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Cut locus structures on graphs

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ABSTRACT

Motivated by a fundamental geometrical object, the cut locus, we introduce and study a new combinatorial structure on graphs.

1. Introduction

The motivation for this work comes from a basic notion in riemannian geometry, that we shortly present in the following. In this paper, by a *surface* we always mean a complete, compact and connected 2-dimensional riemannian manifold without boundary.

The *cut locus* $C(x)$ *of* the point x on the surface S is the set of all extremities (different from x) of maximal (with respect to inclusion) shortest paths (geodesic segments) starting at x; for basic properties and equivalent definitions refer, for example, to [13] or [16]. The notion was introduced by Poincaré [15] and gained, since then, an important place in global riemannian geometry.

For surfaces S it is known that $C(x)$, if not a single point, is a local tree (i.e., each of its points z has a neighborhood V in S such that the component $K_z(V)$ of z in $C(x) \cap V$ is a tree), even a tree if S is homeomorphic to the sphere. A *tree* is a set T any two points of which can be joined by a unique Jordan arc included in T.

All our graphs are finite, connected, undirected, and may have multiple edges or loops.

Myers [14] established that the cut locus of a real analytic surface is (homeomorphic to) a graph, and Buchner [2] extended the result for manifolds of arbitrary dimension. For not analytic riemannian metrics on S, cut loci may be quite large sets, see the work of Hebda [6] and of the first author [9]. Other contributions to the study of this notion were brought, among others, by Buchner [3,4], Gluck and Singer [5], Hebda [7], Itoh [8], Shiohama and Tanaka [17], Zamfirescu [19,20], Weinstein [18].

We show in another paper [10] that *for every graph G there exists a surface S_G and a point x* in S *whose cut locus $C(x)$ is isomorphic to G*; *rephrasing, every graph can be realized as a cut locus.*

If G has an odd number q of generating cycles then any surface S_G realizing G is non-orientable, but if q is even then one cannot generally distinguish, by simply looking at the graph G, whether S_G is orientable or not: explicit examples show that both possibilities can occur [11]. In other words, seen as a graph, *the cut locus does not encode the orientability of the ambient space*.

This is our main motivation to endow graphs with a combinatorial structure—that of cut locus structure (CL-structure). In particular, any such enriched graph does encode the orientability of the ambient space where it lives as a cut locus.

In this paper we treat combinatorial aspects of this new concept: in Section 2 we introduce and discuss this notion, in Section 3 we give two planar representations of CL-structures, and in Section 4 we enumerate all such structures on "small" graphs. The last section presents future work and open questions.

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At the end of this section we recall a few notions from graph theory, in order to fix the notation.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Denote by B the set of all *bridges* in the graph G; i.e., edges whose removal disconnects G. Each non-vertex component of $G \setminus B$ is called a 2-*connected component* of G.

A *k-graph* is a graph all vertices of which have degree k.

The power set $\mathcal{E}$ of E becomes a Z_2-vector space over the two-element field Z_2 if endowed with the symmetric difference as addition. $\mathcal{E}$ can be thought of as the space of all functions $E \to Z_2$, and called the (binary) *edge space* of G. The (binary) *cycle space* is the subspace $\mathcal{Q}$ of $\mathcal{E}$ generated by (the edge sets of) all simple cycles of G. If G is seen as a simplicial complex, $\mathcal{Q}$ is the space of 1-cycles of G with mod 2 coefficients.

Consider the graph G as a simplicial complex. The *cyclic part* of G is the minimal (with respect to inclusion) subgraph G^{cp} of G, to which G is contractible; i.e., the minimal subgraph of G obtained by repeatedly contracting external edges, and for each vertex remaining of degree two (if any) merging its incident edges.

2. Cut locus structures

Definition 2.1. A *G-patch* on the graph G is a topological surface P_G with boundary, containing (a graph isomorphic to) G and contractible to it.

Recall that every boundary component of a patch is homeomorphic to a circle, as a 1-dimensional manifold without boundary.

