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NOTE

A Note on the Vertex-Connectivity Augmentation Problem

Tibor Jordán*

Department of Mathematics and Computer Science, Odense University, Campusvej 55, Odense DK-5230, Denmark

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$k-2$ surplus edges over (a lower bound of) the optimum. Here we introduce two new lower bounds and show that in fact the size of the solution given by (a slightly modified version of) this algorithm differs from the optimum by at most $\lceil (k-1)/2 \rceil$. © 1997 Academic Press

1. INTRODUCTION

A graph $G=(V, E)$ is called *k-connected* if $|V| \geqslant k+1$ and the deletion of any $k-1$ or fewer vertices leaves a connected graph. Given a graph $G=(V, E)$ and an integer l, the connectivity augmentation problem is to find a smallest set F of new edges for which $G'=(V, E \cup F)$ is l-connected. The complexity of this problem is still an exciting open question, even if the graph G to be augmented is k-connected and $l=k+1$. (For $l \leqslant 4$ the problem is known to be polynomially solvable. See [2] for a survey of this area.)

In [4] a polynomial algorithm was given which makes a k-connected graph $(k+1)$-connected by adding at most $k-2$ edges over (a lower bound of) the optimum. The goal of this note is to introduce two new lower bounds on the size of an optimal augmentation and to prove that (a slightly modified version of) our algorithm from [4] produces a solution of size at most $\lceil (k-1)/2 \rceil$ more than the improved lower bound. Our new gap $\lceil (k-1)/2 \rceil$ is sharp in the sense that for every $k \geqslant 3$ there exists an

* Part of this work was done at CWI, Kruislaan 413, 1098 SJ, Amsterdam, The Netherlands. Partially supported by the Danish Natural Science Research Council Grant No. 28808. E-mail: tibor@imada.ou.dk.

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infinite family of graphs for which the gap between the optimum value and the size of the solution is $\lceil (k-1)/2 \rceil$. Moreover, there exists an infinite family for which the gap between the optimum and the lower bound is $\lfloor (k-1)/2 \rfloor$.

In the rest of the introduction we introduce the necessary definitions and briefly summarize the results from [4] we shall rely on in this paper. A vertex $v \in V - X$ is a *neighbour* of $X \subset V$ in the graph $G = (V, E)$ if there exists a vertex $u \in X$ such that $uv \in E$. Let $\Gamma(X)$ denote the set of neighbours of $X \subset V$. It is well-known and easy to check that the function $|\Gamma| : 2^V \to Z_+$ is *submodular*, that is

$$|\Gamma(X)| + |\Gamma(Y)| \geqslant |\Gamma(X \cup Y)| + |\Gamma(X \cap Y)| \qquad \text{for every} \quad X, Y \subseteq V. \quad (1)$$

Let $G = (V, E)$ be a k-connected graph. For a set $K \subset V$ of size k, $b_K(G)$ (or simply b_K) denotes the number of components in $G - K$. Let $b(G) = \max\{b_K(G) : K \subset V, |K| = k\}$. If $b_K \geqslant 2$, the set K is a *cut* of G. A set $P \subset V$ is called *tight* if $|\Gamma(P)| = k$ and $|V - P| \geqslant k + 1$. The maximum number of pairwise disjoint tight sets in G is denoted by $t(G)$. We say that $S \subseteq V$ is a *tight-set cover* of G if $S \cap P \neq \emptyset$ for every tight set P. Let $\tau_t(G)$ denote the size of a smallest tight-set cover of G. $M(G)$ denotes the number $\max\{b(G) - 1, \lceil t(G)/2 \rceil\}$. Suppose that the (for inclusion) minimal tight sets of G are pairwise disjoint. Then for a minimal tight set D_i $(1 \leqslant i \leqslant t(G))$ we define S_i to be the union of those tight sets which include D_i but which are disjoint from every other minimal tight set D_j $(i \neq j)$, see also [4, p. 14].

It is easy to see that $M(G)$ is a lower bound for the minimum number $m(G)$ of new edges which make G $(k+1)$-connected. It is known that for $k = 1$ and $k = 2$ the equality $M(G) = m(G)$ holds. This is not always the case for $k \geqslant 3$. For example $m(K_{k,k}) - M(K_{k,k}) = k - 2$ for the complete bipartite graphs $K_{k,k}$. However, as the main result of [4] shows, the gap between $M(G)$ and $m(G)$ cannot be bigger.

