

(III) The gauge group SU(2) is compact.

These three properties are essential for the whole LQG framework. Without them, LQG would not exist. Properties (I) and (II) imply that there is a sufficiently simple $*$-algebra $\mathfrak{A}$ of functions on phase space separating its points so that one has a chance to find non-trivial Hilbert space representations thereof. Property (III) implies that the holonomies of A are valued in a compact set. It is therefore possible to construct a probability measure on the space of (distributional) connections. The Hilbert space is then an L_2 space of functions of connections which makes sense due to (1) because on such a space the connection acts by multiplication which is only consistent with the algebra if the connection is Poisson self-commuting.

Put together, this enabled to develop a rigorous kinematical mathematical framework [14,15] and to find a background independent representation of the holonomy–flux $*$-algebra $\mathfrak{A}$ which was later shown to be the unique one [16,17] when one insists on a unitary representation of the spatial diffeomorphism group. Moreover, this representation is also well adapted to the quantum dynamics as one may rigorously implement the spatial diffeomorphism constraint [18] and the Hamiltonian constraint [19] including standard matter [20,21] without anomalies.[3]

It transpires that LQG heavily relies on a connection formulation with the properties listed above. This connection can be found by two independent methods. The first stays purely within the Hamiltonian framework [11] and uses an extension of the ADM phase space of General Relativity [22] to the afore mentioned Yang–Mills phase space which is subjected to the additional su(2) Gauß constraint in order that its symplectic reduction recovers the ADM phase space. The core of the proof of this so-called *symplectic reduction theorem* is the observation, that the spin connection of the triad $e^a_j = E^a_j/\sqrt{|\det(E)|}$ has a potential, that is, there exists a functional $F[E]$ such that $\Gamma_a^j(x) = \delta F/\delta E^a_j(x)$. The second method uses the Lagrangian framework and starts from the Holst generalisation [23,24] of the Palatini action in order to accommodate the Immirzi parameter. This is actually an SO(1,3) Yang–Mills theory phase space in Lorentzian signature, however, it is subject to second class constraints which in particular imply that the connection is not self-commuting with respect to the corresponding Dirac bracket [25]. Therefore, properties (II) and (III) listed above are not satisfied and no kinematical Hilbert space representation of the Dirac bracket algebra has been found so far. In order to obtain the SU(2) Yang–Mills phase space without second class constraints, one therefore imposes the so-called time gauge which fixes the boosts of SO(1,3) and solves the second class constraints. What remains from the so(1,3) connection is the afore mentioned su(2) connection whose Dirac brackets are the Poisson brackets (1).

One would now guess that one can simply repeat either of these methods in dimensions $D+1 > 4$. However, this is not the case. As has been shown in [26], the second route leads to a SO(D) gauge theory in the time gauge but it is not a theory with a connection. In order to obtain a connection formulation one would therefore need the analog of a topological Holst term, but such a term is not available in all dimensions. Without time gauge, one does have an SO(1,D) connection formulation but subject to a Dirac bracket which leads to the same complication as in $D+1=4$. Moreover, the gauge group SO(1,D) is not compact so that the functional analytic and measure theoretic tools mentioned before do not apply. The first route also meets difficulties:

In D spatial dimensions, the metric of the ADM phase space has $D(D+1)/2$ degrees of freedom while an so(D) connection has $D^2(D-1)/2$ degrees of freedom of which the so(D)

[2] In the present treatment, it is necessary to have a real representations of the Lorentzian Clifford algebra. We thus have to restrict to such dimensions, however, the most interesting cases $D+1 = 4, 10, 11$ are covered.

[3] As in the $(3+1)$-dimensional treatment [4], anomalies are absent in the following sense: The commutator of two Hamiltonian constraint operators acting on a spin network state is annihilated by diffeomorphism invariant distributions, which mimics the classical Dirac algebra relation $\{C, C\} = \vec{C}$. In which sense an off-shell closure is possible is presently unknown, since due to the non-continuous representation used, an operator corresponding to infinitesimal spatial diffeomorphism does not exist, and only a representation of finite diffeomorphisms can be defined in the quantum theory.

Gauß constraint fixes $D(D-1)/2$. The symplectic reduction of the Yang–Mills phase space by the Gauß constraint therefore leaves $D^2(D-1)/2 - D(D-1)/2 = D(D-1)^2/2$ degrees of freedom which equals $D(D+1)/2$ precisely for $D=3$. If one wants to go beyond $D=3$ one therefore must add more constraints and/or change the gauge group in order to match the correct amount of ADM degrees of freedom and these constraints should better be first class in order to avoid complicated Dirac brackets and associated non-commuting connections.

