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Nonlinear Relaxed Cocoercive Variational Inclusions Involving (A, η)-Accretive Mappings in Banach Spaces

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Abstract—In this paper, we introduce a new concept of (A, η)-accretive mappings, which generalizes the existing monotone or accretive operators. We study some properties of (A, η)-accretive mappings and define resolvent operators associated with (A, η)-accretive mappings. By using the new resolvent operator technique, we also construct a new perturbed iterative algorithm with mixed errors for a class of nonlinear relaxed cocoercive variational inclusions involving (A, η)-accretive mappings and study applications of (A, η)-accretive mappings to the approximation-solvability of this class of nonlinear relaxed cocoercive variational inclusions in q-uniformly smooth Banach spaces. Our results improve and generalize the corresponding results of recent works.

Keywords—(A, η)-accretive mapping, Resolvent operator technique, Nonlinear variational inclusion with relaxed cocoercive mapping, Perturbed iterative algorithm with mixed errors, Convergence and stability.

1. INTRODUCTION

Very recently, in order to study extensively variational inequalities and variational inclusions, which are providing mathematical models to some problems arising in economics, mechanics,

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and engineering science, Ding [1], Huang and Fang [2], Fang and Huang [3], Verma [4,5], Fang and Huang [6,7], Huang and Fang [8], Fang *et al.* [9] introduced the concepts of η-subdifferential operators, maximal η-monotone operators, generalized monotone operators (named H-monotone operators), A-monotone operators, (H,η)-monotone operators in Hilbert spaces, H-accretive operators, generalized m-accretive mappings and (H,η)-accretive operators in Banach spaces and their resolvent operators, respectively. Further, by using the resolvent operator technique, which is a very important method to find solutions of variational inequality and variational inclusion problems, a number of of nonlinear variational inclusions and many systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years. See, for example, [1–16].

Motivated and inspired by the above works, in this paper we introduce a new concept of (A,η)-accretive mappings, which provides a unifying framework for maximal monotone operators [17], m-accretive operators, η-subdifferential operators [1,15], maximal η-monotone operators [2], H-monotone operators [3], generalized m-accretive mappings [8,18], H-accretive operators [7], (H,η)-monotone operators [6], A-monotone mappings [4,5], and (H,η)-accretive operators [9]. We study some properties of (A,η)-accretive mappings and define the resolvent operators associated with (A,η)-accretive mappings which include the existing resolvent operators as special cases. By using the new resolvent operator technique, we develop a new perturbed iterative algorithm with errors to solve a class of nonlinear relaxed cocoercive variational inclusions associated with (A,η)-accretive mappings in q-uniformly smooth Banach spaces and prove the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm.

2. PRELIMINARIES

Let X be a real Banach space with dual space X^*, $\langle\cdot,\cdot\rangle$ be the dual pair between X and X^*, 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^q,\ \|f^*\| = \|x\|^{q-1} \right\}, \qquad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$, and J_q is single-valued if X^* is strictly convex. In the sequel, we always suppose that X is a real Banach space such that J_q is single-valued and $\mathcal{H}$ is a Hilbert space. If $X = \mathcal{H}$, then J_2 becomes the identity mapping on $\mathcal{H}$.

The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\left\{ \frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1,\ \|y\| \le t \right\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_X(t) \le c t^q, \qquad q > 1.$$

Remark that J_q is single-valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [19] proved the following result.

LEMMA 2.1. *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

In the sequel, we give some concept and lemmas needed later.

DEFINITION 2.1. *Let $T, A : X \to X$ be two single-valued mappings. T is said to be*

(i) *accretive if*

$$\langle T(x) - T(y),\ J_q(x-y)\rangle \geq 0, \qquad \forall x, y \in X;$$

(ii) *strictly accretive if T is accretive and*

$$\langle T(x) - T(y),\ J_q(x-y)\rangle = 0$$

if and only if $x = y$;

(iii) *r-strongly accretive if there exists a constant $r > 0$ such that*

$$\langle T(x) - T(y),\ J_q(x-y)\rangle \geq r\|x-y\|^q, \qquad \forall x, y \in X;$$

(iv) *γ-strongly accretive with respect to A if there exists a constant $\gamma > 0$ such that*

$$\langle T(x) - T(y),\ J_q(A(x) - A(y))\rangle \geq \gamma\|x-y\|^q, \qquad \forall x, y \in X;$$