THEOREM 1.1 [4, Theorem 1.1]. *Let G be a k-connected graph for some $k \geqslant 2$. Then $M(G) \leqslant m(G) \leqslant M(G) + k - 2$.*

The proof of Theorem 1.1 was based on the next two theorems.

THEOREM 1.2 [4, Theorem 2.4]. *Suppose that $b(G) \geqslant k + 1$ and $b(G) - 1 \geqslant \lceil t(G)/2 \rceil$ in a k-connected graph G. Then $m(G) = M(G)$.*

An edge $e = xy$ $(x, y \in V(G))$ is *saturating* for a k-connected graph G if $t(G + e) = t(G) - 2$ holds.

THEOREM 1.3 [4, Theorem 3.1]. *Let $G=(V,E)$ be a k-connected graph and suppose that $b(G)-1<\lceil t(G)/2\rceil$, $t(G)\geqslant k+3$ and $|V|\geqslant 2k+1$. Then there exists a saturating edge for G.*

One can easily check that if we skip the last one and a half paragraphs in the proof of [4, Theorem 3.1] and observe that the conditions $|V|\geqslant 2k+1$ and $b(G)-1<\lceil t(G)/2\rceil$ are not used in the proof elsewhere, we obtain the following result.

THEOREM 1.3b. *Let $G=(V,E)$ be a k-connected graph with $t(G)\geqslant k+3$ such that there exists no saturating edge for G. Then for any two sets S_i, S_j $1\leqslant i\neq j\leqslant t(G)$ either*

(a) $\Gamma(S_i)=\Gamma(S_j)$ *or*

(b) $V-\Gamma(S_i)\subseteq\Gamma(S_j)$ *holds.*

If case (a) holds for every pair S_i, S_j, we obtain $b(G)=t(G)$. Case (b) may hold only if $|V|\leqslant 2k$. Also recall that every set S_i induces a connected subgraph and $t(G)\geqslant k+3$ implies that S_i is tight and $S_i\cap S_j=\emptyset$ for every $1\leqslant i\neq j\leqslant t(G)$. Two further observations will also be cited from [4].

LEMMA 1.4 [4, Lemma 3.4]. *Let $G=(V,E)$ be a k-connected graph. Then $m(G)\leqslant\tau_t(G)-1$.*

The proof of [4, Lemma 2.1] and [4, Lemma 3.5, Case II] give:

LEMMA 1.5. *If $t(G)\geqslant k+1$ in a k-connected graph G then the minimal tight sets are pairwise disjoint. If $t(G)\leqslant k+1$ then $\tau_t(G)\leqslant k+1$ holds.*

2. THE NEW LOWER BOUNDS AND THE ALGORITHM

Let $t^*(G)$ denote the number of minimal tight sets in the k-connected graph $G=(V,E)$. Let $S\subseteq V$ be a minimal tight-set cover.

LEMMA 2.1. *$t(G)\leqslant|S|\leqslant t^*(G)$ and for $t(G)\geqslant k+1$ the equality $t(G)=t^*(G)$ holds. Furthermore, $\lceil t^*(G)/2\rceil\leqslant m(G)$.*

Proof. The inequality $t(G)\leqslant|S|$ is obvious. The minimality of S implies that for any $s_i\in S$ there exists a minimal tight set P_i for which $P_i\cap S=\{s_i\}$. Therefore $|S|\leqslant t^*(G)$. Lemma 1.5 implies that if $t(G)\geqslant k+1$, then equality holds everywhere.

To prove that $\lceil t^*(G)/2\rceil$ is a lower bound for the size of an optimal augmentation suppose that G can be made $(k+1)$-connected by adding a set F of new edges with $|F| \leqslant \lceil t^*(G)/2 \rceil - 1$. Since F is an augmenting set, for every tight set X of G there exists an edge $e' = x'y'$ in F for which one end-vertex, say x', is contained by X and y' belongs to $V - X - \Gamma(X)$. Thus our assumption implies that there exists an edge $e = xy \in F$ such that in $G \cup F$ the vertex y is a new neighbour of (at least) two different minimal tight sets P_1 and P_2 of G. More precisely, $x \in P_1 \cap P_2$ and $y \in V - (P_1 \cup P_2 \cup \Gamma(P_1 \cup P_2))$ hold. This implies that $\Gamma(P_1 \cup P_2)$ separates y from the set $P_1 \cup P_2$. By the k-connectivity of G this gives $|\Gamma(P_1 \cup P_2)| \geqslant k$. Applying (1) we obtain

$$k + k = |\Gamma(P_1)| + |\Gamma(P_2)| \geqslant |\Gamma(P_1 \cap P_2)| + |\Gamma(P_1 \cup P_2)| \geqslant k + k,$$

from which we conclude that equality holds everywhere. Thus $P_1 \cap P_2$ is also tight, contradicting the minimality of P_1. ∎