The general analysis of the first route has been started in [27,28]. We take an unbiased viewpoint and consider a general Yang–Mills phase space with gauge group G, connection A_a^α and conjugate momentum π_α^a where $\alpha = 1, \ldots, N = \dim(G)$ denotes the Lie algebra index. There are DN degrees of freedom in the connection of which the Gauß constraint removes N. If there are no other constraints then we must have $(D-1)N = D(D+1)/2$. The only positive integer solutions to this equation are $(D, N) = (2, 3)$, $(3, 3)$ which again corresponds to 3-dimensional or 4-dimensional gravity respectively with gauge groups $\mathrm{SO}(1,2)$ and $\mathrm{SO}(3)$ respectively (with Lorentzian signature). Thus, necessarily more constraints are required.

As has been demonstrated in [27], one can obtain GUT theories by varying G. We are for the time being only interested in General Relativity and ask for the group G of minimal dimension that accomplishes all our requirements. To constrain the possible choices we try to follow as closely as possible the treatment of [11,27] in $D=3$ and consider a "square root" e_a^I of the spatial metric $q_{ab} = \eta_{IJ} e_a^I e_b^J$ where $I, J, K, \ldots = 1, \ldots, n \geqslant D$. Here η defines a G invariant metric of signature (p, q) with $q \geqslant D$ which is necessary for q_{ab} to be positive definite. This constrains the gauge group to be $\mathrm{SO}(p, q)$, $n = p + q$ which has dimension $N = n(n-1)/2$. The basic idea of the proof of equivalence with the ADM formulation of General Relativity will be to define a gauge theory subject to certain constraints, the degrees of freedom of which will be related to the ADM degrees of freedom. One then needs to show that the symplectic reduction of the gauge theory coincides with the ADM phase space, i.e. the Poisson brackets of the ADM degrees of freedom coincide with the Poisson brackets derived form the gauge theory.

Next to the spatial metric, we require an expression for the ADM momentum $P^{ab} := \sqrt{\det(q)}\, q^{ac}[K_c{}^b - \delta_c^b K_d{}^d]$, where K_{ab} is the extrinsic curvature. For this, we need to construct from e_a^I the so-called hybrid connection Γ_{aIJ} defined by

$$D_a e_b^I = \partial_a e_b^I - \Gamma_{ab}^c e_c^I + \Gamma_a{}^I{}_J e_b^J = 0 \tag{2}$$

and, in close analogy to the $(3+1)$-dimensional case, we set

$$\sqrt{\det(q)}\, K_a{}^b := [A_{aIJ} - \Gamma_{aIJ}]\pi^{bIJ}, \tag{3}$$

where A is the $\mathrm{so}(p, q)$ connection and π its conjugate momentum. The logic behind this definition becomes clear only after the symplectic reduction to the ADM phase space has been performed: K_{ab} will be shown to coincide with the usual extrinsic curvature from the ADM formalism on the constraint surface and the definition is thus consistent with the usual one. Notice that (3) is meaningful since $A - \Gamma$ transforms as a Lie algebra valued one form and not as a connection under $\mathrm{SO}(p, q)$. The question is of course whether Γ exists. The fact that $\Gamma_{a(IJ)} = 0$ leads to the consistency condition (all internal indices are moved with η)

$$e_{(c|I}\partial_{a|}e_{b)}^I - \Gamma_{(c|a|b)} = 0, \tag{4}$$

which can be shown to be identically satisfied. Therefore of the $D^2 n$ equations (2) for the $Dn(n-1)/2$ coefficients Γ_{aIJ} only $D^2 n - D^2(D+1)/2$ are independent. Requiring that Γ_{aIJ} can be uniquely solved for leads to a quadratic equation with the 2 roots $n = D$, $n = D+1$, that is, either $p = 0$, $q = D$ for $n = D$, or $p = 1$, $q = D$ and $p = 0$, $q = D+1$ for $n = D+1$.

