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NOTE

NATURAL SPANNING TREES OF $\mathbb{Z}^d$ ARE RECURRENT

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We show that the simple random walk on the natural spanning tree of $\mathbb{Z}^d$ is recurrent for every d ($=1,2,3,\ldots$) and determine the asymptotic behaviour of the probability of returning to the origin in n steps ($n\to\infty$). This is in contrast to a result of Polya [6]: $\mathbb{Z}^d$ is recurrent for $d=1$, 2 and transient for $d\geq 3$.

The result

The simple random walk on a graph is defined as a Markov chain with one-step transition probabilities

$$p(x,y)=\begin{cases}\dfrac{1}{d(x)} & \text{if } x,\ y \text{ are adjacent vertices,}\\ 0 & \text{otherwise,}\end{cases}$$

where $d(x)=$ degree of the vertex x. We write $p^n(x,y)$ for the corresponding n-step transition probabilities:

$$p^n(x,y)=\sum_z p(x,z)p^{n-1}(z,y)$$

(see [2, 3] or [7] for more information about random walks on graphs).

An old result of Polya [6] says that the simple random walk on $\mathbb{Z}^d$ is recurrent for $d=1$, 2 and transient for $d\geq 3$. More precise results are known (e.g. [8]) namely

$$p^{2n}(O,O)\underset{n\to\infty}{\simeq} c_d\cdot n^{-d/2}\quad \text{on } \mathbb{Z}^d.$$

The natural spanning tree T_d of $\mathbb{Z}^d$ connects the origin $O=(0,0,\ldots,0)$ to every other vertex $(a_1,a_2,\ldots,a_d)$ (in some cartesian coordinate system) by geodesics (see Fig. 1) through the points $(a_1,0,\ldots,0)$, $(a_1,a_2,0,\ldots,0)$,

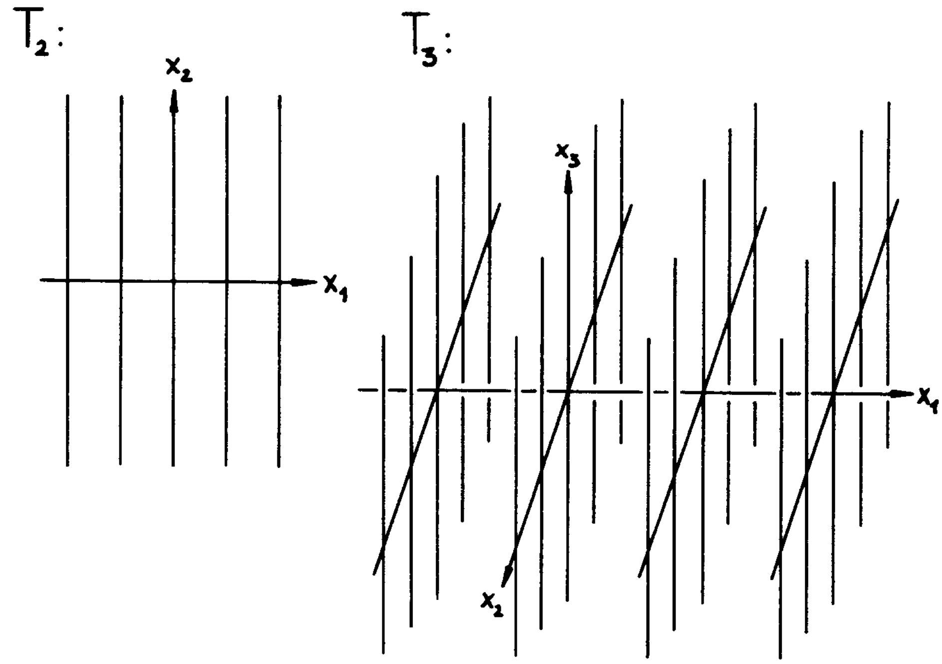

Fig. 1.

$\dots, (a_1, a_2, \dots, a_{d-1}, 0)$. The distances from the origin to every other vertex are the same in $\mathbb{Z}^d$ and T_d. We prove the following

Theorem. *If* $p^n(O, O) = \text{Prob}$(*to walk from O to O in n steps*), *then*

$$p^{2n+1}(O, O) = 0,$$

$$p^{2n}(O, O) \underset{n\to\infty}{\sim} \frac{d \cdot 2^{-1+2^{1-d}}}{\Gamma(2^{-d})} \cdot n^{-1+2^{-d}} \quad \text{on } T_d \ (d \geq 1).$$

Corollary. *The simple random walk on every tree* T_d ($d = 1, 2, 3, \dots$) *is recurrent.*

Some remarks

(i) Call a graph *recurrent* (*transient*) if the simple random walk on it is recurrent (transient). To find out for a given graph if it is recurrent or transient is called the *type problem*. This is in many cases much easier to solve for trees. So one is led to the

Question 1. Can the type problem for a graph be solved by considering its spanning subtrees?