(v) *m-relaxed cocoercive with respect to A if, there exists a constant $m > 0$ such that*

$$\langle T(x) - T(y),\ J_q(A(x) - A(y))\rangle \geq -m\|T(x) - T(y)\|^q, \qquad \forall x, y \in X;$$

(vi) *(α, ξ)-relaxed cocoercive with respect to A if, there exist constants $\alpha, \xi > 0$ such that*

$$\langle T(x) - T(y),\ J_q(A(x) - A(y))\rangle \geq -\alpha\|T(x) - T(y)\|^q + \xi\|x-y\|^q, \qquad \forall x, y \in X;$$

(vii) *s-Lipschitz continuous if there exists a constant $s > 0$ such that*

$$\|T(x) - T(y)\| \leq s\|x-y\|, \qquad \forall x, y \in X.$$

REMARK 2.1. When $X = \mathcal{H}$, (i)–(iv) of Definition 2.1 reduce to the definitions of monotonicity, strict monotonicity, strong monotonicity, and strong monotonicity with respect to A, respectively (see [3,6]).

DEFINITION 2.2. *A single-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that*

$$\|\eta(x,y)\| \leq \tau\|x-y\|, \qquad \forall x, y \in X.$$

DEFINITION 2.3. *Let $\eta : X \times X \to X$ and $A, H : X \to X$ be single-valued mappings. Then set-valued mapping $M : X \to 2^X$ is said to be*

(i) *accretive if*

$$\langle u - v,\ J_q(x-y)\rangle \geq 0, \qquad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(ii) *η-accretive if*

$$\langle u - v,\ J_q(\eta(x,y))\rangle \geq 0, \qquad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(iii) *strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;*

(iv) *r-strongly η-accretive if there exists a constant $r > 0$ such that*

$$\langle u - v,\ J_q(\eta(x,y))\rangle \geq r\|x-y\|^q, \qquad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(v) *α-relaxed η-accretive if there exists a constant $\alpha > 0$ such that*

$$\langle u - v,\ J_q(\eta(x,y))\rangle \geq -\alpha\|x-y\|^q, \qquad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(vi) *m-accretive if M is accretive and $(I + \rho M)(X) = X$ for all $\rho > 0$, where I denotes the identity operator on X;*

(vii) *generalized m-accretive if M is η-accretive and $(I + \rho M)(X) = X$ for all $\rho > 0$;*

(viii) *H-accretive if M is accretive and $(H + \rho M)(X) = X$ for all $\rho > 0$;*

(ix) *(H, η)-accretive if M is η-accretive and $(H + \rho M)(X) = X$ for every $\rho > 0$.*

In a similar way, we can define strictly η-accretivity and strongly η-accretivity of the single-valued mapping A.

REMARK 2.2.

(1) The class of generalized m-accretive operators was first introduced by Huang and Fang [18], and includes that of m-accretive operators as a special case. The class of H-accretive operators was first introduced and studied by Fang and Huang [7], and also includes that of m-accretive operators as a special case.
(2) When $X = \mathcal{H}$, (i)–(ix) of Definition 2.3 reduce to the definitions of monotone operators, η-monotone operators, strictly η-monotone operators, strongly η-monotone operators, relaxed η-monotone operators, maximal monotone operators, maximal η-monotone operators, H-monotone operators and (H, η)-monotone operators, respectively.

DEFINITION 2.4. *Let $T : X \to X$ be a single-valued mapping. For all $x, y \in X$, the mapping $N : X \times X \to X$ is called to be ϵ-Lipschitz continuous with respect to the first argument, if there exists a constant $\epsilon > 0$ such that*

$$\|N(x, \cdot) - N(y, \cdot)\| \le \epsilon \|x - y\| \qquad \forall\, x, y \in X.$$

In a similar way, we can define Lipschitz continuity of the mapping $N(\cdot, \cdot)$ with respect to the second argument.

LEMMA 2.2. *Let r and s be two nonnegative real numbers. Then*

$$(r + s)^q \le 2^q (r^q + s^q).$$

PROOF. $(r + s)^q \le (2 \max\{r, s\})^q = 2^q (\max\{r, s\})^q \le 2^q (r^q + s^q)$.

3. (A, η)-ACCRETIVE MAPPINGS AND RESOLVENT OPERATORS

In this section we introduce a new concept of (A, η)-accretive mappings, which provides a unifying framework for the existing monotone operators in Hilbert spaces and accretive operators in Banach spaces. We study some properties of (A, η)-accretive mappings and define the resolvent operators associated with (A, η)-accretive mappings. We also establish the Lipschitz continuity of resolvent operators associated with (A, η)-accretive mappings under suitable conditions.