Definition 2.2. A *G-strip* (or a strip on G, or simply a *strip*, if the graph is clear from the context), is a G-patch with a 1-component boundary; i.e., whose boundary is one topological circle; see Fig. 1(a).

The next remark gives the geometrical background for the notion of cut locus structure.

Remark 2.3. Consider a point x on a surface S, and a geodesic segment $\gamma : [0, l] \to S$ parameterized by arclength, with $\gamma(0) = x$ and $\gamma(l) \in C(x)$. For $\varepsilon > 0$ smaller than the injectivity radius at x, and hence smaller than l, the point $\gamma(l - \varepsilon)$ is well defined. Since $S \setminus C(x)$ is contractible to x along geodesic segments, and thus homeomorphic to an open disk, the union over all γs of those points $\gamma(l - \varepsilon)$ is homeomorphic to the unit circle, and therefore the set $\bigcup_\gamma \{\gamma(l - \mu) : 0 \le \mu \le \varepsilon\}$ is a $C(x)$-strip.

Definition 2.4. A *cut locus structure* (a *CL-structure* for short) on the graph G is a strip on the cyclic part G^{cp} of G.

We show in another paper [10], with geometrical tools, the converse to Remark 2.3: every CL-structure can be obtained (with some suitable surface and point on the surface) as described in Remark 2.3.

Each G-strip defines a circular order around each vertex of G, and thus a *rotation system*. Conversely, one can alternatively define a G-strip as the graph associated to a rotation system, together with a 2-cell embedding having precisely one face. We choose not to follow this way, and to keep in our presentation as much as possible of the geometrical intuition.

Definition 2.5. An *elementary strip* is an edge-strip (arc-strip) or a point-strip; i.e., a strip defined by the graph with precisely one edge (arc) of different extremities, respectively by the graph consisting of one single vertex.

Definition 2.6. An *elementary decomposition* of a G-patch P_G is a decomposition of P_G into elementary strips such that: each edge-strip corresponds to precisely one edge of G; each point-strip corresponds to precisely one vertex of G; see Fig. 1(b) and (c).

Our notion of "G-patch" is equivalent to that of "*fibered surface*" introduced by Bestvina and Handel: "*a fibered surface is a compact surface F with boundary which is decomposed into arcs and into polygons that are modeled on k-junctions*, $k = 1, 2, 3, \ldots$. *The components of the subsurface fibered by arcs are strips. Shrinking the decomposition elements to points produces a graph G, where vertices (of valence k) correspond to (k-) junctions and strips to edges. We can think of G as being embedded in F, representing the spine of F*" [1]. We choose the most (in our opinion) appropriate name for our purpose, and thus different from theirs.

In order to easily handle a CL-structure, we associate to it an object of combinatorial nature. To this goal, denote by $\mathcal{P}$ and $\mathcal{A}$ the set of all point-strips, respectively edge-strips, of a CL-structure $\mathcal{C}$ on the graph G.

Definition 2.7. Consider an elementary decomposition of the G-strip P_G such that each elementary strip has a distinguished face, labeled $\bar{0}$. The face opposite to the distinguished face will be labeled $\bar{1}$. Here, $\bar{0}$ and $\bar{1}$ are the elements of the 2-element group $(Z_2, \oplus)$.

To each pair $(v, e) \in V \times E$ consisting of a vertex v and an edge e incident to v, we associate the Z_2-sum $\bar{s}(v, e)$ of the labels of the elementary strips $\nu \in \mathcal{P}$, $\varepsilon \in \mathcal{A}$ associated to v and e; i.e., $\bar{s}(v, e) = \bar{0}$ if the distinguished faces of ν and ε agree with each other, and $\bar{1}$ otherwise. Therefore, to any cut locus structure $\mathcal{C}$ we can associate a function $s_{\mathcal{C}} : E \to \{\bar{0}, \bar{1}\}$,

$$s_{\mathcal{C}}(e) = \bar{s}(v, e) \oplus \bar{s}(v', e), \tag{1}$$

where v and v' are the vertices of the edge $e \in E$.