Thus a better lower bound for $m(G)$ is $M^*(G) = \max\{b(G) - 1, \lceil t^*(G)/2 \rceil\}$. The following example shows that there can be a gap between $m(G)$ and $M^*(G)$ as well for any $k \geqslant 3$ and for arbitrarily high number of vertices. Take a complete bipartite graph $K_{k,k} = (A, B; E)$ and replace some vertex $v \in B$ by a copy of K_r for some $r \geqslant k+1$ and give different end-vertices to the k edges entering the K_r. This graph H_k^r is k-connected with $M^*(H_k^r) = k$ and $m(H_k^r) = k + \lfloor (k-1)/2 \rfloor$. (A smallest augmenting set consists of $k-1$ edges connecting the k components of $H_k^r - A$ and a set of $\lceil k/2 \rceil$ edges which cover the vertices of A.) On the other hand we shall prove that this is (almost) the biggest possible gap if $|V| \geqslant 2k+1$.

To show a similar gap (and for the analysis of the algorithm) in the case $|V| \leqslant 2k$ we need another new lower bound. We call two cuts K, L *overlapping* if $V - K \subseteq L$ (and hence $V - L \subseteq K$) holds. It is easy to see that for a family $\mathcal{K}'$ of pairwise overlapping cuts $\sum_{K \in \mathcal{K}'} (b_K - 1) \leqslant m(G)$. Let $b^*(G) = \max\{\sum_{K \in \mathcal{K}'} (b_K - 1) : \mathcal{K}'$ is a family of pairwise overlapping cuts of $G\}$. Clearly, overlapping cuts exist only if $|V| \leqslant 2k$. Thus $b^*(G) = b(G) - 1$ if G has at least $2k+1$ vertices.

Now suppose that $|V| \leqslant 2k$, $t(G) \geqslant k+3$ and there exists no saturating edge for G. Let $D_1, ..., D_t$ denote the minimal tight sets in G and $S_1, ..., S_t$ denote the corresponding (tight) sets. Let $\mathcal{K} = \{K_1, ..., K_s\}$ $(s \leqslant t)$ denote the different members of $\{\Gamma(S_1), ..., \Gamma(S_t)\}$. By Theorem 1.3b the family $\mathcal{K}$ consists of pairwise overlapping cuts (and $s \geqslant 2$). Hence $b^*(G) \geqslant \sum_{i=1}^{s} (b_{K_i} - 1)$. Observe that if $D_i \subseteq C$ for some minimal tight set D_i and some component C of $G - \Gamma(S_j)$, then $C = S_i$ must hold by Theorem 1.3b. Thus for each $K_j \in \mathcal{K}$ every component of $G - K_j$ equals S_i for some i and includes precisely one minimal tight set D_i.

Let the augmenting set $F = \bigcup_{i=1}^{s} F_i$ be defined as follows. For every $K_i \in \mathcal{K}$ let F_i contain $b_{K_i} - 1$ edges which make $V - K_i$ connected such that every vertex which is an end-vertex of some edge in F_i is contained by some minimal tight set. Such sets clearly exist and $|F| \leqslant b^*(G)$. The next lemma shows that F is an (optimal) augmenting set.