DEFINITION 3.1. *Let $A : X \to X, \eta : X \times X \to X$ be two single-valued mappings. Then, a multivalued mapping $M : X \to 2^X$ is called (A, η)-accretive if*

(1) *M is m-relaxed η-accretive,*
(2) *$(A + \rho M)(X) = X$ for every $\rho > 0$.*

REMARK 3.1.

(1) If $m = 0$, then Definition 3.1 reduces to the definition of (H, η)-accretive operators due to [9].
(2) The definition of (I, η)-accretive operators is just that of generalized m-accretive operators due to [8,18] when $m = 0$.
(3) When $m = 0$ and $\eta(x, y) = x - y$ for all $x, y \in X$, Definition 3.1 reduces to the definition of H-accretive operators due to [7].
(4) When $m = 0$, $A = I$ and $\eta(x, y) = x - y$ for all $x, y \in X$, Definition 3.1 reduces to the definition of classical m-accretive operators.
(5) When $X = \mathcal{H}$, Definition 3.1 reduces to the definition of (A, η)-monotone operators, which is a new concept.

(6) When $X = \mathcal{H}$ and $\eta(x,y) = x - y$ for all $x, y \in \mathcal{H}$, Definition 3.1 reduces to the definition of A-monotone operators due to [4, 5].
(7) When $X = \mathcal{H}$, $m = 0$, Definition 3.1 reduces to the definition of (H,η)-monotone operators due to [6].
(8) When $X = \mathcal{H}$, $m = 0$ and $A = I$, Definition 3.1 reduces to the definition of maximal η-monotone operators due to [2].
(9) When $X = \mathcal{H}$, $m = 0$, $A = I$ and $\eta(x,y) = x - y$ for all $x, y \in \mathcal{H}$, Definition 3.1 reduces to the definition of classical maximal monotone operators (see [17]).

THEOREM 3.1. *Let $A : X \to X$ be a r-strongly η-accretive mapping, $M : X \to 2^X$ be an (A,η)-accretive mapping, and $x, u \in X$ be given points. If $\langle u - v,\ J_q(\eta(x,y))\rangle \geq 0$ holds for all $(y,v) \in \text{Graph}(M)$, where $\text{Graph}(M) = \{(a,b) \in X \times X : b \in M(a)\}$, then $(x,u) \in \text{Graph}(M)$.*

PROOF. Since M is (A,η)-accretive, $(A + \rho M)(X) = X$ holds for every $\rho > 0$. Then there exists $(x_0, u_0) \in \text{Graph}(M)$ such that

$$A(x_0) + \rho u_0 = A(x) + \rho u. \tag{3.1}$$

Since M is m-relaxed η-accretive and A is r-strongly η-accretive, we have

$$-m\,\|x - x_0\|^q \leq \rho\,\langle u - u_0,\ J_q(\eta(x,x_0))\rangle = -\langle A(x) - A(x_0),\ J_q(\eta(x,x_0))\rangle \leq -r\,\|x - x_0\|^q.$$

This implies that $x = x_0$. From (3.1), we know that $u = u_0$. Thus $(x,u) \in \text{Graph}(M)$. This completes the proof.

REMARK 3.2. Theorem 3.1 generalizes and improves (1) of Theorem 2.1 of [2], Proposition 2.1 of [3], Theorem 2.1 of [7], Theorem 3.1 of [6], and Theorem 3.1 of [9].

THEOREM 3.2. *Let $A : X \to X$ be a r-strongly η-accretive mapping, $M : X \to 2^X$ be an (A,η)-accretive mapping. Then, the operator $(A + \rho M)^{-1}$ is single-valued.*

PROOF. For any given $z \in X$, and $x, y \in (A + \rho M)^{-1}(z)$, it follows that $-A(x) + z \in \rho M(x)$ and $-A(y) + z \in \rho M(y)$. Since M is m-relaxed η-accretive and A is r-strongly η-accretive,

$$\begin{aligned} -m\,\|x - y\|^q &\leq \langle (-A(x) + z) - (-A(y) + z),\ J_q(\eta(x,y))\rangle \\ &= -\langle A(x) - A(y),\ J_q(\eta(x,y))\rangle \leq -r\,\|x - y\|^q. \end{aligned}$$

This implies that $x = y$. Thus $(A + \rho M)^{-1}$ is single-valued. The proof is completed.