Finally, the number of constraints additional to the Gauß constraint needed is given by $Dn(n-1)/2 - n(n-1)/2 - D(D+1)/2$ which equals $D^2[D-3]/2$ for $n = D$ and $D(D+1)[D-2]/2$ for $n = D+1$. The question is of course what these constraints should be. A natural choice is that these constraints should somehow impose that π^{aIJ} is entirely determined by e_a^I. The excess number of degrees of freedom in π^{aIJ} as compared to e_a^I is given by $Dn(n-1)/2 - Dn$ which equals precisely the number of additional constraints needed for both $n = D$ and $n = D+1$. In order to write π^{aIJ} purely in terms of e_a^I we require an internal vector n^I built from e_a^I such that

$$\pi^{aIJ} = 2\sqrt{\det(q)}\, q^{ab} n^{[I} e_b^{J]}. \tag{5}$$

There is no way to construct n^I out of e_a^I algebraically for $n = D$ so that in this case we must resort to $D = 3$. For $n = D+1$, however, we may build the unit normal

$$n_I := \frac{1}{D!\sqrt{\det(q)}} \epsilon_{IJ_1..J_D} \epsilon^{a_1..a_D} e_{a_1}^{J_1} .. e_{a_D}^{J_D} \tag{6}$$

satisfying $n_I n^I = -1$ for $\mathrm{SO}(1, D)$ or $n_I n^I = 1$ for $\mathrm{SO}(D+1)$.

We conclude that if we want to obtain a connection representation with compact gauge group for *Lorentzian General Relativity*, a natural choice is to consider the phase space of an $\mathrm{SO}(D+1)$ Yang–Mills theory subject to an $\mathrm{so}(D+1)$ Gauß constraint and additional *simplicity constraints* which impose that π^{aIJ} is determined by a generalised D-bein e_a^I via (5) where $n_I e_a^I = n_I n^I - 1 = 0$ is the unit normal determined by e_a^I. In fact, one might have almost guessed that:

If one performs the Hamiltonian analysis of the Lorentzian Palatini action in $D+1$ dimensions (see e.g. [29] and references therein) then one obtains a primary constraint precisely of the form (5). However, as secondary constraints one obtains an $\mathrm{so}(1, D)$ Gauß constraint next to spatial diffeomorphism and Hamiltonian constraints plus one more constraint. This last constraint, let us call it D-constraint, is a second class constraint partner to the simplicity constraint S. Thus, the Hamiltonian analysis of the $D+1$ Palatini action for *Lorentzian gravity* fails to deliver a canonical theory of connections with the properties (I), (II) and (III) listed above for two reasons:

1. The gauge group is $\mathrm{SO}(1, D)$ rather than $\mathrm{SO}(D+1)$ and thus non-compact.
2. The theory suffers from second class constraints and thus leads to Dirac bracket non-commuting connections.

As it turns out [29], the second problem can be circumvented by employing the machinery of *gauge unfixing* [30,31]. Under certain conditions, which are satisfied for our second class pair (S, D), it is possible to trade the second class system under consideration for an equivalent first class system equipped with the original Poisson bracket rather than the Dirac bracket so that the connection remains Poisson commuting. However, the first problem cannot be overcome starting from the Palatini action for Lorentzian General Relativity. Therefore, the canonical theory that we are about to describe does not have an obvious Lagrangian origin (other than by backwards Legendre transform).

The first step [32] consists in writing both the hybrid connection and the simplicity constraint purely in terms of π^{aIJ}. The guideline for doing this consists in replacing the solution $\Gamma_{aIJ}[e]$ of (2), which can be computed explicitly, by a function $\Gamma_{aIJ}[\pi]$ of π alone such that it reduces to $\Gamma_{aIJ}[e]$ when $\pi^{aIJ} = 2n^{[I} E^{a|J]}$

where $E^{aI} = \sqrt{\det(q)}\, q^{ab} e^I_b$. This has been done explicitly in [32] which leads to a rather complicated expression which can be displayed as a rational homogeneous function of degree zero in terms of π and its first partial derivatives and which transforms as an $\mathrm{so}(D+1)$ connection.

Next, one shows [32] that the condition that π^{aIJ} is of the form $2n^{[I}E^{a|J]}[E]$ is equivalent to the condition

$$S^{aIJ;bKL} := \pi^{a[IJ}\,\pi^{|b|KL]} = 0 \tag{7}$$

provided that for any non-zero vector n the object

$$Q^{ab} = \pi^{aIK}\pi^{bJL}\delta_{IJ} n_K n_L$$

is non-degenerate. The proof follows closely the seminal paper [33] in which the possibility of a higher dimensional, canonical version of LQG is also contemplated. The non-degeneracy of Q^{ab} is equivalent to the non-degeneracy of q_{ab}, which is a standard restriction in canonical General Relativity, and we will keep this restriction in the classical theory in order to ensure equivalence with General Relativity.