We call the function $s_{\mathcal{C}}$ defined by (1) the *companion function* of $\mathcal{C}$.

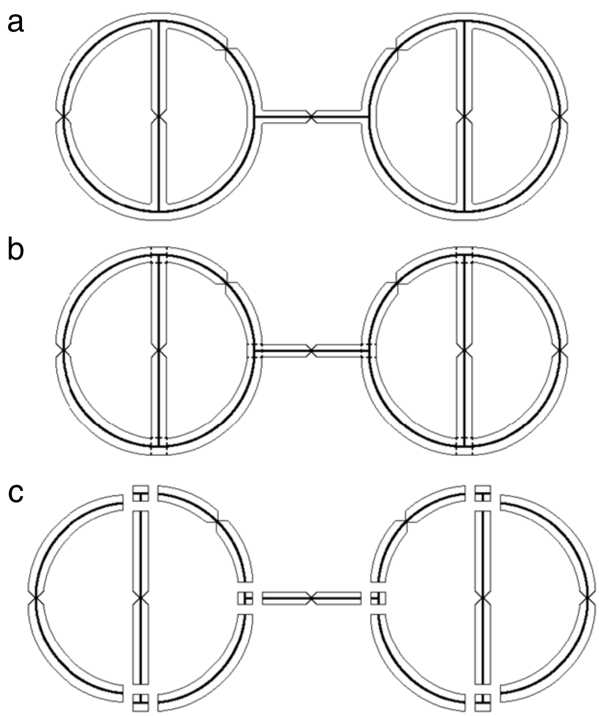

Fig. 1. A strip and its elementary decomposition.

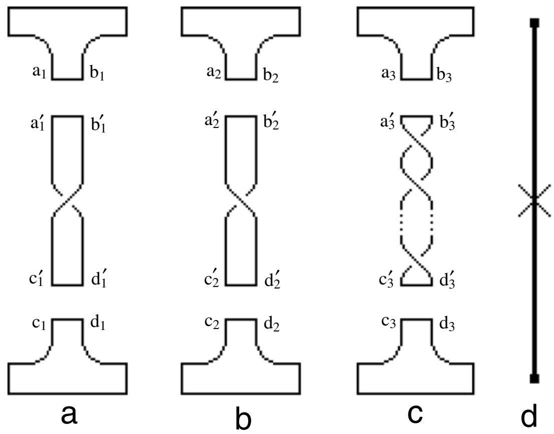

Fig. 2. Equivalent CL-structures (a), (b) and (c), and schematic representation (d). The edge-strip at (a) corresponds to a rectangular band whose base is π-rotated "to the left" with respect to the top; the edge-strip at (b) corresponds to a rectangular band whose base is π-rotated "to the right" with respect to the top; the edge-strip at (c) corresponds to a rectangular band whose base is $(2k+1)\pi$-rotated "to the left" with respect to the top.

The value $s_{\mathcal{C}}(e)$ above can be thought of as the *switch* of the edge e.

Definition 2.8. Assume first that the graph G is 2-connected. Two *CL-structures* $\mathcal{C}$, $\mathcal{C}'$ on G are called *equivalent* if their *companion functions* are *equivalent*: i.e., $s_{\mathcal{C}}$ and $s_{\mathcal{C}'}$ are equal, up to a simultaneous change of the distinguished face for all elementary strips in G (i.e., either $s_{\mathcal{C}} = s_{\mathcal{C}'}$, or $s_{\mathcal{C}} = \bar{1} \oplus s_{\mathcal{C}'}$).

If G is not 2-connected, the *CL-structures* $\mathcal{C}$, $\mathcal{C}'$ on G are called *equivalent* if their companion functions are equivalent on every 2-connected component of G. See Fig. 2.