REMARK 3.3. Theorem 3.2 generalizes and improves (2) of Theorem 2.1 of [2], Theorem 2.1 of [3], Theorem 2.2 of [7], Theorem 3.2 of [6], Proposition 2 of [16], and Theorem 3.2 in [9].

Based on Theorem 3.2, we can define the resolvent operator $R^{\rho,A}_{\eta,M}$ associated with an (A,η)-accretive mapping M as follows.

DEFINITION 3.2. *Let $A : X \to X$ be a strictly η-accretive mapping and $M : X \to 2^X$ be an (A,η)-accretive mapping. The resolvent operator $R^{\rho,A}_{\eta,M} : X \to X$ is defined by*

$$R^{\rho,A}_{\eta,M}(u) = (A + \rho M)^{-1}(u), \qquad \forall\, u \in X. \tag{3.2}$$

REMARK 3.4. Resolvent operators associated with (H,η)-accretive mappings include as special cases the corresponding resolvent operators associated with (H,η)-monotone operators [6], H-accretive operators [7], generalized m-accretive operators [8,18], maximal η-monotone operators [2], H-monotone operators [3], A-monotone operators [4], η-subdifferential operators [1,15], the classical m-accretive and maximal monotone operators [17].

The following theorem establishes Lipschitz continuity of resolvent operators associated with (A,η)-accretive mappings, which plays a prominent role in the resolvent operator technique associated with an (A,η)-accretive mapping.

THEOREM 3.3. *Let $\eta : X \times X \to X$ be τ-Lipschitz continuous, $A : X \to X$ be a r-strongly η-accretive mapping and $M : X \to 2^X$ be an (A, η)-accretive mapping. Then the resolvent operator $R^{\rho,A}_{\eta,M} : X \to X$ is $\tau^{q-1}/(r - \rho m)$-Lipschitz continuous, i.e.,*

$$\|R^{\rho,A}_{\eta,M}(x) - R^{\rho,A}_{\eta,M}(y)\| \le \frac{\tau^{q-1}}{r - \rho m}\|x - y\|, \qquad \forall x, y \in X,$$

where $\rho \in (0, r/m)$ is a constant.

PROOF. For given $x, y \in X$, from (3.2), we have

$$R^{\rho,A}_{\eta,M}(x) = (A + \rho M)^{-1}(x), \qquad R^{\rho,A}_{\eta,M}(y) = (A + \rho M)^{-1}(y).$$

It follows that

$$\frac{1}{\rho}\left(x - A\left(R^{\rho,A}_{\eta,M}(x)\right)\right) \in M\left(R^{\rho,A}_{\eta,M}(x)\right), \qquad \frac{1}{\rho}\left(y - A\left(R^{\rho,A}_{\eta,M}(y)\right)\right) \in M\left(R^{\rho,A}_{\eta,M}(y)\right).$$

Since M is m-relaxed η-accretive, we get

$$\begin{aligned}-m\left\|R^{\rho,A}_{\eta,M}(x) - R^{\rho,A}_{\eta,M}(y)\right\|^q &\le \frac{1}{\rho}\left\langle x - A\left(R^{\rho,A}_{\eta,M}(x)\right) - \left(y - A\left(R^{\rho,A}_{\eta,M}(y)\right)\right), J_q\left(\eta\left(R^{\rho,A}_{\eta,M}(y), R^{\rho,A}_{\eta,M}(y)\right)\right)\right\rangle \\ &= \frac{1}{\rho}\left\langle x - y - \left(A\left(R^{\rho,A}_{\eta,M}(x)\right) - A\left(R^{\rho,A}_{\eta,M}(y)\right)\right), J_q\left(\eta\left(R^{\rho,A}_{\eta,M}(x), R^{\rho,A}_{\eta,M}(y)\right)\right)\right\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\tau^{q-1}\|x - y\| \cdot \left\|R^{\rho,A}_{\eta,M}(x) - R^{\rho,A}_{\eta,M}(y)\right\|^{q-1} \\ &\ge \|x - y\| \cdot \left\|\eta\left(R^{\rho,A}_{\eta,M}(x), R^{\rho,A}_{\eta,M}(y)\right)\right\|^{q-1} \\ &\ge \left\langle x - y, J_q\left(\eta\left(R^{\rho,A}_{\eta,M}(x), R^{\rho,A}_{\eta,M}(y)\right)\right)\right\rangle \\ &\ge \left\langle A\left(R^{\rho,A}_{\eta,M}(x)\right) - A\left(R^{\rho,A}_{\eta,M}(y)\right), J_q\left(\eta\left(R^{\rho,A}_{\eta,M}(x), R^{\rho,A}_{\eta,M}(y)\right)\right)\right\rangle \\ &\quad - \rho m\left\|R^{\rho,A}_{\eta,M}(x) - R^{\rho,A}_{\eta,M}(y)\right\|^q \\ &\ge (r - \rho m)\left\|R^{\rho,A}_{\eta,M}(x) - R^{\rho,A}_{\eta,M}(y)\right\|^q.\end{aligned}$$

