Polynomial Sequences of Integral Type and Recursive Matrices

M. BARNABEI
Department of Mathematics, University of Bologna
Piazza di Porta S. Donato, 5, 40127 Bologna, Italy
barnabei@dm.unibo.it

Abstract—We show how the theory of recursive matrices—bi-infinite matrices in which each row can be recursively computed from the previous one—can be used to formulate a version of the umbral calculus that is also suited for the study of polynomials \( p(x) \) taking integer values when the variable \( x \) is an integer. In this way, most results of the classical umbral calculus—such as expansion theorems and closed formulas—can be seen as immediate consequences of the two main properties of recursive matrices, namely, the Product Rule and the Double Recursion Theorem. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords—Integral polynomials. Laurent series. Recursive matrices. Umbral calculus.

1. INTRODUCTION

Recursive matrices, which are bi-infinite matrices in which each row can be recursively computed from the previous one, were introduced and studied in the previous paper [1], and later utilized in [2] in order to formulate a characteristic-free version of the umbral calculus. Nevertheless, in the above works, the necessary need for technical precision negatively affected their understanding, making it difficult to justify the reason for their study.

Formulations of similar theories were later given by other authors (see [3–5]), with applications to various combinatorial problems, like lattice path enumeration. Recently, the recursive matrix machinery proved to be a useful tool in digital signal processing, and precisely in the study of algebraic aspects of \( M \)-channel filter banks. As a matter of fact, it turns out that (see [6,7]) linear operators that appear in the construction of filter banks are associated to special recursive matrices, and by making use of their properties, as proved in [1], it is possible both to find a formal explanation of properties of the above operators and to show new results.

The discovery of this application of the theory of recursive matrices gave rise to a need for a more understandable exposition, which is the goal of the present paper. To do so, we decided to start from the idea that originated the work in the first place (see [8]): that is, how to formulate an umbral calculus—as stated in [9,10]—for polynomials \( p(x) \) taking integer values when the variable \( x \) is an integer. It was shown that for the \( \mathbb{Z} \)-module of these polynomials, the sequence of powers \( x^n \) is not a basis, whether the Newton Expansion Theorem shows that a natural basis is the so-called "binomial basis", that is, the sequence of the polynomials \( \beta_n(x) - \binom{x}{n} \). Making
use of this basis, which does not satisfy the binomial recurrence, but the following:

$$\beta_n(x + y) = \sum_{k=0}^{n} \beta_k(x)\beta_{n-k}(y),$$

it comes natural to replace sequences of binomial type with integral polynomial sequences satisfying the recurrence

$$p_n(x + y) = \sum_{k=0}^{n} p_k(x)p_{n-k}(y).$$ \hfill (1)

It is immediately seen that sequences of real polynomial satisfying recursion (1) correspond bijectively to sequences of binomial type, via the map \((p_n(x))_{n\in\mathbb{N}} \rightarrow (n!p_n(x))_{n\in\mathbb{N}}\). However, recursion (1) allows us to highlight some properties of polynomial sequences which are not evident in the binomial case, mainly the fact that polynomial sequences of integral type can be characterized by means of the sequence of their coefficients with respect to the basis \((\beta_n(x))\), which satisfies the same kind of recursion.

If we decide to list in two infinite matrices the values of one of the above sequences on one side, and on the other the coefficients, it is seen that these two matrices share a common property, which is equivalent to recurrence (1); that is, each row can be calculated from the preceding one, by multiplying its generating function by a fixed formal power series. As a natural consequence, we studied the more general class of recursive matrices, namely, bi-infinite matrices which are characterized by such a property. These matrices have been defined allowing their elements to belong to a commutative ring, with unity, of any characteristic, and their properties hold in this general setting. However, for the sake of simplicity, we decided to state in the present paper definitions and results in the particular case of the ring of integers, which in a certain way can be considered the universal commutative ring.

Once the main properties of recursive matrices are studied, it is possible to formulate the basic notions of umbral calculus in a more general setting. This way, we are enabled to state the fact that the class of Sheffer sequences is the intersection of two wider classes, the one of recursive sequences (whose coefficient matrix with respect to the binomial basis is recursive), and the one of sequences that are associated to particular linear operators (whose coefficient matrix with respect to the binomial basis is the transpose of a recursive matrix). A fundamental property of recursive matrices, namely, the Double Recursion Theorem, which gives the conditions under which a certain matrix is recursive by rows and columns at the same time, and is shown to be equivalent to the Lagrange inversion formula, characterizes the intersection of the above classes. The same theorem, from a different point of view, recovers two key results of umbral calculus, stated in [10,11]: the first of the two yields the connection intercurring between the indicators of a delta operator and of the Sheffer sequences associated to it, while the second (the Transfer Formula) is a closed formula which allows us to compute directly the \(n^{th}\) polynomial of any Sheffer sequence associated with a given delta operator.

2. INTEGRAL POLYNOMIALS

We will be concerned with the \(\mathbb{Z}\)-module \(P_I\) of integral polynomials, namely, real polynomial functions \(p(x)\) such that \(p(x)\) is an integer for every \(x \in \mathbb{Z}\).

First of all, we point out that \(P_I\) does not coincide with the \(\mathbb{Z}\)-module \(\mathbb{Z}[x]\) of polynomials with integer coefficients. For example, for every nonnegative integer \(n\), consider the polynomial

$$\beta_n(x) - \binom{x}{n} - \frac{x(x-1)\cdots(x-n+1)}{n!}.$$
Note that for every $n \geq 1$, $\beta_n(x)$ is an integral polynomial of degree $n$, whose roots are the integers $0, 1, \ldots, n - 1$. Except for $\beta_0(x) = 1$ and $\beta_1(x) = x$, these polynomials cannot be written as a linear combination with integer coefficients of the powers $x^i$.

Hence, the standard basis $(x^n)_{n \in \mathbb{N}}$ of the $\mathbb{Z}$-module $\mathbb{Z}[x]$ is not a basis for the $\mathbb{Z}$-module of integral polynomials. We will see that a basis for $\mathcal{P}_I$ is given by the sequence of polynomials $(\beta_n(x))_{n \in \mathbb{N}}$.

Denote by $\mathcal{F}$ the set of all functions $f : \mathbb{Z} \to \mathbb{Z}$. The forward difference operator $\Delta : \mathcal{F} \to \mathcal{F}$ is defined as

$$\Delta f(x) = f(x + 1) - f(x).$$

Note that

$$\Delta \beta_n(x) = \beta_{n-1}(x),$$
$$\Delta \beta_0(x) = 0.$$

Integral polynomials can be characterized in terms of the operator $\Delta$, by means of the following classical result.