Definition 2.9. An edge-strip P_e (or simply an edge e) in a CL-structure $\mathcal{C}$ is called *switched* if $s_{\mathcal{C}}(e) = \bar{1}$.

Proposition 2.10. *If two CL-structures on the same graph G are equivalent then the corresponding G-strips are homeomorphic surfaces.*

Proof. We may assume that G is cyclic.

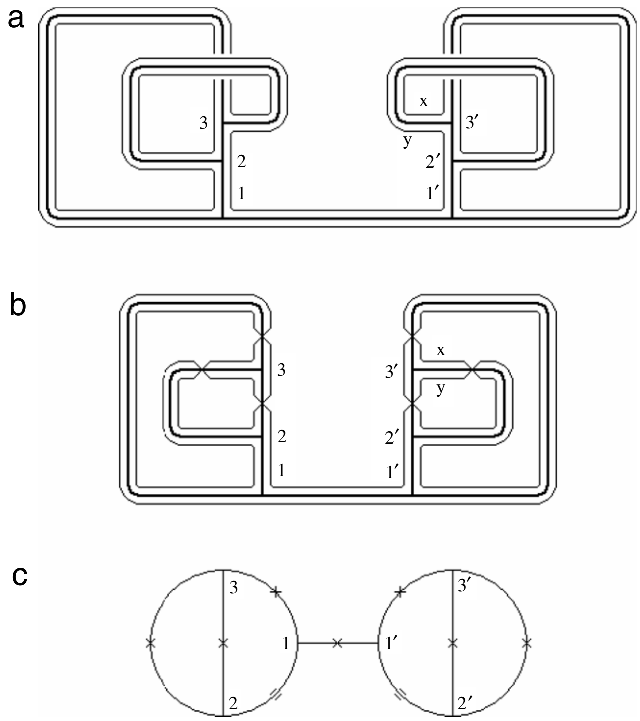

Fig. 3. Representations of CL-structures. (a) *Graph representation* of a strip. (b) Intermediate step to obtain (c). (c) *Natural representation* for the strip at (a). Additional points x, y are indicated to make clear the transformation.

Assume, moreover, that we have two CL-structures on G, whose companion functions are equivalent on every 2-connected component of G. The desired homeomorphism can be constructed inductively, extending it with each new "gluing" of an elementary strip, see Fig. 2. □

3. Representations of CL-structures

We propose two ways to represent in the plane a CL-structure $\mathcal{C}$ on the graph G.

Definition 3.1. The *graph representation* of $\mathcal{C}$ starts with some planar representation of G, and afterward points out the CL-structure, see Fig. 3(a).

Definition 3.2. The *natural representation* of $\mathcal{C}$ starts by representing in the plane each vertex-strip such that its distinguished face is "up", and afterward connects the vertex-strips by edge-strips. The idea is illustrated by Fig. 3(b) and (c).

Consider the natural representation of a CL-structure on a cubic graph. We shall overwrite an "x" to the drawn image of an edge if its strip is switched, and a " $=$" to the drawn image of an edge if its strip is non-switched. See Figs. 3 and 4.

To represent the edge-strip switching property, one could as well use only one marker, "x", and consider the edges without this mark as non-switched. We choose the two-marker-system in order to keep an aesthetic balance between the representations of switched and non-switched edges.

Notice that neither the natural representation, nor the graph representation, of a CL-structure on a graph is unique.

Proposition 3.3. *For any planar cubic graph G and any CL-structure on G, the natural representation and the graph representation coincide, up to planar homeomorphisms.*

Proof. This follows from the definitions above. □

Example 3.4. If the 3-graph G is not planar, Proposition 3.3 is not true. An easy example, obtained from a flat torus of rectangular fundamental domain (see the procedure described in Remark 2.3), is illustrated by Fig. 5.

Directly from the definitions we have the following.