Therefore,

$$\left\|R^{\rho,A}_{\eta,M}(x) - R^{\rho,A}_{\eta,M}(y)\right\| \le \frac{\tau^{q-1}}{r - \rho m}\|x - y\|, \qquad \forall x, y \in X.$$

This completes the proof.

REMARK 3.5. Theorem 3.3 extends Theorem 3.3 of [9] and Lemma 2 of [5], and so extends Theorem 2.2 of [2], Theorem 2.2 of [3], Theorem 2.3 of [7], Theorem 3.3 of [6], Theorem 2.2 of [1], and Lemma 3 of [15].

4. NONLINEAR VARIATIONAL INCLUSIONS

In this section, by using resolvent operator technique associated with (A, η)-accretive mappings, we shall develop a new perturbed iterative algorithm with mixed errors for solving the class of nonlinear relaxed cocoercive variational inclusion problems in Banach spaces and prove the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm.

Let $A, S, V : X \to X$, $F : X \times X \to X$ be single-valued mappings and $M : X \to 2^X$ be an (A, η)-accretive mapping. For any given $a \in X$, $\lambda > 0$, we consider the problem of finding $x \in X$ such that

$$a \in F(S(x), V(x)) + \lambda M(x). \tag{4.1}$$

If $a = 0$, $\lambda = 1$ and $F(S(x), V(x)) = T(x)$ for all $x \in X$, where $T : X \to X$ is a single-valued mapping, then the problem (4.1) can be replaced to finding $x \in X$ such that

$$0 \in T(x) + M(x), \tag{4.2}$$

which is considered by Bi *et al.* [12].

If $X = X^* = \mathcal{H}$, $\eta(x, y) = x - y$ and $M = \Delta\varphi$, where $\Delta\varphi$ denotes the subdifferential of a proper convex lower semicontinuous function φ on $\mathcal{H}$, then the problem (4.2) becomes the following classical variational inequality.

Find $x \in X$ such that

$$\langle T(x), y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \qquad \forall y \in X.$$

DEFINITION 4.1. *Let S be a self-map of X, $x_0 \in X$, and let $x_{n+1} = h(S, x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^{\infty}$ in X. Suppose that $\{x \in X : Sx = x\} \neq \emptyset$ and $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point x^* of S. Let $\{u_n\} \subset X$ and let $\epsilon_n = \|u_{n+1} - h(S, u_n)\|$. If $\lim \epsilon_n = 0$ implies that $u_n \to x^*$, then the iteration procedure defined by $x_{n+1} = h(S, x_n)$ is said to be S-stable or stable with respect to S.*

LEMMA 4.1. *(See [20].) Let $\{a_n\}, \{b_n\}, \{c_n\}$ be three nonnegative real sequences satisfying the following condition: there exists a natural number n_0 such that*

$$a_{n+1} \leq (1 - t_n) a_n + b_n t_n + c_n, \qquad \forall n \geq n_0,$$

where $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} b_n = 0$, $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0 (n \to \infty)$.

From Definition 3.2, we can obtain the following conclusion.

LEMMA 4.2. *Let $A : X \to X$ be r-strongly η-accretive, $M : X \to 2^X$ is (A, η)-accretive and $F : X \times X \to X$ and $S, V : X \to X$ be any nonlinear mappings. Then, an element x is a solution to the problem (4.1) if and only if x satisfies*

$$x = R_{\eta,M}^{\rho\lambda,A}(A(x) - \rho(F(S(x), V(x)) - a)), \tag{4.3}$$

where $R_{\eta,M}^{\rho\lambda,A} = (A + \rho\lambda M)^{-1}$ and $\rho >$ is a constant.