**Theorem 2.1. Newton Expansion.** A function $f \in \mathcal{F}$ is an integral polynomial whenever

$$\Delta^k f(x) = 0,$$

for some integer $k \geq 0$. If this is the case, then the following expansion formula holds:

$$f(x) = \sum_{i \geq 0} (\Delta^i f)(0) \beta_i(x).$$

If $f$ is an integral polynomial such that $\Delta^{n+1} f(x) = 0$, while $\Delta^n f(x) \neq 0$, then the integer $n$ is called the degree of $f$, and denoted by $\text{deg}(f)$.

Theorem 2.1 shows that the sequence of integral polynomials $(\beta_n(x))_{n \in \mathbb{N}}$ is a basis for the $\mathbb{Z}$-module $\mathcal{P}_I$, called the binomial basis.

### 3. Polynomial Sequences of Integral Type

As we have seen, the binomial basis plays a central role in the theory of integral polynomials. Such a basis satisfies the recursion

$$\beta_n(x + y) = \sum_{k=0}^{n} \beta_n(x) \beta_{n-k}(y).$$

Every other basis of the $\mathbb{Z}$-module $\mathcal{P}_I$ satisfying the same kind of recursion is called a polynomial sequence of integral type. More precisely, a sequence of integral polynomials $(p_n(x))_{n \in \mathbb{N}}$ is a polynomial sequence of integral type whenever it satisfies the following conditions.

- **Basis Conditions.** The sequence $(p_n(x))_{n \in \mathbb{N}}$ is a basis of the $\mathbb{Z}$-module $\mathcal{P}_I$, namely:
  (i) for every nonnegative integer $n$, the polynomial $p_n(x)$ has degree $n$, and
  (ii) $p_n(x)$ is monic (i.e., the leading coefficient in its expansion with respect to the basis $(\beta_k(x))$ equals 1 or $-1$).
Recursion. For every integer \( x \) and \( y \),

\[
p_n(x + y) = \sum_{k=0}^{n} p_k(x)p_{n-k}(y).
\]

As an immediate consequence of the definition, we get that if \( (p_n(x)) \) is a polynomial sequence of integral type, then \( p_0(x) = 1 \) for every \( x \in \mathbb{Z} \). Moreover, it can be easily proved by induction that \( p_n(0) = 0 \) for \( n \geq 1 \).

**Example 3.1.** The sequence \( (\gamma_n(x))_{n \in \mathbb{N}} \), where

\[
\gamma_n(x) = \binom{x + n - 1}{n} = (-1)^n \binom{-x}{n} = \frac{x(x+1)\cdots(x+n-1)}{n!},
\]

is a polynomial sequence of integral type. We recall that, when \( x \) is a natural number, \( \gamma_n(x) \) gives the number of multisets of length \( n \) on a set of \( x \) elements.

It is now worthwhile to make some considerations. First of all, it is immediately seen that if \( (p_n(x))_{n \in \mathbb{N}} \) is a polynomial sequence of integral type, then the sequence \( (n! \ p_n(x))_{n \in \mathbb{N}} \) satisfies the binomial recursion, and is, therefore, of binomial type. Conversely, if \( (q_n(x))_{n \in \mathbb{N}} \) is a polynomial sequence of binomial type, the sequence \( \left(\frac{q_n(x)}{n!}\right)_{n \in \mathbb{N}} \) satisfies recursion (1). Hence, sequences of real polynomial satisfying recursion (1)—which will be called real sequences of integral type—correspond bijectively to sequences of binomial type. However, on the one hand, recursion (1) seems to be more suited to the study of an analog of the umbral calculus for integral polynomials. On the other hand, it allows us to highlight some properties of polynomial sequences which become clear as soon as one considers the “integral” recursion and the natural basis related thereby, namely, the basis \( (\beta_n(x))_{n \in \mathbb{N}} \) of binomial coefficients.

To begin with, we analyze the behavior of the coefficients of a polynomial sequence of integral type \( (p_n(x))_{n \in \mathbb{N}} \) with respect to the binomial basis. For every nonnegative integer \( n \), set

\[
p_n(x) = \sum_{k=0}^{n} c_{k,n} \beta_k(x), \quad c_{k,n} \in \mathbb{Z}.
\]

Since \( p_0(x) = 1 = \beta_0(x) \), and \( p_n(0) = 0 \) for \( n \geq 1 \), we have

\[
c_{0,0} = 1, \quad c_{n,0} = 0, \quad \text{for } n \geq 1,
\]

and, for every \( n > 1 \),

\[
c_{0,n} = 0, \quad c_{1,n} = p_n(1).
\]

The next result—proved in [8]—shows that polynomial sequences of integral type can be characterized by means of the sequence of their coefficients with respect to the basis \( (\beta_n(x)) \), which satisfies the same kind of recursion.

**Theorem 3.1.** A sequence \( (p_n(x))_{n \in \mathbb{N}} \) of integral, monic polynomials, with \( \deg(p_n) = n \) for every \( n \geq 0 \), and

\[
p_n(x) = \sum_{k=0}^{n} c_{k,n} \beta_k(x), \quad c_{k,n} \in \mathbb{Z},
\]

is a polynomial sequence of integral type if and only if the sequence \( (c_{k,n}) \) of coefficients satisfies the following recursion:

\[
c_{h+k,n} = \sum_{i=0}^{h} c_{h,i} c_{k,n-i}.
\]
By Theorem 3.1, both coefficients $c_{k,n}$ and values $p_n(x)$ of a polynomial sequence of integral type can be recursively computed starting from the values at 1 of each polynomial, as follows:

$$c_{1+k,n} = \sum_{i=1}^{n} c_{1,i} c_{k,n-i}, \quad (4)$$

$$p_n(x + 1) = \sum_{i=0}^{n} p_i(1)p_{n-i}(x) = p_n(x) + \sum_{i=1}^{n} c_{1,i} p_{n-i}(x). \quad (5)$$

This implies that the values at 1 of the polynomials of a sequence of integral type—or, equivalently, the first coefficient in the expansion of each polynomial of the sequence in terms of the basis $(\beta_k(x))$—can be arbitrarily chosen.