Lemma 3.5. *In any natural representation of a strip, each cycle-patch contains at least one switched edge-strip.*

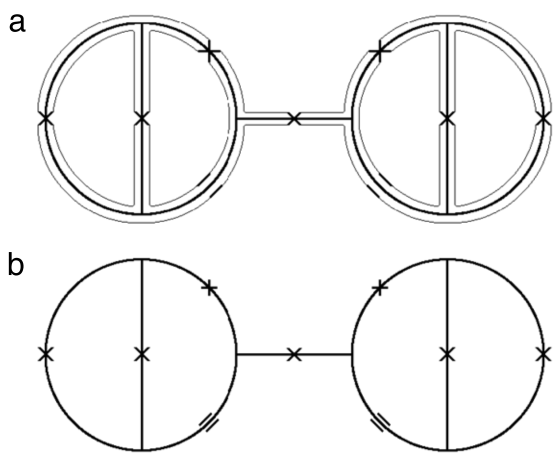

Fig. 4. Schematic representation of the strip in Fig. 1(a).

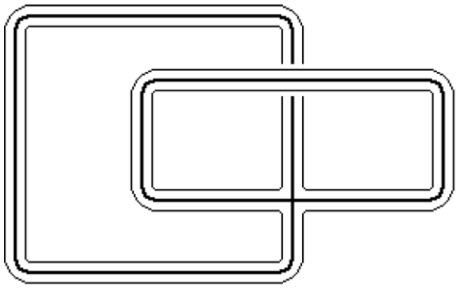

Fig. 5. CL-structure obtained from a flat torus of rectangular fundamental domain.

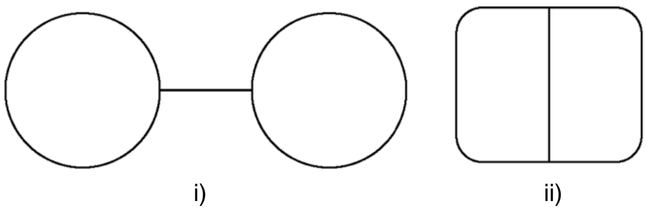

Fig. 6. All 3-graphs with 2 generating cycles.

4. CL-structures on small graphs

We present in this section all distinct cut locus structures on 3-graphs with $q = 2, 3$ generating cycles.

The following statement can be obtained by straightforward inductive constructions.

Lemma 4.1. *There are precisely* 2, *respectively* 6, *distinct* 3*-graphs with* 2, *respectively* 3, *generating cycles, see* Figs. 6 and 7.

Theorem 4.2. (a) *There are precisely* 3 *non-equivalent CL-structures on the* 3*-graphs with* 2 *generating cycles, see* Figs. 8 and 9.
(b) *There are precisely* 17 *non-equivalent CL-structures on the* 3*-graphs with* 3 *generating cycles, see* Figs. 10–15.

Proof. We employ the natural representation of CL-structures. It is straightforward to generate all patches on the graphs in Figs. 6 and 7, to keep only the strips (by the use of Lemma 3.5), and to use Definition 2.8 and the symmetries of the graphs to identify equivalent CL-structures. □

Our last result shows that the case of CL-structures on 3-graphs is, in some sense, sufficient. For this goal, we define the *degree of a graph* as the maximal degree of its vertices.

Theorem 4.3. *Any CL-structures on a graph with q generating cycles and degree larger than* 3 *can be obtained from CL-structures on* 3*-graphs with q generating cycles, by contracting non-switched edge-strips.*