REMARK 4.1. The equality (4.3) can be written as

$$\begin{aligned} z &= A(x) - \rho(F(S(x), V(x)) - a), \\ x &= R_{\eta,M}^{\rho\lambda,A}(z), \end{aligned}$$

where $\rho, \lambda > 0$ are constants. This fixed point formulation enables us to suggest the following iterative algorithm.

ALGORITHM 4.1. *For any given $z_0 \in X$, we choose $x_0 \in X$ such that*

$$x_0 = R_{\eta,M}^{\rho\lambda,A}(z_0).$$

Let

$$z_1 = (1 - \alpha_0) z_0 + \alpha_0 [A(x_0) - \rho(F(S(x_0), V(x_0)) - a)] + \alpha_0 e_0 + f_0.$$

For z_1, we take $x_1 \in X$ such that

$$x_1 = R_{\eta,M}^{\rho\lambda,A}(z_1).$$

Let

$$z_2 = (1 - \alpha_1) z_1 + \alpha_1 [A(x_1) - \rho(F(S(x_1), V(x_1)) - a)] + \alpha_1 e_1 + f_1.$$

Continuing this way, we can obtain sequence $\{x_n\}$ *satisfying*

$$\begin{aligned} x_n &= R_{\eta,M}^{\rho\lambda,A}(z_n), \\ z_{n+1} &= (1-\alpha_n) z_n + \alpha_n [A(x_n) - \rho(F(S(x_n), V(x_n)) - a)] + \alpha_n e_n + f_n, \end{aligned} \tag{4.4}$$

where $\rho, \lambda > 0$ *are constants,* $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *with* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and* $e_n, f_n \in X$ $(n \geq 0)$ *are errors to take into account a possible inexact computation of the resolvent operator point satisfying the following conditions.*

(i) $e_n = e'_n + e''_n$;
(ii) $\lim_{n\to\infty} \|e'_n\| = 0$;
(iii) $\sum_{n=0}^{\infty} \|e''_n\| < \infty$, $\sum_{n=0}^{\infty} \|f_n\| < \infty$.

Furthermore, let $\{u_n\}$ *be a sequence in* X *such that sequences* $\{\varepsilon_n\}$ *and* $\{u_n\}$ *satisfy*

$$\begin{aligned} \varepsilon_n &= \|u_{n+1} - \{(1-\alpha_n) u_n + \alpha_n [A(t_n) - \rho(F(S(t_n), V(t_n)) - a)] + \alpha_n e_n + f_n\}\|, \\ t_n &= R_{\eta,M}^{\rho\lambda,A}(u_n). \end{aligned} \tag{4.5}$$

Now, we prove the existence of a solution of problem (4.1) and the convergence of Algorithm 4.1.

THEOREM 4.1. *Let* X *be a* q*-uniformly smooth Banach space and* $A : X \to X$ *be* r*-strongly* η*-accretive and* σ*-Lipschitz continuous, respectively. Let* $S, V : X \to X$ *be* ξ*-Lipschitz continuous and* ζ*-Lipschitz continuous, respectively. Suppose that* $\eta : X \times X \to X$ *be* τ*-Lipschitz continuous and* $M : X \to 2^X$ *be an* (A,η)*-accretive. Let* $F : X \times X \to X$ *be* (s,ι)*-relaxed cocoercive with respect to* A *and* δ*-Lipschitz continuous in the first argument,* (d,ϱ)*-relaxed cocoercive with respect to* A *and* ϵ*-Lipschitz continuous in the second argument, respectively. If there exists a constant* $\rho > 0$ *such that*

$$\sigma^q - q\rho(\iota + \varrho) + q\rho(s\delta^q\xi^q + d\epsilon^q\zeta^q) + 2^q c_q \rho^q (\delta^q\xi^q + \epsilon^q\zeta^q) < \tau^{q(1-q)} (r - \rho\lambda m)^q, \qquad r > \rho\lambda m, \tag{4.6}$$

where c_q *is the constant as in Lemma 2.1, then*

(1) *the problem (4.1) has a unique solution* x^*;
(2) *the iterative sequence* $\{x_n\}$ *generated by Algorithm 4.1 converges strongly to* x^*;
(3) *if, in addition, there exists a* $\alpha > 0$ *such that* $\alpha_n \geq \alpha$ *for all* $n \geq 0$*, then*