**Corollary 3.1.** Let $c_1 = \pm 1, c_2, c_3, \ldots$ be a given sequence of integers. There exists a unique polynomial sequence of integral type $(p_n(x))_{n \in \mathbb{N}}$ such that $p_n(1) = c_n$ for every $n$. 

The preceding results can be equivalently expressed in matrix notation, as follows. If $(p_n(x))$ is a sequence of integral polynomials, with $p_n(x) = \sum_{k=0}^{n} c_{k,n} \beta_k(x)$, we associate to $(p_n(x))$ two infinite matrices with integer elements, namely,

- the value tableau $V$ of $(p_n(x))$, defined as $V := [p_j(i)]$, $i \in \mathbb{Z}$, $j \in \mathbb{N}$, and
- the canonical matrix $C$ of $(p_n(x))$, defined as $C := [c_{i,j}]$, $i, j \in \mathbb{N}$.

We recall that, given a matrix $M = [m_{ij}]$, the generating function of the $i^{th}$ row of $M$ is defined as the formal series

$$M(i) := \sum_{i} m_{ij} t^j,$$

where $t$ is a formal variable.

Consider now the generating function of the $i^{th}$ row of the matrix $C$, $i \geq 1$, namely, the formal power series

$$C(i) = \sum_{j \geq 0} c_{i,j} t^j.$$

If $(p_n(x))$ is a polynomial sequence of integral type, by identity (4), we get

$$C(i) = \sum_{j \geq 0} \left( \sum_{k=1}^{j} c_{1,k} c_{i-1,j-k} \right) t^j,$$

and this is the convolution product between the two power series $C(1)$ and $C(i - 1)$. Hence, for every positive index $i$, we get

$$C(i) = (C(1))^i.$$

Similarly, by (5), the generating function of the $i^{th}$ row of $V$ is

$$V(i) = \sum_{k \geq 0} p_k(i) t^k = \sum_{k \geq 0} \left( \sum_{j=0}^{k} p_j(1)p_{n-j}(i-1) \right) t^k.$$

This is the convolution product between $V(1)$ and $V(i - 1)$. Since, by formulas (2) and (3), we can write $V(1) = 1 + \sum_{k \geq 1} c_{1,j} t^k = 1 + C(1)$, this gives

$$V(i) = (1 + C(1))^i.$$

The formal power series $\phi := C(1)$ is called the indicator of the sequence $(p_n(x))$. We point out that, by (3), the indicator $\phi$ is of the form

$$\phi = \sum_{k \geq 1} c_{k,1} t^k = \sum_{k \geq 1} p_k(1) t^k = \pm t + \sum_{k \geq 2} p_k(1) t^k.$$
EXAMPLE 3.2. The indicator of the sequence \((\beta_n(x))\) is

\[
\phi = \sum_{k \geq 1} \beta_k(1)t^k = t.
\]

EXAMPLE 3.3. The indicator of the sequence \((\gamma_n(x))\) is

\[
\phi = \sum_{k \geq 1} \gamma_k(1)t^k = t + t^2 + t^3 + \cdots = \frac{t}{1-t}.
\]

The preceding considerations allow us to state the following matrix characterization of polynomial sequences of integral type.

**Theorem 3.2.** A sequence \((p_n(x))\) of integral polynomials with value tableau \(V\) and canonical matrix \(C\) is a polynomial sequence of integral type whenever

- every entry in the 0th column of the matrix \(V\) equals 1,
- every entry in the 0th column of the matrix \(C\) equals 0, except for the 0th entry, which equals 1, and
- both matrices \(V\) and \(C\) satisfy the following conditions:
  1. the generating function of the 0th row is the series 1;
  2. if \(\alpha\) denotes the generating function of the first row, then the generating function of the \(i\)th row equals \(\alpha^i\) for every index \(i\);
  3. the \((1,1)\)-entry equals \(\pm 1\).

4. **RECURSIVE MATRICES**

In this section, we introduce the notion of recursive matrix, whose definition is based upon the main property shared by the value tableau and canonical matrix of a polynomial sequence of integral type, namely, the fact that the generating function of each row can be obtained by the product of the preceding row with a fixed series. For the sake of simplicity, we will develop our consideration over the ring \(\mathbb{Z}\) of integers, but we remark that \(\mathbb{Z}\) can be replaced by any commutative ring, with unity, as shown in [1].

Let \(\mathcal{L}^+\) be the ring of Laurent series over \(\mathbb{Z}\), endowed with the usual sum and convolution product

\[
\mathcal{L}^+ = \left\{ \alpha = \sum_{n \geq d} a_n t^n \mid a_n \in \mathbb{Z}, \ a_d \neq 0 \right\}
\]

The integer \(d\) will be called the “plus” degree of the Laurent series \(\alpha\) and denoted by the symbol \(\deg^+(\alpha)\).

Obviously, a Laurent series of nonnegative degree is nothing but a formal power series. It is well known that a series \(\alpha \in \mathcal{L}^+\) is invertible with respect to the convolution product whenever its first coefficient equals 1 or \(-1\). The set of such series will be denoted by \(\mathcal{R}^+\).

We will also need to consider the ring \(\mathcal{L}^-\) of inverse Laurent series

\[
\mathcal{L}^- = \left\{ \psi = \sum_{n \leq h} a_n t^n \mid a_n \in \mathbb{Z}, \ a_h \neq 0 \right\}
\]

The integer \(h\) will be called the “minus” degree of the inverse Laurent series \(\psi\) and denoted by the symbol \(\deg^- (\psi)\). The star operator

\[
\alpha = \sum_n a_n t^n \mapsto \alpha^* = \sum_n a_{-n} t^n
\]

maps a Laurent series into an inverse Laurent series, and conversely.
Denote by $\mathcal{M}$ the set of bi-infinite matrices over the field $K$, namely, matrices $M : \mathbb{Z} \times \mathbb{Z} \rightarrow K$. Given two nonzero Laurent series $\alpha = \sum_{j \geq d} a_j t^j$, $\beta = \sum_{j \geq h} b_j t^j$, with $\alpha \in \mathbb{R}^+$, the unique matrix $M \in \mathcal{M}$ such that, for every integer $i$, the generating function $M(i)$ of its $i$th row has the following expression:

$$M(i) = a_i / \beta$$

is called the $(\alpha, \beta)$-recursive matrix, and denoted by the symbol $R(\alpha, \beta)$. The series $\alpha$ is called the recurrence rule of $M$, while $\beta$ is called the boundary value of $M$.