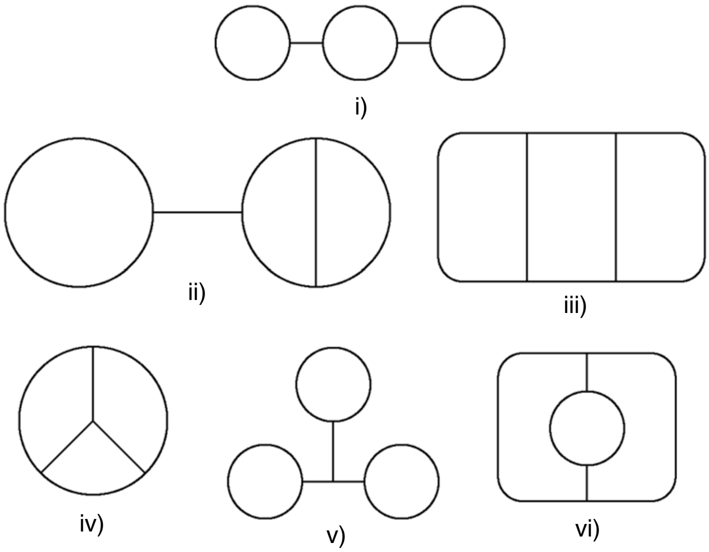

Fig. 7. All 3-graphs with 3 generating cycles.

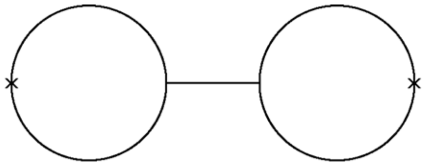

Fig. 8. Unique CL-structure on the graph in Fig. 6(i).

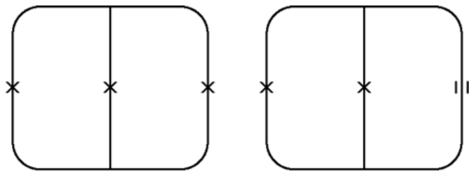

Fig. 9. 2 CL-structures on the graph in Fig. 6(ii).

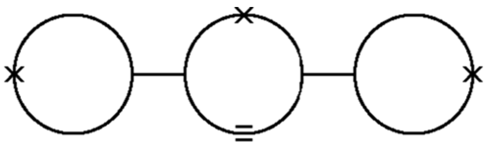

Fig. 10. Unique CL-structure on the graph in Fig. 7(i).

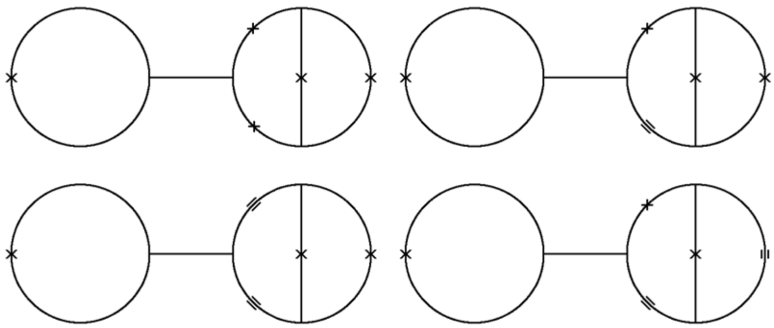

Fig. 11. 4 CL-structures on the graph in Fig. 7(ii).

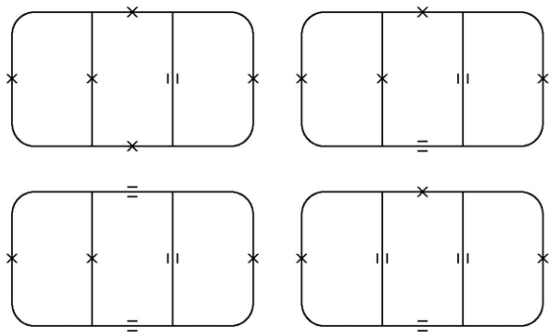

Fig. 12. 4 CL-structures on the graph in Fig. 7(iii).

Fig. 13. 3 CL-structures on the graph in Fig. 7(iv).

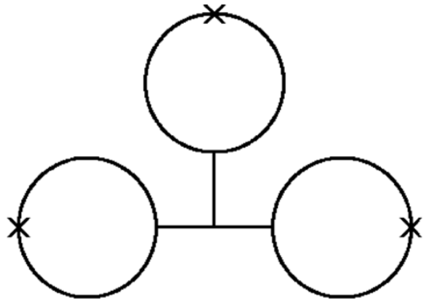

Fig. 14. Unique CL-structure on the graph in Fig. 7(v).