LEMMA 2.2. *$G' = (V, E \cup F)$ is $(k+1)$-connected.*

Proof. Suppose that $|\Gamma(X)| = k$ and $X^* = V - X - \Gamma(X) \neq \emptyset$ for some $X \subset V$ in G'. Clearly $\Gamma(X) = \Gamma(X^*)$. Now $t(G) \geqslant k+3$ and $|\Gamma(X)| = k$, hence there exist (at least three) sets among the pairwise disjoint sets S_i $(1 \leqslant i \leqslant t(G))$ which are included in $X \cup X^*$. Since there are no edges between X and X^* and each S_i induces a connected subgraph, we may assume (possibly by changing the role of X and X^*) that $S_j \subseteq X$ holds for some $1 \leqslant j \leqslant t(G)$. Let $\mathscr{D} = \{D_l : D_l \subseteq X^*$ is a minimal tight set in $G\}$. Focus on some $D_l \in \mathscr{D}$. By Theorem 1.3b either $\Gamma(S_j) = \Gamma(S_l)$ or $D_l \subseteq S_l \subseteq \Gamma(S_j)$ holds. Since there are no neighbours of S_j in X^*, the latter case is impossible. Thus $\Gamma(S_j) = \Gamma(S_l)$. This shows that there are no neighbours of S_l in X^*, thus $S_l - X - \Gamma(X)$ is a component of $G - \Gamma(X)$. This implies that every vertex of $\Gamma(X) - S_l$ is a neighbour of S_l. Hence for any minimal tight set D_p for which $D_p \cap \Gamma(X) \neq \emptyset$ (and hence $p \neq l$ and $D_p \cap S_l = \emptyset$) we obtain $\Gamma(S_l) \cap S_p \neq \emptyset$ and hence $\Gamma(S_l) \neq \Gamma(S_p)$ holds.

Wlog let $K_j = \Gamma(S_j)$. The above facts imply that $K_j = \Gamma(S_r)$ holds for any $D_r \in \mathscr{D}$. Let $D = \bigcup_{D_i \in \mathscr{D}} D_i$. By definition, $D \subseteq X^*$. By the choice of F_j there exists an edge $e \in F_j$ connecting D to some D_u, where $D_u \cap D = \emptyset$ and $\Gamma(S_u) = K_j$. We saw that $D_u \cap \Gamma(X) \neq \emptyset$ would imply for an arbitrary $D_l \in \mathscr{D}$ that $\Gamma(S_u) \neq \Gamma(S_l) = K_j$. $D_u \subseteq X^*$ is also impossible by the definition of $\mathscr{D}$. Thus e connects some vertex of $D \subseteq X^*$ to a vertex of X, a contradiction. ∎

THEOREM 2.3. *For any k-connected $(k \geqslant 3)$ graph $G = (V, E)$ the inequality $m(G) \leqslant \max\{M^*(G), b^*(G)\} + \lceil (k-1)/2 \rceil$ holds.*

Proof. First suppose that $|V| \geqslant 2k+1$. In this case we prove the inequality by induction on $t(G)$. First observe that if $t(G) \leqslant k+2$ then we have $\tau_t(G) \leqslant k+2$ by Lemma 1.5. Thus by Lemmas 1.4 and 2.1 we get $m(G) - M^*(G) \leqslant m(G) - \lceil t^*(G)/2 \rceil \leqslant \tau_t(G) - 1 - \lceil \tau_t(G)/2 \rceil \leqslant \lceil (k-1)/2 \rceil$, as required.

Let us assume now that $t(G) \geqslant k+3$. (In this case $t(G) = t^*(G)$ by Lemma 1.5.) If there exists no saturating edge for G then we are done, since by Theorem 1.3b we obtain $t(G) = b(G)$, hence $m(G) = M(G) = M^*(G)$ holds by Theorem 1.2.

Now consider the case where there exists a saturating edge e for G. If $b(G) - 1 \leqslant \lceil t(G)/2 \rceil - 1$ then $\lceil t(G)/2 \rceil - 1 = \lceil t(G+e)/2 \rceil = M^*(G+e)$ and $m(G+e) \geqslant m(G) - 1$ hold, thus the addition of e does not decrease the value $m - M^*$. Since the required inequality holds for $G + e$ by the induction hypothesis, we are done. If $b(G) - 1 \geqslant \lceil t(G)/2 \rceil$, we distinguish two subcases. In the first subcase $b(G) \geqslant k + 1$, where $m(G) = M(G) = M^*(G)$ holds by Theorem 1.2, and the inequality follows.

In the second subcase $b(G) - 1 \geqslant \lceil t(G)/2 \rceil$ and $b(G) \leqslant k$. Now let G' be a maximal supergraph of G for which $k + 1 \leqslant t(G') \leqslant t(G) - 2$ and $2(|E(G')| - |E(G)|) = t(G) - t(G')$ holds. There exists a saturating edge for G, therefore such a G' exists. If we can reach $t \leqslant k + 2$ by adding saturating edges, that is, $t(G') \leqslant k + 2$, then we have $m(G) - M^*(G) \leqslant m(G) - \lceil t^*(G)/2 \rceil \leqslant m(G') - \lceil t^*(G')/2 \rceil \leqslant \lceil (k-1)/2 \rceil$ by the induction hypothesis (and using the fact that the addition of a saturating edge does not decrease $m(G) - \lceil t(G)/2 \rceil$). If $t(G') \geqslant k + 3$ then (since there is no saturating edge for G') by Theorem 1.3b we obtain that $k + 3 \leqslant t(G') = b(G') \leqslant b(G)$, contradicting the assumption $b(G) \leqslant k$. This proves the theorem in the case $|V| \geqslant 2k + 1$.