Equivalently, the matrix $M = [m_{ij}]$ is the $(\alpha, \beta)$-recursive matrix whenever, for every integer $j$, we have

$$m_{0j} = b_j,$$

and for every pair of integers $i, j$, the element $m_{i,j}$ can be recursively computed from the elements of the preceding row of $M$ as follows:

$$m_{i,j} = \sum_{s \geq d} a_s m_{i-1,j-s},$$

or, equivalently, it can be computed from the elements of the following row of $M$ in this way:

$$m_{i,j} = \sum_{s \geq -d} \bar{a}_s m_{i+1,j+s},$$

where the $\bar{a}_s$ are the coefficients of the series $\alpha^{-1}$, the inverse of $\alpha$. Note that the bi-infinite identity matrix is the recursive matrix with recurrence rule $t$ and boundary value $1$, namely, $R(t, 1)$.

If $R(\alpha, \beta) = [m_{ij}]$, $i, j \in \mathbb{Z}$, is a recursive matrix, we will denote by $VR(\alpha, \beta)$ the east submatrix of $R(\alpha, \beta)$, namely, the matrix $[m_{ij}]$, $i \in \mathbb{Z}$, $j \in \mathbb{N}$, and by $MR(\alpha, \beta)$ the southeast minor of $R(\alpha, \beta)$, namely, the matrix $[m_{ij}]$, $i, j \in \mathbb{N}$.

If $(p_n(x))$ is a polynomial sequence of integral type, by Theorem 3.2, its value tableau is the east submatrix of the recursive matrix $R(1 + \phi, 1)$, while its canonical matrix is the southeast minor of the recursive matrix $R(\phi, 1)$, where $\phi$ is the indicator of the sequence. Conversely, if $\alpha$ is a Laurent series of degree 1, whose first coefficient equals 1 or $-1$, then $VR(1 + \alpha, 1)$ is the value tableau of some polynomial sequence of integral type $(p_n(x))$, with canonical matrix $MR(\alpha, 1)$.

**Example 4.1.** The Pascal triangle is the value tableau of the polynomial sequence $(p_n(x))$, and it is, therefore, the east submatrix of the recursive matrix $R(1 + t, 1)$, namely, the following matrix, whose $0^{th}$ row and column are marked by a double bar:

```
[ ... ... ... ... ... ... ... ... ... ... ... ... ... ... ... ... ]
[ ... 0 0 1 -4 10 -20 35 -56 84 ... ]
[ ... 0 0 1 -3 6 -10 15 -21 28 ... ]
[ ... 0 0 1 -2 3 -4 5 -6 7 ... ]
[ ... 0 0 1 -1 1 -1 1 -1 1 ... ]
[ ... 0 0 1 0 0 0 0 0 0 ... ]
[ ... 0 0 1 1 0 0 0 0 0 ... ]
[ ... 0 0 1 2 1 0 0 0 0 ... ]
[ ... 0 0 1 3 3 1 0 0 0 ... ]
[ ... 0 0 1 4 6 4 1 0 0 ... ]
[ ... 0 0 1 5 10 10 5 1 0 ... ]
[ ... 0 0 1 6 15 20 15 6 1 ... ]
[ ... 0 0 1 7 21 35 35 21 7 ... ]
[ ... 0 0 1 8 28 56 70 56 28 ... ]
[ ... ... ... ... ... ... ... ... ... ... ... ... ... ... ... ... ]
```
EXAMPLE 4.2. The indicator of the polynomial sequence \( (\gamma_n(x)) \) is the series \( t/(1-t) \), as shown in Example 3.3. Hence, its value tableau is the east submatrix of the recursive matrix \( R(1/(1-t), 1) \),

\[
\begin{bmatrix}
\ldots & \
\ldots & \
\ldots & \
\ldots & \
\ldots & \
\ldots & \\
\ldots & 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & -3 & 3 & -1 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \ldots \\
\ldots & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
\ldots & 0 & 0 & 1 & 3 & 6 & 10 & 15 & 21 & 28 & \ldots \\
\ldots & 0 & 0 & 1 & 4 & 10 & 20 & 35 & 56 & 84 & \ldots \\
\ldots & 0 & 0 & 1 & 5 & 15 & 37 & 70 & 126 & 210 & \ldots \\
\ldots & 0 & 0 & 1 & 6 & 21 & 56 & 126 & 252 & 462 & \ldots \\
\ldots & 0 & 0 & 1 & 7 & 28 & 84 & 210 & 462 & 924 & \ldots \\
\ldots & 0 & 0 & 1 & 8 & 36 & 120 & 330 & 792 & 1716 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

EXAMPLE 4.3. Several kinds of bi-infinite matrices which are widely used in numerical applications turn out to be recursive matrices. We give here two examples.

(a) If \( \beta = \sum_{i \geq d} b_i t^i \) is any Laurent series, the recursive matrix \( R(t, \beta) \) is the Toeplitz matrix with generating function \( \beta \),

\[
\begin{bmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \ldots \\
\ldots & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \ldots \\
\ldots & b_{-3} & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\
\ldots & b_{-4} & b_{-3} & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\
\ldots & b_{-5} & b_{-4} & b_{-3} & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & \ldots \\
\ldots & b_{-6} & b_{-5} & b_{-4} & b_{-3} & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

(b) If \( \beta = \sum_{i \geq d} b_i t^i \) is any Laurent series, the recursive matrix \( R(t^{-1}, \beta) \) is the Hankel matrix with generating function \( \beta \),

\[
\begin{bmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & b_{-6} & b_{-5} & b_{-4} & b_{-3} & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & \ldots \\
\ldots & b_{-5} & b_{-4} & b_{-3} & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & \ldots \\
\ldots & b_{-4} & b_{-3} & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\
\ldots & b_{-3} & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \ldots \\
\ldots & b_{-2} & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \ldots \\
\ldots & b_{-1} & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

The main property of recursive matrices is the fact that—under suitable conditions—they can be multiplied, and the product is again a recursive matrix. More precisely, we have (see \([1]\)) the following theorem.