We are now in the position to establish the relation with the ADM phase space. We postulate the following non-trivial Poisson brackets

$$\left\{A_{aIJ}(x), \pi^{bKL}(y)\right\} := 2\beta\, G\, \delta^b_a\, \delta^K_{[I}\, \delta^L_{J]}\, \delta(x,y), \tag{8}$$

where β is a free real parameter and consider the following quantities

$$\det(q) q^{ab} := -\mathrm{Tr}\left(\pi^a \pi^b\right),$$
$$\sqrt{\det(q)}\, K_a{}^b := -\frac{1}{\beta}\mathrm{Tr}\left([A_a - \Gamma_a]\pi^b\right) \tag{9}$$

which play the role of the intrinsic D metric and the extrinsic curvature. Then one can show [32] that with the usual formula $P^{ab} := \sqrt{\det(q)}\, q^{ac}[K_c{}^b - \delta^b_c\, K_d{}^d]$ one obtains the following non-trivial Poisson brackets

$$\left\{P^{ab}(x), q_{cd}(y)\right\} = -G\, \delta^{(a}_c\, \delta^{b)}_d\, \delta(x,y) \tag{10}$$

modulo terms that vanish on the joint constraint surface defined by the simplicity constraint (7) and the $\mathrm{so}(D+1)$ Gauß constraint

$$G^{IJ} = \partial_a \pi^{aIJ} + [A_a, \pi]^{IJ}. \tag{11}$$

This is non-trivial and heavily relies on the fact that the hybrid connection $\Gamma_{aIJ}[\pi]$ has a *weak potential*, that is, there is a functional $F[\pi]$ such that $\Gamma_{aIJ}(x) = \delta F/\delta\pi^{aIJ}(x)$ modulo terms that vanish on the simplicity constraint surface. Without this property, the bracket $\{P^{ab}(x), P^{cd}(y)\}$ would not vanish on the constraint surface. It is also not difficult to show that the algebra of the simplicity and Gauß constraints is first class and that the variables (9) are weak Dirac observables with respect to both constraints.

Thus, as with the usual LQG variables [11], the (weak) integrability of the spin (hybrid) connection is central to show that the symplectic reduction of the $\mathrm{SO}(D+1)$ Yang–Mills phase space by Gauß and simplicity constraints results in the ADM phase space. That the ADM spatial diffeomorphism constraints C_a and Hamiltonian constraint C, when expressed in terms of (9), weakly Poisson commute with $S^{aIJ;bKL}$, G^{IJ} and that their algebra among themselves, as compared to the ADM phase space, is unchanged modulo terms vanishing when $S^{aIJ;bKL} = G^{IJ} = 0$ is a simple corollary. We conclude that we have found a connection formulation for Lorentzian General Relativity in $D+1$ dimensions with $D \geqslant 2$ with all the desired properties, in particular, all four types of constraints form a first class algebra.

Remarkably, all we have said so far can be performed for all four combinations of the space–time and internal signatures (s,ζ) respectively where $s = \pm 1$ for Euclidean and Lorentzian General Relativity respectively and $\zeta = \pm 1$ for $\mathrm{SO}(D+1)$ and $\mathrm{SO}(1,D)$ respectively.

Similar to the situation with the usual variables [11], the constraints take a simple form when written in terms of the curvature F_{abIJ} of A_{aIJ}. One finds modulo Gauß and simplicity constraints [32]

$$C_a = -\mathrm{Tr}\left(F_{ab}\pi^b\right),$$
$$\begin{aligned}\sqrt{\det(q)}\, C = {}& -\zeta\, \mathrm{Tr}\left(F_{ab}\pi^a\pi^b\right)\\ &+ \frac{1}{(D-1)^2}\left[\zeta - \frac{s}{\beta^2}\right]\left[K_a{}^b K_b{}^a - \left(K_c{}^c\right)^2\right]\\ &- K^{TT}_{aIJ}\, T^{aIJ;bKL}\, K^{TT}_{bKL}.\end{aligned} \tag{12}$$