The theorem (corollary) shows that one has to be careful.

(ii) It can be shown (see [4] or [7]) that $\mathbb{Z}^d$ has transient (spanning) subtrees for $d \geq 3$, while every subtree of $\mathbb{Z}^1$ or $\mathbb{Z}^2$ is recurrent. So one is led to the

Question 2. Which subtrees of $\mathbb{Z}^d$ ($d \geq 3$) are transient?

Intuitively a tree is transient if it has many branching vertices (i.e., vertices of degree ≥ 3), but to be a subtree of $\mathbb{Z}^d$ implies that there cannot be too many branchings (slow growth). How can this be made precise?

(iii) In general the answer to Question 1 is no. There is an example by Peter Doyle of a transient graph without transient subtree but in some sense this graph (although transient) is similar to a recurrent tree. This leads to

Question 3. Which (transient) graphs have transient subtrees?

This is true for $\mathbb{Z}^d$ ($d \geq 3$) and perhaps for all transient Cayley graphs (of groups).

The proof

1) Let O be the origin of $\mathbb{Z}^d$ and write

$$p_d^n(O, O) = \text{Prob}\left(O \xrightarrow[\text{in } T_d]{\text{in } n \text{ steps}} O\right),$$

$$f_d^n(O, O) = \text{Prob}\left(O \xrightarrow[\text{in } T_d]{\text{in } n \text{ steps, but not earlier}} O\right),$$

$$q_d^n(O, O) = \text{Prob}\left(O \xrightarrow[\text{the first step is in direction } x_1;\text{ in } T_d]{\text{in } n \text{ steps, but not earlier}} O\right).$$

All these probabilities are zero for n odd. The corresponding generating functions are

$$G_d(x) = \sum_{n\geq 0} p_d^{2n}(O, O)x^n, \qquad F_d(x) = \sum_{n\geq 1} f_d^{2n}(O, O)x^n,$$

$$Q_d(x) = \sum_{n\geq 1} q_d^{2n}(O, O)x^n.$$

A flow chart analysis (see e.g. [5]) shows that for $d \geq 1$

$$F_d = \frac{d-1}{d} F_{d-1} + \frac{1}{d} Q_d, \tag{1}$$

$$Q_d = \frac{\dfrac{1}{2d}x}{1 - \dfrac{d-1}{d}F_{d-1} - \dfrac{1}{2d}Q_d}, \tag{2}$$

and

$$G_d = (1 - F_d)^{-1}. \tag{3}$$

(From here the corollary follows immediately since $F_d(1) = 1$.)

2) Eliminating Q_d from (1) and (2) gives

$$d \cdot F_d = d - ((d-(d-1)F_{d-1})^2 - x)^{\frac{1}{2}}.$$

From this and (3) we infer, writing $G_d = d\gamma_d$,

$$\gamma_d = \gamma_{d-1}(1 + 2\gamma_{d-1} + (1-x)\gamma_{d-1}^2)^{-\frac{1}{2}}. \tag{4}$$

Since $\gamma_1 = G_1 = (1-x)^{-\frac{1}{2}}$, all the functions γ_d and G_d have a singularity at $x = 1$ (and this is the only singularity on the circle of convergence $|x| = 1$). From (4) we find the local behaviour of γ_d (and G_d) at the singularity $x = 1$, namely

$$\gamma_d = (1-x)^{-2^{-d}} g_d((1-x)^{2^{1-d}}), \tag{5}$$

where $g_d(t)$ is analytic in t and $g_d(0) = 2^{-1+2^{1-d}}$ (>0). This follows by recurrence after some calculations.

3) If we write down (5) more explicitely it yields

$$G_d(x) = (1-x)^{-2^{-d}} h_1(x) + (1-x)^{2^{-d}} h_2(x) + \cdots + (1-x)^{(2^d-3)2^{-d}} h_{2^{d-1}}(x),$$

where $h_i(x)$ are analytic near $x = 1$ and

$$h_1(1) = d \cdot g_d(0) = d \cdot 2^{-1+2^{1-d}}.$$

Now we may apply the method of Darboux (see e.g. [1]) to each summand and infer that

$$p_d^{2n}(O, O) \underset{n\to\infty}{\sim} \frac{h_1(1)}{\Gamma(2^{-d})} n^{-1+2^{-d}}.$$

This completes the proof. □

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