**Theorem 4.1. Product Rule.** Let \( \alpha, \beta, \delta \) be nonzero Laurent series, and let \( \gamma \) be a Laurent series of positive degree. Then,

\[
R(\alpha, \beta) \times R(\gamma, \delta) = R(\alpha \circ \gamma, (\beta \circ \gamma)\delta),
\]

where \( \alpha \circ \gamma \) denotes the functional composition of the series \( \alpha, \gamma \).

\[ \tag{9} \]
We recall that a Laurent series $\alpha$ admits an inverse $\tilde{\alpha}$ with respect to functional composition whenever $\alpha \in \mathcal{R}^+$ and $\deg^+(\alpha) = 1$. Hence, by the Product Rule, if $\alpha, \beta \in \mathcal{R}^+$ and $\deg^+(\alpha) = 1$, the recursive matrix $R(\alpha, \beta)$ is invertible, and its inverse is the recursive matrix

$$R(\alpha, \beta)^{-1} = R\left(\tilde{\alpha}, (\beta \circ \tilde{\alpha})^{-1}\right). \tag{10}$$

## 5. RECURSIVE SEQUENCES OF INTEGRAL POLYNOMIALS

Polynomial sequences of integral type are associated with recursive matrices whose boundary value is the series 1. In this section, we define and study recursive sequences, namely, sequences of integral polynomials which are associated with more general recursive matrices.

More precisely, a sequence $(f_n(z))_{n \in \mathbb{N}}$ of integral polynomials will be called a recursive sequence if there exists a formal power series $\alpha = \sum_{n \geq 0} a_n t^n$, with $a_0 = 1$, such that

$$f_n(z + 1) = \sum_{k=0}^n a_k f_{n-k}(z). \tag{11}$$

It is easily checked that if $(f_n(z))$ is a recursive sequence, formula (11) implies that $f_0(z)$ is a constant polynomial.

The Laurent series $\phi := \alpha - 1$ is called the indicator of $(f_n(z))$, while the series $\delta := \sum_{n \geq 0} f_n(0) t^n$ is its boundary value.

As done for polynomial sequences of integral type, we associate to any recursive sequence $(f_n(z))$ two matrices:

- the value tableau $V : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z}$, whose $(i, j)^{\text{th}}$ entry is $f_j(i)$;
- the canonical matrix $C : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$, whose $(i, j)^{\text{th}}$ entry is the $i^{\text{th}}$ coefficient of $f_j(x)$ with respect to the basis $(\beta_n(z))$.

By formula (11), the value tableau of $(f_n(z))$ is the east submatrix $V R(\phi + 1, \delta)$ of the recursive matrix $R(\phi + 1, \delta)$, where $\phi$ is the indicator of $(f_n(z))$, and $\delta$ its boundary value.

Moreover, by the Product Rule, we get

$$R(\phi + 1, \delta) = R(1 + t, 1) \times R(\phi, \delta). \tag{12}$$

Since $R(1 + t, 1)$ is the value tableau of the basis $(\beta_i(z))$, this implies that the canonical matrix of $(f_n(z))$ is the southeast minor $MR(\phi, \delta)$ of the recursive matrix $R(\phi, \delta)$.

A polynomial sequence of integral type $(p_n(z))$ is a recursive sequence with boundary value 1 and indicator of degree 1. Moreover, its indicator belongs to $\mathcal{R}^+$, since its first coefficient equals $\pm 1$.

We define a homogeneous recursive sequence to be any recursive sequence whose boundary value is the series 1. Polynomial sequences of integral type are therefore particular homogeneous sequences. We point out that homogeneous recursive sequences do not—in general—satisfy the requirement $\deg(f(z)) = n$, as shown in Example 5.1 below. However, they can be characterized by means of the same recursion of polynomial sequences of integral type. In fact, their definition implies immediately the following theorem.

**Theorem 5.1.** A sequence $(f_n(z))$ of integral polynomials is a recursive homogeneous sequence whenever $f_0(z) = 1$ and, for every integer $x$ and $y$,

$$f_n(x + y) = \sum_{k=0}^n f_k(x) f_{n-k}(y). \quad \blacksquare$$

Given a recursive sequence $(f_n(z))$, the associated homogeneous sequence $(p_n(z))$ is defined as the (unique) homogeneous recursive sequence with the same indicator $\phi$ as $(f_n(z))$. This implies that the value tableau of $(p_n(z))$ is the matrix $VR(\phi + 1, 1)$. The next result shows that the notion of recursive sequence is strictly related to the notion of Sheffer sequence.
Theorem 5.2. A sequence \((f_n(x))\) of integral polynomials is a recursive sequence whenever there exists a homogeneous recursive sequence \((p_n(x))\) such that, for every integer \(x\) and \(y\),

\[
f_n(x + y) = \sum_{k=0}^{n} p_k(x)f_{n-k}(y).
\]

Proof. It is sufficient to remark that, by the Product Rule, we get

\[
R(\phi \circ 1, \delta) = R(\phi \circ 1, 1) \times R(t, \delta).
\]

Example 5.1. Consider the sequence \((f_n(x))\) defined as follows:

\[
\begin{align*}
    f_0(x) &= 1, \\
    f_{2n}(x) &= \beta_n(x) + \beta_{n+1}(x), \quad n \geq 1, \\
    f_{2n+1}(x) &= \beta_n(x), \quad n \geq 0.
\end{align*}
\]

The canonical matrix of \((f_n(x))\) is

\[
\begin{bmatrix}
    1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
    0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

This is the southeast minor of the recursive matrix \(R(t^2, 1 + t + t^2)\). Hence, \((f_n(x))\) is a recursive sequence, with indicator \(t^2\) and boundary value \(1 + t + t^2\). The associated homogeneous sequence \((p_n(x))\) is

\[
\begin{align*}
    p_{2n}(x) &= \beta_n(x), \quad n \geq 1, \\
    p_{2n+1}(x) &= 0, \quad n \geq 0.
\end{align*}
\]

As we have already remarked, the \(n\)th polynomial of a recursive sequence does not have, in general, degree exactly \(n\), and hence, the recursive sequence is not a basis of \(\mathcal{P}_t\). However, any recursive sequence spans the \(\mathbb{Z}\)-module \(\mathcal{P}_t\), provided that its indicator and boundary value belong to \(\mathbb{R}^+\). In fact, we have the following theorem.

Theorem 5.3. Let \((f_n(x))\) be a recursive sequence with indicator \(\phi\) and boundary value \(\delta\). Then, for every nonnegative integer \(d\), there is at least one index \(n_d\) such that \(\deg(f_{n_d}) = d\). Therefore, \((f_n(x))\) is a spanning set for the \(\mathbb{Z}\)-module \(\mathcal{P}_t\) whenever both \(\phi\) and \(\delta\) belong to \(\mathbb{R}^+\).