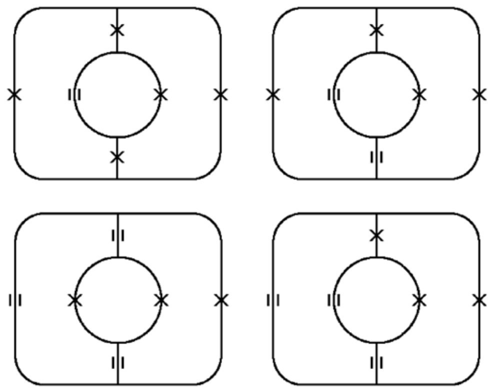

Fig. 15. 4 CL-structures on the graph in Fig. 7(vi).

Proof. Fix q; we consider only graphs with q generating cycles, and proceed by induction over the number of vertices of degree larger that 3. Denote by $D(G)$ this number for the graph G.

Assume the cyclic graph G has $D(G) \geq 1$, and choose a vertex v in G with $\deg(v) = d > 3$.

Let $\mathcal{C}$ be a CL-structure on G, and denote by $v_1, \ldots, v_d$ the neighbors of v in G, and by T the subtree of G rooted at v, with leaves $v_1, \ldots, v_d$. Let G^- be the complement of T in G, and $\mathcal{C}^-$ be the union of patches naturally induced by $\mathcal{C}$ on G^-. Let $s_{\mathcal{C}}^-$ be the restriction of the companion function $s_{\mathcal{C}}$ of $\mathcal{C}$ to G^-.

Replace T in G by a tree T_3 of leaves $v_1, \ldots, v_d$, all of whose internal vertices have degree 3 (T_3 is generally not unique), and denote by G^v the new graph. Now complete $\mathcal{C}^-$ to a CL-structure $\mathcal{C}^v$ on G^v, by extending $s_{\mathcal{C}}^-$ on the internal edges of T_3 with $\bar{0}$, and on the external edges of T_3 with the original values of $s_{\mathcal{C}}$. Observe that $\mathcal{C}^v$ is indeed a CL-structure on G^v, and $D(G^v) = D(G) - 1$, so the proof is complete. □

5. Future work and open questions

We prove in another paper [10] that every CL-structure corresponds to a cut locus on a surface, while in a subsequent one [11] we consider the orientability of the surfaces realizing CL-structures as cut loci. An upper bound on the number of (*orientable*) CL-structures on a graph G is given in [12].

In this paper, we defined the CL-structures for connected graphs. We see two possible ways to extend this definition for a *disconnected* graph G.

The simplest one is to define a CL-structure on each component of G, and to notice that this enriched graph can be realized as the cut loci of points (as many as the components of G) on distinct surfaces, a point on each surface.

For a second way, notice that any cut locus on a complete, compact and connected 2-dimensional riemannian manifold without boundary is a connected local tree. For non-compact surfaces, the cut locus may be disconnected as well, thus naturally defining CL-structures on disconnected graphs.

We leave for future work the relationship between these two possible definitions.

Our approach leaves several questions unanswered, of which we find the following three most interesting.

Question 5.1. *Characterize the companion functions of CL-structures in the set $\mathcal{S} = \{s : E \to \{\bar{0}, \bar{1}\}\}$, see Definition 2.7.*

Question 5.2. *A planar graph is, by definition, a graph which can be represented in the plane without crossings (self-intersections). As we have seen in Example 3.4, there are CL-structures on (not cubic) planar graphs whose natural representations in the plane necessarily produce crossings. What is the minimal number of such crossings which guarantees a planar natural representation?*

The same question can be asked for non planar graphs too, where the (minimal number of necessary) crossings of a graph is a new parameter.

Question 5.3. *Which graphs with q generating cycles have the largest number of different CL-structures?*

Acknowledgements

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