The spatial diffeomorphism constraint takes unsurprisingly a form analogous to the formulation in $D = 3$. Also the first two terms in C look familiar. However, the third term in the expression for C is new. Here $K_{aIJ} = (A_{aIJ} - \Gamma_{aIJ})/\beta$ and K^{TT}_{aIJ} is transversal $n^I K^{TT}_{aIJ} = 0$ and trace free $E^{aI} K^{TT}_{aIJ} = 0$ on the solution $\pi^{aIJ} = 2n^{[I}E^{a|J]}$ of the simplicity constraint, it can be written as $P_{aIJ}{}^{bKL}[\pi]\, K_{bKL}$ where $P[\pi]$ is a transverse tracefree projector and just depends on π. Also $T^{aIJ;bKL}[\pi]$ is a tensor constructed entirely from π. The significance of this third term is that it removes the dependence of the curvature on the transverse tracefree components of K_{aIJ} which are pure gauge with respect to the simplicity constraints and on which the ADM variables do not depend. Notice that for matching internal and space–time signature $\zeta = s$ and $\beta = 1$ the second term in C vanishes and the Hamiltonian constraint simplifies, a feature that also is familiar from the usual formulation. In this case one can arrive at (12) also starting from the Palatini action by the method of gauge unfixing [29] where now the third term is generated when making the Hamiltonian constraint invariant under the gauge transformations generated by the simplicity constraint so that the afore mentioned second class partner can be dropped.

In order to quantise the Hamiltonian constraint, we observe that the terms quadratic in K_{aIJ} can be written, as in the $D = 3$ situation [19] as [34]

$$K_{aIJ}(x) \propto \left\{A_{aIJ}(x), \{V, C_E\}\right\}, \tag{13}$$

where $V = \int d^D x \sqrt{\det(q)}$ is the total volume and

$$\sqrt{\det(q)}\, C_E := -\mathrm{Tr}\left(F_{ab}\pi^a\pi^b\right) \tag{14}$$

is the "Euclidean piece" of the Hamiltonian constraint. Similar remarks apply to the tensors T, P which can be treated by analogous Poisson bracket identities as displayed in [19,20]. Notice that the kinematical Hilbert space techniques [14–17] as well as the treatment of the spatial diffeomorphism constraint [18] have been formulated for canonical theories of connections for compact gauge groups in any dimension and thus can be applied to our situation without further effort. In particular, the holonomy–flux algebra now consists of $\mathrm{SO}(D+1)$-valued holonomies of A along piecewise analytic (or semianalytic) 1-dimensional paths and $\mathrm{so}(D+1)$-valued fluxes of π through semianalytic $D-1$ surfaces.

The following remark is due at this point for the case of $D = 3$:

In the usual formulation the connection A^{LQG} is an SU(2) connection and is related to the extrinsic curvature by

$$A^{\mathrm{LQG}}_{ajk} - \Gamma^{\mathrm{SPIN}}_{ajk}[E] = \gamma\, \epsilon_{jkl} K^l_a;$$
$$A^{\mathrm{LQG}}_{a0j} = \Gamma^{\mathrm{SPIN}}_{a0j}[E] \equiv 0, \tag{15}$$

where $K_{ab} = K_a^j e_{bj}$ and where $\Gamma^{\mathrm{SPIN}}_{ajk}[E]$ is the spin connection of the co-triad e_a^j built from E^a_j. Neither A^{LQG} nor Γ^{SPIN} have a "boost" part, the information about the extrinsic curvature is encoded in the rotational part of A^{LQG} and depends on the Immirzi parameter γ. On the other hand, in the "time gauge" $n^I = \delta^I_0$

$$A^{\mathrm{NEW}}_{ajk} - \Gamma^{\mathrm{HYB}}_{ajk}[\pi] \approx S - \text{gauge},$$
$$A^{\mathrm{NEW}}_{a0j} - \Gamma^{\mathrm{HYB}}_{a0j}[\pi] = \beta K_{aj}, \qquad (16)$$

where $\Gamma^{\mathrm{HYB}}[\pi]$ is the hybrid connection built from π. The trace-free rotational components of the new connection are pure gauge and the trace piece of the rotational components vanishes by the boost part of the Gauß constraint. Therefore the information about the extrinsic curvature is encoded in the boost part of the new connection and depends on the parameter β. It transpires that the Immirzi parameter γ and β have nothing to do with each other and that the new formulation is a rather different extension of the ADM phase space as compared to the LQG formulation even in $D = 3$ which nevertheless have the same symplectic reduction, namely the ADM phase space. In [29] we show that for $D = 3$ one can generalise the exposition given so far by considering a 2-parameter family of connection formulations depending on both γ, β. The essential features remain the same, the connection remains Poisson commuting.