$$\lim_{n\to\infty} u_n = x^* \qquad \text{if and only if } \lim_{n\to\infty} \varepsilon_n = 0,$$

where ε_n *is defined by (4.5).*

PROOF. From Lemma 4.2, for every $x \in X$, take

$$P(x) = R_{\eta,M}^{\rho\lambda,A}(A(x) - \rho(F(S(x), V(x)) - a)). \tag{4.7}$$

Then x^* is the unique solution of problem (4.1) if and only if x^* is the unique fixed point of P. In fact, it follows from the assumptions, (4.7), Theorem 3.3, and Lemmas 2.1 and 2.2 that

$$\begin{aligned} &\|P(x) - P(y)\| \\ &\quad = \left\| R_{\eta,M}^{\rho\lambda,A}(A(x) - \rho(F(S(x), V(x)) - a)) - R_{\eta,M}^{\rho\lambda,A}(A(y) - \rho(F(S(y), V(y)) - a)) \right\| \\ &\quad \leq \frac{\tau^{q-1}}{r - \rho m} \|A(x) - A(y) - \rho[F(S(x), V(x)) - F(S(y), V(y))]\|, \\ &\|A(x) - A(y) - \rho[F(S(x), V(x)) - F(S(y), V(y))]\|^q \\ &\quad \leq \|A(x) - A(y)\|^q + 2^q c_q \rho^q \|F(S(x), V(x)) - F(S(y), V(x))\|^q \\ &\qquad + 2^q c_q \rho^q \|F(S(y), V(x)) - F(S(y), V(y))\|^q \\ &\qquad - q\rho \langle F(S(x), V(x)) - F(S(y), V(x)), J_q(A(x) - A(y)) \rangle \\ &\qquad - q\rho \langle F(S(y), V(x)) - F(S(y), V(y)), J_q(A(x) - A(y)) \rangle \\ &\quad \leq [\sigma^q + 2^q c_q \rho^q (\delta^q\xi^q + \epsilon^q\zeta^q) - q\rho(\iota + \varrho - s\delta^q\xi^q - d\epsilon^q\zeta^q)] \|x - y\|^q. \end{aligned} \tag{4.8}$$

Thus,

$$\|P(x) - P(y)\| \leq \theta \|x - y\|,$$

where

$$\theta = \frac{\tau^{q-1}}{r - \rho m} \cdot \sqrt{\sigma^q + 2^q c_q \rho^q (\delta^q \xi^q + \epsilon^q \zeta^q) - q\rho (\iota + \varrho - s\delta^q \xi^q - d\epsilon^q \zeta^q)}.$$

It follows from (4.6) that $0 < \theta < 1$ and so $P : X \to X$ is a contractive mapping, i.e., P has a unique fixed point in X.

Next, let $z^* = A(x^*) - \rho(F(S(x^*), V(x^*)) - a)$ and $x^* = R_{\eta,M}^{\rho\lambda,A}(z^*)$. Then, by (4.4) and the proof of (4.8), we know that

$$\begin{aligned}
\|z_{n+1} - z^*\| &\leq (1 - \alpha_n) \|z_n - z^*\| + \alpha_n (\|e_n'\| + \|e_n''\|) + \|f_n\| \\
&\quad + \alpha_n \|A(x_n) - A(x^*) - \rho (F(S(x_n), V(x_n)) - F(S(x^*), V(x^*)))\| \\
&\leq (1 - \alpha_n) \|z_n - z^*\| + \alpha_n \|e_n'\| + (\|e_n''\| + \|f_n\|) \\
&\quad + \alpha_n \sqrt{\sigma^q + 2^q c_q \rho^q (\delta^q \xi^q + \epsilon^q \zeta^q) - q\rho (\iota + \varrho - s\delta^q \xi^q - d\epsilon^q \zeta^q)} \|x_n - x^*\|.
\end{aligned} \tag{4.9}$$

On the other hand, we find that

$$\|x_n - x^*\| \leq \left\|R_{\eta,M}^{\rho\lambda,A}(z_n) - R_{\eta,M}^{\rho\lambda,A}(z^*)\right\| \leq \frac{\tau^{q-1}}{r - \rho\lambda m} \|z_n - z^*\|. \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\|z_{n+1} - z^*\| \leq (1 - \alpha_n (1 - \theta)) \|z_n - z^*\| + \alpha_n (1 - \theta) \cdot \frac{1}{1 - \theta} \|e_n'\| + (\|e_n''\| + \|f_n\|). \tag{4.11}$$

Since $\sum_{n=0}^{\infty} \alpha_n = \infty$, it follows from Lemma 4.1 and (4.11) that $\|z_n - x^*\| \to 0 (n \to \infty)$. Hence, by (4.10), we know that the sequence $\{x_n\}$ converges to x^*.