Proof. It is sufficient to recall that the indicator \(\phi\) of a recursive sequence \((f_n(x))\) is a formal power series of positive degree, and its boundary value \(\delta\) is a nonzero formal power series. This implies that, in the recursive matrix \(R(\phi, \delta)\), for every nonnegative integer \(d\), there is at least one column whose last nonzero element is in row \(d\); namely, the corresponding polynomial has degree \(d\).

As a consequence of the preceding result, a recursive sequence \((f_n(x))\) is a basis of \(\mathcal{P}_t\) whenever \(\phi, \delta \in \mathbb{R}^+\) and \(\deg^+(\phi) = 1\). In this case, \((f_n(x))\) is called a recursive basis. Equivalently, \((f_n(x))\) is a recursive basis whenever its canonical matrix is the southeast minor of an invertible recursive matrix.

Needless to add, recursive bases are the integral analogs of Sheffer sequences, and the Umbral Composition Theorem [10, Theorem 7] can now be seen as an immediate consequence of the Product Rule for recursive matrices.
6. RECURSIVE OPERATORS

The classical umbral calculus is based upon the interplay between sequences of binomial type and two classes of linear operators, namely, umbral operators and shift-invariant operators. In the present setting, we can show how both these classes can be recovered under the more general notion of recursive operator, which is once again strictly related to that of recursive matrix. As before, we develop our considerations for integral polynomials, but our definitions and results are still valid in the case of real or complex polynomials.

A linear operator \( L : \mathcal{P}_I \rightarrow \mathcal{P}_I \) such that the sequence \( (L\beta_n(x)) \) is a recursive sequence is called a recursive operator. Equivalently, a linear operator \( L \) is a recursive operator whenever the canonical matrix of \( L \), namely, the matrix associated with \( L \) with respect to the binomial basis \( (\beta_n(x)) \), is the southeast minor of a recursive matrix.

We recall that an umbral operator is defined as a linear operator which maps any polynomial sequence of binomial type into a basis of the vector space \( \mathbb{R}[x] \) satisfying the same recursion. Since a sequence \( (p_n(x)) \) is of binomial type if and only if \( (p_n(x)/n!) \) is a real sequence of integral type, we get that every umbral operator \( U : \mathcal{P}_I \rightarrow \mathcal{P}_I \) is a recursive operator.

Note that, if \( L \) is a recursive operator with canonical matrix \( MR(\alpha, \beta) \), then \( \alpha \) and \( \beta \) are the indicator and the boundary value of the recursive sequence \( (L\beta_n(x)) \), respectively. As a consequence, by Theorem 5.3, the recursive operator \( L \) is invertible (and will be called an integral umbral operator) whenever \( \alpha, \beta \in \mathcal{R}^+ \) and \( \text{deg}^+(\alpha) = 1 \).

**EXAMPLE 6.1.**

(a) The shift operator \( E \)

\[
Ef(x) = f(x + 1)
\]

has canonical matrix \( MR(t, 1 + t) \), and is, therefore, an integral umbral operator.

(b) The duality operator \( L \)

\[
Lf(x) = f(-x)
\]

is an integral umbral operator, with canonical matrix \( MR(-t/(1 + t), 1) \), since

\[
L\beta_n(x) = \binom{-x}{n} = (-1)^n \gamma_n(x).
\]

(c) Another example of an integral umbral operator is given by the operator \( S \), defined as follows:

\[
S\beta_n(x) = (-1)^n \beta_n(x),
\]

whose canonical matrix is \( MR(-t, 1) \).

(d) As a consequence, the operator \( LS \)—namely, the functional composition of the operators \( L, S \) previously defined—is an integral umbral operator, and its canonical matrix is given by

\[
MR\left(\frac{-t}{1 + t}, 1\right) \times MR(-t, 1) = MR\left(\frac{t}{1 + t}, 1\right),
\]

which is also the canonical matrix of the sequence \( (\gamma_n(x)) \). This implies that

\[
LS\beta_n(x) = \gamma_n(x).
\]

In the classical umbral calculus, a central role is played by shift-invariant operators, defined as those operators commuting with the shift operator \( E \), or, equivalently, with the forward difference operator \( \Delta \), which can be written as

\[
\Delta = E - I.
\]

Shift-invariant operators can now be equivalently described as those recursive operators whose canonical matrix is a Toeplitz matrix \( MR(t, \delta) \), where \( \delta \) is a formal power series. In fact, we have the following result.
THEOREM 6.1. Let \( L \) be a recursive operator with canonical matrix \( MR(t, \delta) \), \( \delta := \sum_{h \geq 0} d_h t^h \). Then we have
\[
L = \delta(\Delta) = \sum_{h \geq 0} d_h \Delta^h.
\]
As a consequence, \( L \) commutes with the operator \( \Delta \). Conversely, if \( L \) is a shift-invariant operator, then its canonical matrix is a Toeplitz matrix.

PROOF. Suppose that the canonical matrix \( [m_{ij}] \) of \( L \) is the Toeplitz matrix \( MR(t, \delta) \), \( \delta := \sum_{h \geq 0} d_h t^h \). Recalling that, in this case, \( m_{ij} = d_j^i \), for every integer \( n \), we have
\[
L \beta_n(x) = \sum_{k=0}^{n} m_{kn} \beta_k(x) = \sum_{k=0}^{n} d_{n-k} \beta_k(x) = \sum_{k=0}^{n} d_{n-k} \Delta^{n-k} \beta_n(x).
\]
Conversely, suppose that \( L \) is a shift-invariant operator. If we denote by \( [m_{ij}] \) the canonical matrix of \( L \), we get
\[
\sum_{i=0}^{n} m_{in} \beta_i(x) = S \beta_n(x) = S \Delta \beta_{n+1}(x) = \Delta S \beta_{n+1}(x)
\]
\[
= \Delta \sum_{k=0}^{n+1} m_{k,n+1} \beta_k(x) = \sum_{k=0}^{n+1} m_{k,n+1} \beta_{k-1}(x) = \sum_{i=0}^{n} m_{i+1,n+1} \beta_i(x).
\]
This implies that \( m_{in} = m_{i+1,n+1} \) for every \( i, n \in \mathbb{N} \); namely, \( [m_{ij}] \) is a Toeplitz matrix.

The preceding theorem shows that a shift-invariant operator \( L \) is a recursive operator such that the recurrence rule of its canonical matrix is the series \( t \). \( L \) is therefore determined by the boundary value, which, according to the classical language, will be called the S-indicator of \( L \).