The price to pay for the higher dimensional connection formulation are the additional simplicity constraints. On the other hand, this makes this canonical formulation appear much closer to the spin foam models [35] which aim to be a path integral formulation of LQG in $D + 1 = 4$ dimensions. For instance, we expect that the $\beta = 1$, $\zeta = s$ theory with arbitrary γ corresponds to the "new spin foam models" [36,37] because both can be obtained from the Holst modification of the Palatini action without imposing the time gauge. Notice that our formulation is in the continuum rather than on a fixed triangulation. In particular, we must define the simplicity constraint operators on spin network functions for all graphs not only simplicial ones. The necessity of doing that in order to make contact with the canonical formulation has been recently emphasised also in [38].

The quadratic, canonical quantum simplicity constraints are naturally quantised by smearing the two factors of π in (7) over two independent $D - 1$ surfaces in order to obtain flux operators which are then shrunk to a point x [34]. Then two types of simplicity constraints arise: Either x is an interior point of an edge (edge simplicity constraints) or a vertex (vertex simplicity constraints). Taking advantage of the flexibility in the choice of the surfaces available in the continuum formulation as well as taking linear combinations one can show that one can reduce the simplicity constraints to the following building blocks

$$S^{IJKL}_{e,e'} := X_e^{[IJ} X_{e'}^{KL]} = 0 \qquad (17)$$

for all pairs of edges e, e' of the given graph which share a vertex. Here X_e^{IJ} is right invariant vector field on the copy of $\mathrm{SO}(D+1)$ associated with the edge e. The edge simplicity constraints arise for $e = e'$, the vertex simplicity constraints for $e \neq e'$. These are the analogs of the diagonal and cross simplicity constraints appearing in the spin foam literature which involve either one face or two faces sharing a "temporal edge". These faces arise from our edges by "time evolving" them into faces.

The edge simplicity constraints enforce the irreducible $\mathrm{SO}(D+1)$ representations on the edges of the $\mathrm{SO}(D+1)$ spin network functions to be simple representations which have already been described in great detail in [33] for $\mathrm{SO}(D+1)$. Despite the first class nature of the simplicity constraint in the classical, non-distributional theory,[4] the vertex simplicity constraints become second class in the quantum theory due to the non-commutativity of the fluxes, which results in an anomaly and strong imposition leads to the higher dimensional analogue of the Barrett–Crane intertwiner also described already in [33]. To avoid the anomaly one can invent several strategies, none of which are entirely satisfactory. Either one can try to construct a vertex master constraint [34] for each vertex whose solutions, however, are beyond any analytic control at the moment. Another possibility is to consider for each vertex a preferred recoupling scheme which then selects a maximal commuting subset of vertex simplicity constraints which in turn select a so-called simple intertwiner [39] which appears to establish a unitary map between the old and new formulation in $D + 1 = 4$. The unnatural feature here is that the notion of simple intertwiner depends on the chosen recoupling scheme. Yet another option in the spirit of the "improved spin foam models" [36,37] is to consider linear simplicity constraints [39] instead. These arise in the canonical framework by considering the time normal n^I as an independent quantum field subject to the normalisation constraint $\mathcal{N} = n^I n_I - 1$ and to replace the quadratic constraints (7) by the linear constraints

$$S^{aIJK} = \pi^{a[IJ} n^{K]}. \qquad (18)$$

The framework described so far can be adapted to this formulation of the simplicity constraint and all essential features are preserved. This formulation automatically avoids the topological sector for $D = 3$. The then needed Hilbert space representation for the time normal field is supplied in [40]. Again, it is natural to smear the π appearing in (18) over arbitrary surfaces. Notice that now the wave functions depend on both A and n where A is located along the edges and n along the vertices. The building blocks of the simplicity constraint are now the operators

$$X^{[IJ}_{e,x} n^{K]}(x) \qquad (19)$$

for each point x on the edge e and each edge e of the graph. Here $X_{e,x}$ denotes the right invariant vector field on the copy of $\mathrm{SO}(D+1)$ corresponding to segment of e which starts at x and ends at the end point of e. The nice thing is that these constraints are non-anomalous because they generate the Lie algebra of the $\mathrm{SO}(D)$ subgroup of $\mathrm{SO}(D+1)$ stabilising n, however, they enforce that at each point of an edge there should be a specific n dependence which maps out of the space of spin network functions described above. While some strategies for matching linear and quadratic simplicity constraints are spelled out in [39] based on ideas from group averaging, none of them can be considered as complete so far and they deserve further research.