Now, we prove the conclusion (3). By (4.5), we know

$$\begin{aligned}
&\|u_{n+1} - x^*\| \\
&\leq \|(1 - \alpha_n) u_n + \alpha_n [A(t_n) - \rho (F(S(t_n), V(t_n)) - a)] + \alpha_n e_n + f_n - x^*\| + \varepsilon_n.
\end{aligned} \tag{4.12}$$

As the proof of inequality (4.11), we have

$$\begin{aligned}
&\|(1 - \alpha_n) u_n + \alpha_n [A(t_n) - \rho (F(S(t_n), V(t_n)) - a)] + \alpha_n e_n + f_n - x^*\| \\
&\leq (1 - \alpha_n (1 - \theta)) \|u_n - z^*\| + \alpha_n (1 - \theta) \cdot \frac{1}{1 - \theta} \|e_n'\| + (\|e_n''\| + \|f_n\|).
\end{aligned} \tag{4.13}$$

Since $0 < \alpha \leq \alpha_n$, it follows from (4.12) and (4.13) that

$$\|u_{n+1} - x^*\| \leq (1 - \alpha_n (1 - \theta)) \|u_n - z^*\| + \alpha_n (1 - \theta) \cdot \frac{1}{1 - \theta} \left(\|e_n'\| + \frac{\varepsilon_n}{\alpha}\right) + (\|e_n''\| + \|f_n\|).$$

Suppose that $\lim \varepsilon_n = 0$. Then from $\sum_{n=0}^{\infty} \alpha_n = \infty$ and Lemma 4.1, we have $\lim u_n = x^*$.

Conversely, if $\lim u_n = x^*$, then we get

$$\begin{aligned}
\varepsilon_n &= \|u_{n+1} - \{(1 - \alpha_n) u_n + \alpha_n [A(t_n) - \rho (F(S(t_n), V(t_n)) - a)] + \alpha_n e_n + f_n\}\| \\
&\leq \|u_{n+1} - x^*\| + \|(1 - \alpha_n) u_n + \alpha_n [A(t_n) - \rho (F(S(t_n), V(t_n)) - a)] + \alpha_n e_n + f_n - x^*\| \\
&\leq \|u_{n+1} - x^*\| + (1 - \alpha_n (1 - \theta)) \|u_n - z^*\| + \alpha_n \|e_n'\| + (\|e_n''\| + \|f_n\|) \to 0
\end{aligned}$$

as $n \to \infty$. This completes the proof.

REMARK 4.1. If $e_n = 0$ or $f_n = 0$ $(n \geq 0)$ in Algorithm 4.1, then the conclusions of Theorem 4.1 also hold. The results of Theorem 4.1 improve and generalize the corresponding results of [1,20]. For other related works, we refer to [13,14,16].

REMARK 4.2. If X is 2-uniformly smooth Banach space and there exists $\rho \in (0, r/(\lambda m))$ such that

$$k = \delta^2\xi^2 + \epsilon^2\zeta^2 > \frac{\lambda^2 m^2}{4c_2\tau^2},$$

$$h = \iota + \varrho - s\delta^2\xi^2 - d\epsilon^2\zeta^2 > \frac{\sqrt{(\tau^2\sigma^2 - r^2)(4c_2k\tau^2 - \lambda^2m^2)} + \lambda m}{\tau^2},$$

$$\left|\rho - \frac{\tau^2 h - \lambda m}{4c_2k\tau^2 - \lambda^2m^2}\right| < \frac{\sqrt{(\tau^2h - \lambda m)^2 - (\tau^2\sigma^2 - r^2)(4c_2k\tau^2 - \lambda^2m^2)}}{4c_2k\tau^2 - \lambda^2m^2},$$

then (4.6) holds. We note that Hilbert space and L_p (or l_p) $(2 \leq p < \infty)$ spaces are 2-uniformly smooth Banach spaces.

REMARK 4.3. If M is a (H, η)-accretive operator, then we can obtain the corresponding results of Theorem 4.1. Our results improve and generalize the corresponding results of recent works.

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