EXAMPLE 6.2.
(a) The forward difference operator \( \Delta \) and the shift operator \( E \) are obviously shift-invariant operators, whose S-indicators are the series \( t \) and \( 1 + t \), respectively.
(b) The backwards difference operator \( \nabla \) can be written as
\[
\nabla = I - F^{-1} = F^{-1} \Delta = \frac{\Delta}{I + \Delta} = \Lambda - \Lambda^2 + \Lambda^3 - \ldots,
\]
and is, therefore, a shift-invariant operator, with S-indicator \( t/(1 + t) \)

7. ASSOCIATED SEQUENCES

In this section, we will define and examine a new class of polynomial sequences called associated sequences, namely, sequences \( (f_n(x)) \) of integral polynomials such that there exists a shift-invariant operator mapping the polynomial \( f_n(x) \) into \( f_{n-1}(x) \) for every positive index \( n \). In the classical case, this kind of property is typical of Sheffer sequences. We will see that this is indeed a different class of sequences, which properly contains the class of recursive bases.

Let \( Q = \rho(\Delta) \) be a shift-invariant operator, with \( d := \deg(\rho) \geq 1 \). A sequence \( (f_n(x))_{n \in \mathbb{N}} \) of integral polynomials is said to be associated with \( Q \) whenever the following condition holds:
\[
Q f_n(x) = f_{n-1}(x), \quad \text{for every positive integer } n.
\] (13)
Condition (13) implies that
\[
\deg(f_n) = d + \deg(f_{n-1}).
\]
As a consequence, the set \( \{f_n(x) \mid n \in \mathbb{N} \} \) is linearly independent.
EXAMPLE 7.1.
(a) The sequence \((\beta_n)_{n \in \mathbb{N}}\) is associated with the forward difference operator \(\Delta\).
(b) The sequence \((\gamma_n)_{n \in \mathbb{N}}\) is associated with the backwards difference operator \(\nabla\), since
\[
\nabla \gamma_n(x) = (-1)^n \binom{-x}{n} - (-1)^n \binom{-x - 1}{n} = (-1)^{n-1} \binom{-x}{n-1} = \gamma_{n-1}(x).
\]
(c) Set \(Q = \Delta^2\). The sequence \((f_n(x))_{n \in \mathbb{N}}\), such that
\[
f_n(x) = \beta_0(x) + \beta_2(x) + \cdots + \beta_{2n}(x)
\]
is associated with the operator \(Q\).

We remark that the associated sequences of the first two examples are recursive sequences, while the last one is not. In fact, a sequence of integral polynomials associated with some shift-invariant operator is not, in general, a recursive sequence. However, its canonical matrix presents some other kind of recurrence. In order to describe it, we need the following definition.

Given two nonzero inverse Laurent series \(\rho = \sum_{j \leq d} j t^j\), \(\sigma = \sum_{j \leq h} j t^j\), with \(\phi \in \mathcal{R}^-\), the \((\rho, \sigma)\)-column recursive matrix is the unique matrix \(M \in \mathcal{M}\) such that, for every integer \(j\),
\[
M[j] = \rho^j \sigma,
\]
where \(M[j] = \sum_i m_{ij} t^i\) denotes the generating function of the \(j\)th column of \(M\). Such a matrix will be denoted by the symbol \(C(\rho, \sigma)\).

Column recursive and row recursive matrices are strictly related to each other. In fact, consider the operator \(T\) acting on the \(\mathbb{Z}\)-module of bi-infinite matrices
\[
T : \mathcal{M} \to \mathcal{M},
\]
defined as follows: if \(M = [m_{ij}]\) is a matrix in \(\mathcal{M}\), then
\[
TM := [m_{-j,-i}].
\]
The operator \(T\) is called a pivoting operator, and maps row recursive matrices into column recursive matrices and conversely, as shown in the next result whose proof is straightforward.

THEOREM 7.1. Let \(\alpha, \beta\) be two Laurent series. Then,
\[
TR(\alpha, \beta) = C \left((\alpha^{-1})^*, \beta^*\right).
\]

Column recursive matrices turn out to be related to associated sequences. In fact, we have the following theorem.

THEOREM 7.2. Let \(\alpha\) be a Laurent series of positive degree, \(\alpha \in \mathcal{R}^+\), and consider the shift-invariant operator \(Q := \alpha(\Delta)\). A polynomial sequence \((f_n(x))\) is associated with the operator \(Q\) if and only if the canonical matrix of \((f_n(x))\) is the southeast minor of a column recursive matrix, with recurrence rule \((\alpha^{-1})^*\).

As an immediate consequence of the preceding result, every shift-invariant operator whose \(S\)-indicator has positive degree and belongs to \(\mathcal{R}^+\) has infinitely many associated sequences.

EXAMPLE 7.2. Consider the polynomial sequence \((f_n(x))\) defined in Example 7.1(c). According to Theorem 7.2, it is easily checked that its canonical matrix is the southeast minor of the column
recursive matrix $C(t^2, \sum_{n \geq 0} t^{-2n})$, namely, the matrix

$$
\begin{array}{cccccccccccc}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
$$

8. RECURSIVE BASES

We have introduced so far the two notions of recursive sequence and associated sequence. As we have seen, a recursive sequence is a spanning set of the $\mathbb{Z}$-module $\mathcal{P}_I$, and its canonical matrix is the southeast minor of a row recursive matrix, while an associated sequence is an independent set of $\mathcal{P}_I$ such that its canonical matrix is the southeast minor of some column recursive matrix. Our next goal is to characterize the intersection of these two classes of sequences, namely, the class of those recursive sequences which are also associated sequences with respect to some shift-invariant operator, or, equivalently, bases of $\mathcal{P}_I$ whose canonical matrix is the southeast minor of a bi-infinite matrix which is both row and column recursive.

Such matrices are characterized by the next result, originally stated and proved in [1].

**THEOREM 8.1. DOUBLE RECURSION.** Let $\alpha$ be a Laurent series of degree 1, whose first coefficient equals 1 or $-1$. For every Laurent series $\beta$, the row recursive matrix $R(\alpha, \beta)$ is also column recursive. More precisely, we have

$$R(\alpha, \beta) = C\left( (\bar{\alpha}^{-1})^*, (t \bar{\alpha}^{-1} D\bar{\alpha} (\beta \circ \bar{\alpha}))^* \right),$$

(15)

or, equivalently,

$$TR(\alpha, \beta) = R\left( \bar{\alpha}, t \frac{D\bar{\alpha}}{\alpha} (\beta \circ \bar{\alpha}) \right),$$

(16)

where $\bar{\alpha}$ denotes the inverse of $\alpha$ with respect to composition, and $D\bar{\alpha}$ is the formal derivative of $\alpha$.