Notice that especially in $D = 3$ we are now faced with a real challenge and opportunity to make progress with the simplicity constraint:

We have two classically equivalent connection formulations, the one based on the Ashtekar–Barbero–Immirzi SU(2) connection with SU(2) Gauß constraint and the new SO(4) connection formulation with SO(4) Gauß and simplicity constraint. One would therefore expect the corresponding quantum theories to be unitarily or at least semiclassically equivalent. Thus, the usual LQG

[4] I.e. before using the singular smearing to construct holonomies and fluxed from the gauge and frame fields. In fact, the second class nature of the vertex simplicity constraints is already present in the classical theory when using a singular smearing, i.e. holonomies and fluxes. The wording "anomaly" however still seems to be appropriate, because the singular smearing is needed for the quantum theory, but not for the classical one.

formulation may help us to find a consistent and non-anomalous definition of the simplicity constraint as argued in [39].

So far we have only considered the purely gravitational degrees of freedom. We now turn to matter. As far as standard matter is concerned, all the machinery developed for $D = 3$ straightforwardly generalises to arbitrary D as far as gauge bosons and scalars are concerned [20]. Slightly more attention is required when considering fermionic matter because the Lorentzian theory uses Dirac matrices for the $\mathrm{SO}(1, D)$ Clifford algebra which we would like to recast in terms of the Dirac matrices for the $\mathrm{SO}(D+1)$ Clifford algebra in order that matter and gravitational degrees of freedom transform under the same gauge group $\mathrm{SO}(D+1)$. As far as Dirac fermions are concerned, one can proceed similarly [41] as in the case $D = 3$ in [20]: Consider the Lorentzian theory first in time gauge $n^I = \delta^I_0$ which breaks the $\mathrm{SO}(1, D)$ internal group down to the $\mathrm{SO}(D)$ subgroup preserving the time gauge. In particular, the theory is now formulated in terms of E^a_j, K^j_a as in [26]. Now extend all constraints and variables in an $\mathrm{SO}(D + 1)$ covariant way such that they reduce to the previous expressions in the time gauge. This can be done using that $E^{ai} = \pi^{ai0}$ in time gauge when the simplicity constraint holds, by subtracting terms proportional to K^{TT}_{aIJ} as we did for the gravitational degrees of freedom and by replacing the Lorentzian γ^0_L Dirac matrix by the Euclidean one $\gamma_0 := \gamma^0_E := i\gamma^0_L$ wherever it appears. For instance, the term $E^a_j\Psi^\dagger\gamma^0\gamma_j\nabla_a\Psi$ where Ψ are the Dirac fermions and ∇_a is the covariant derivative defined by the $\mathrm{SO}(D)$ spin connection of E^a_i, can be written as $\pi^{aIJ}\Psi^\dagger\gamma_{[I}\gamma_{J]}\mathcal{D}_a\Psi$ plus terms involving K_{aIJ} where $\mathcal{D}_a$ is the covariant derivative defined by A_{aIJ}. Here we used the fact that the $\mathrm{SO}(D)$ spin connection is the $\mathrm{SO}(D+1)$ hybrid connection in time gauge. The K_{aIJ} terms then need to be written as above in double Poisson brackets involving volume and Euclidean piece of the gravitational contribution to the Hamiltonian constraint. This produces 4-fermion terms, see [41] for details but apart from this one can copy the Hilbert space representation from the case $D = 3$ for Dirac fermions given in [21]. We conclude that standard matter can be treated just as in $D = 3$.