**EXAMPLE 8.1.** For every Toeplitz matrix $M := R(t, \beta)$, the pivoted matrix $TM$ is row recursive. More precisely, by Theorem 8.1, we get

$$TR(t, \beta) = R(t, \beta);$$

namely, the pivoting operator maps every Toeplitz matrix into itself.

We remark that, in the special case $\beta = D\alpha$, identity (16) yields

$$TR(\alpha, D\alpha) = R\left( \bar{\alpha}, t \bar{\alpha}^{-1} \right).$$

(17)

Hence, in the second row of $TR(\alpha, D\alpha)$, we can read the coefficients of the series $\bar{\alpha}$, which implies that the Double Recursion Theorem is a matrix equivalent of the Lagrange inversion formula.
EXAMPLE 8.2. Consider the series \( \alpha := t/(1-t) = t + t^2 + t^3 + \cdots \), whose inverse with respect to composition is \( \tilde{\alpha} = t/(1 + t) = t - t^2 + t^3 - \cdots \). By identity (17), we get

\[
TR(\alpha, D\alpha) = TR \left( \frac{t}{1 - t}, \frac{1}{(1 - t)^2} \right) = R \left( \frac{1}{1 + t}, 1 + t \right).
\]

In other terms, in the 0th column of the matrix \( R(t/(1 - t), (1/(1 - t)^2)) \)—which is marked by two vertical bars in the figure below—one can read from bottom to top the coefficients of the series \( 1 + t \), while the \((-1)^{th}\) column contains the coefficients of \((1 + t)\tilde{\alpha} = t\), and so on.

\[
\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccccc
Theorem 8.4. Let $\alpha \in \mathbb{R}^+$, $\deg^+ (\alpha) = 1$. For any given series $\delta \in \mathbb{R}^+$, $\deg (\delta) = 0$, with $\delta = \sum_{n \geq 0} d_n t^n$, the unique recursive basis $(f_n(x))$ associated with the delta operator $Q = \alpha (\Delta)$ such that

$$f_n(0) = d_n$$

is given by

$$f_n(x) = T^n Q' P^{n-1} \beta_n(x),$$

where

$$T = \delta (Q), \quad Q' = (D\alpha)(\Delta), \quad P = (t^{-1}\alpha)(\Delta).$$

As a final example of application of the recursive matrix approach to the umbral calculus, we give a matrix proof of the analog for recursive bases of the Umbral Expansion Theorem.

Theorem 8.5. Umbral Expansion. Let $(f_n(x))$ be a recursive basis with indicator $\alpha$ and boundary value $\delta$. For every integral polynomial $p(x)$ of degree $n$, we have

$$p(x) = \sum_{i=0}^{n} a_i f_i(x),$$

where

$$a_i = \left( \tilde{\alpha}^i (\delta \circ \tilde{\alpha})^{-1} \right) (\Delta)(0).$$

In particular, if $(f_n(x))$ is a homogeneous recursive basis, we get

$$p(x) = \sum_{i=0}^{n} (Q' p)(0) f_i(x),$$

where $Q = \tilde{\alpha} (\Delta)$ is the delta operator associated with $(f_n(x))$.

Proof. Denote by $[p]_\beta, [p]_\mathcal{F}$ the column vectors of the coefficients of the polynomial $p(x)$ with respect to the bases $(\beta_n(x)), (f_n(x))$, respectively. We have

$$[p]_\mathcal{F} = C^{-1} \times [p]_\beta,$$

where $C = R(\alpha, \delta)$ is the canonical matrix of the basis $(f_n(x))$. The assertion now follows immediately from the Newton Expansion Theorem and formula (10).

Example 8.3. For every integer $a$, the Gould operator relative to $a$ is defined as the shift-invariant operator

$$G_a = E^{-a} \Delta = \frac{\Delta}{(I + \Delta)^a} = \alpha (\Delta),$$

with

$$\alpha = \frac{t}{(1 + t)^a}.$$

Every polynomial sequence associated with the Gould operator has indicator

$$\tilde{\alpha} = \frac{t}{(1 - t)^a}.$$

Hence, the unique homogeneous recursive basis $(g_{a,n}(x))$ associated with $G_n$—the sequence of Gould polynomials—is given by

$$g_{a,n}(x) = \sum_{k=0}^{n} c_{k,n} \beta_k(x),$$
where \( c_{k,n} \) is the \( n \)th coefficient of the series \( \hat{a}^k = t^k / (1-t)^a \), namely

\[
c_{k,n} = \binom{n-1+k(a-1)}{n-k}.
\]

Note that \( g_{1,n}(x) = \gamma_n(x) \). The Umbral Expansion Theorem gives, for every polynomial \( p(x) \),

\[
p(x) = \sum_{k=0}^{n} (E^{-a} \Delta^k p)(0) g_{a,k}(x) = \sum_{k=0}^{n} (\Delta^k p)(-ak) g_{a,k}(x),
\]
or, recalling that \( \nabla = E^{-1} \Delta \), and hence, the operator \( G_a \) can be also written as \( G_a = E^{1-a} \Delta \),

\[
p(x) = \sum_{k=0}^{n} (\nabla^k p)(k - ak) g_{a,k}(x).
\]

**Example 8.4.** Let \( (f_n(x)) \) be the recursive basis whose canonical matrix is \( MR(t(1-t)^{-1}, (1-t)^{-2}) \), namely, the southeast minor of the recursive matrix of Example 8.2. Since the compositional inverse of the series \( \alpha := t(1-t)^{-1} \) is \( \hat{\alpha} = t(1+t)^{-1} \), by Theorem 8.2, the basis \( (f_n(x)) \) is associated with the delta operator \( \Delta(I+\Delta)^{-1} = \nabla \), whose associated homogeneous basis is \( \gamma_n(x) \). Hence, setting \( \beta = (1-t)^{-2} \), by Theorem 8.3, we get

\[
f_n(x) = (\beta \circ \hat{\alpha})(\Delta) \gamma_n(x) = (I+\Delta)^2 \gamma_n(x) = (I-\nabla)^{-2} \gamma_n(x)
\]

or equivalently, recalling that \( I+\Delta = E \),

\[
f_n(x) = E^{-a+1} \beta_n(x) = \binom{x+n+1}{n}.
\]

**REFERENCES**