The reason why this worked so well is that both $\mathrm{SO}(D+1)$ and $\mathrm{SO}(1, D)$ act on the same *complex* representation spaces. When we turn to Supergravity theories, this is no longer helps because Supergravity multiplets contain the Rarita–Schwinger field (gravitino) as a *Majorana fermion* and these belong to *real* representation spaces. In particular, the Lorentzian Dirac matrices (for our mostly plus signature choice in $D + 1 = 4, 10, 11$) are real-valued which is crucial in order that the real vector space V formed by Majorana fermions is preserved under $\mathrm{SO}(1, D)$. Since $\gamma^0_E = i\gamma^0_L$ is then purely imaginary this is no longer the case for $\mathrm{SO}(D+1)$ so that $\mathrm{SO}(D+1)$ does not act on the vector space of Majorana fermions. To see this explicitly, notice that $[\gamma^I, \gamma^J]/8$ are the generators of $\mathrm{so}(1, D)$ or $\mathrm{so}(D+1)$ respectively in the spinor representation. We resolve the arising tension by noticing that while there is no action of $\mathrm{SO}(D + 1)$ on V there is an action on the set of pairs (n, θ) where θ is a Majorana fermion and n is the time normal field. To see this, notice that the action of $g \in \mathrm{SO}(D+1)$ on n is simply given by $n \mapsto g \cdot n$. Let us now build an $\mathrm{SO}(D + 1)$ matrix $A(n)$ from the components of n, for instance the one that is given in [40], which maps n^I to the time gauge normal vector δ^I_0. Then one can show that $gA(n)^{-1} = A(g\cdot n)^{-1}R(g, n)$ where $R(g, n)$ is a rotation preserving the time gauge. Such a rotation is generated in the spinor representation by $[\gamma^j, \gamma^k]/8$, $j, k = 1, \ldots, D$ and thus only involves the *real* Dirac matrices γ^j. We now consider the extension $\theta \mapsto A(n)^{-1} \cdot \theta$ off the time gauge and observe that the $\mathrm{SO}(D + 1)$ orbit of these vectors is preserved as $g \cdot A(n)^{-1} \cdot \theta = A(g \cdot n)^{-1} \cdot R(g, n) \cdot \theta$. In other words, we have the action on pairs $g \cdot (n, \theta) = (g \cdot n, R(g, n) \cdot \theta)$. Starting from time gauge, one then rewrites all constraints in terms of these variables, replaces Lorentzian γ^0 by the Euclidean one and finally extends off the time gauge similar as for Dirac fermions. There is still a remaining complication which arises from the fact that there is a reality condition on the Majorana spinors which effectively gives rise to a non-trivial Dirac antibracket and which prohibits to use the formalism of [21]. We supply the missing details and provide a proper Hilbert space representation of the Dirac antibracket [40].

Supergravity theories not only contain the Rarita–Schwinger field but often additional tensor fields such as the 3-index photon field A in 11d Supergravity. This field is self-interacting due to a Chern–Simons term in the action (the Hamiltonian is a polynomial of fourth order in A and the momentum π conjugate to it) and it is constrained by a twisted Gauß law constraint which is of the form $G = d * \pi - c/2F \wedge F$. Here, $F = dA$ is the curvature of A and c is related to the level of the Chern–Simons theory. The appearance of the $F \wedge F$ term implies that one cannot simply solve the constraint by considering the theory as the 3-form equivalent of Maxwell theory in higher dimensions and for which gauge invariant states would simply be gauge invariant “Spin network states”, i.e. functions of A integrated over closed 3-surfaces. In particular the analog of the LQG representation is inappropriate to solve the constraint. We resolve the tension [42] by considering the Weyl algebra generated by the exponentials of the Dirac observables F, $P = *\pi + cA \wedge F$ with respect to G which can be computed in closed form and show that the resulting $*$-algebra admits a state (positive linear functional) which is discontinuous in both F, P and is thus of the Narnhofer–Thirring type[5] [43]. The resulting background independent Hilbert space representation then follows from the GNS construction [44] and since the contribution of the 3-index Photon to the Hamiltonian and spatial diffeomorphism constraint can be written just in terms of the observables, they can be quantised in terms of Weyl elements [42].

We hope that the tools provided in our work will turn out to be useful for a rigorous canonical and non-perturbative quantisation, for instance by the methods of LQG, of a wide variety of Supergravity theories, including $d = 4$, $N = 8$, $d = 11$, $N = 1$ and $d = 10$, $N = 1$. An extension to the $d = 10$, $N = 2a$ and $d = 10$, $N = 2b$ theories related to type $2a$ and $2b$ Superstring theory should also be possible, although we are not aware of explicit Hamiltonian formulations of these theories.

In order to attack the issue of comparing LQG techniques to String Theory, it makes sense to start with symmetry reduced models such as cosmology and black holes. The goal however should not just be to compare certain numerical values, like the black hole entropy, to each other, but to understand possible connections between the derivations of these. We hope that new features of higher dimensional (Super)LQG, e.g. supersymmetric effects in cosmological models and the treatment of topologically interesting black holes in higher dimensions (black rings), might yield hints on how to proceed. Also, despite many technical and conceptual hurdles along the way, revisiting the AdS/CFT correspondence [45] and its integrability structure [46] in the light of the proposed techniques seems to be an interesting project for further research.

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[5] This state is different from the Fock state and has the attractive feature of being ghost free while manifestly Poincaré invariant.

motivational discussions about higher dimensional Ashtekar variables and their application to Supergravity and String Theory.

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