Let us assume now that $|V| \leqslant 2k$. If $t(G) \leqslant k + 2$, we get $m(G) - M^*(G) \leqslant \lceil (k-1)/2 \rceil$ as before. Otherwise let us saturate G as long as possible, that is, let G' be a maximal supergraph of G for which $k + 1 \leqslant t(G')$ and $2(|E(G')| - |E(G)|) = t(G) - t(G')$ holds. If $t(G') \leqslant k + 2$, we obtain $m(G) - M^*(G) \leqslant m(G) - \lceil t^*(G)/2 \rceil \leqslant m(G') - \lceil t^*(G')/2 \rceil \leqslant \lceil (k-1)/2 \rceil$ using the same argument as above. If $t(G') \geqslant k + 3$, Theorem 1.3b and Lemma 2.2 imply that $m(G') = b^*(G')$. Since $t(G') \geqslant k + 3$, the minimal tight sets are pairwise disjoint in G' (and in G), thus the saturating edges in $E(G') - E(G)$ are pairwise independent. This gives $|E(G')| - |E(G)| \leqslant \lfloor (k-3)/2 \rfloor$ in our case of $|V| \leqslant 2k$. Also observe that $b^*(G) \geqslant b^*(G')$, since overlapping cuts of G' correspond to overlapping cuts in G. Hence $m(G) - b^*(G) \leqslant m(G') + \lfloor (k-3)/2 \rfloor - b^*(G) \leqslant b^*(G') + \lfloor (k-3)/2 \rfloor - b^*(G') \leqslant \lceil (k-1)/2 \rceil$, as required. ∎

Following the steps of the previous proof the algorithm given in [4, Section 4] can be modified easily in such a way that the size of the solution it produces is at most $m(G) + \lceil (k-1)/2 \rceil$. To obtain the better performance guarantee we need the following changes (which do not effect the running time $O(n^5)$). After Phase 1 (and in Phase 3) in the case of $k + 3 \leqslant t(G) \leqslant 2k - 1$ we still need to search for and add saturating edges until it is possible. (But the parameter $b(G)$ need not be computed.) When we reach $t(G') \leqslant k + 2$, we may choose an arbitrary minimal tight-set cover and find a solution as in Phase 5. If $t(G') \geqslant k + 3$ for the graph G' which contains no further saturating edges then either we can identify a cut in G' with $b_K(G) \geqslant b_K(G') \geqslant k + 1$ and hence $b_K(G) - 1 \geqslant \lceil t(G)/2 \rceil$, in which case an optimal augmentation can be found like in Phase 4, or $|V| \leqslant 2k$ and the augmenting set F as defined before Lemma 2.2 makes G' optimally $(k+1)$-connected. To find F it is enough to compute the minimal tight sets

D_i and the corresponding sets S_i in G', which can be done by max-flow computations. The correctness and the preformance guarantee of this modified algorithm follows from the proof of Theorem 2.3 and Lemmas 2.1 and 2.2.

Note, that $t^*(G)$ need not be computed during the algorithm—although it can be computed in polynomial time by max-flow calculations.

3. REMARKS

Observe the slight difference between the gap guaranteed by Theorem 2.3 and the gap of the example graph H_k^r. The sharp value seems to be $\lfloor (k-1)/2 \rfloor$. This would follow if Theorem 1.3 and 1.3b were valid for $t(G) = k+2$, as well. This looks true for $k \geqslant 2$.

The graphs H_k^r can be modified to show that the algorithm may add $\lceil (k-1)/2 \rceil$ surplus edges over the optimum in some cases. For this let r be sufficiently large and attach $2k-1$ new vertices of degree k to the copy of K_r in such a way that their sets of neighbours are pairwise different. For this graph H' we have $m(H') = 2k-1$ since the addition of a set of $2k-1$ independent edges, pairing the new vertices and the "old" vertices of degree k makes $H'(k+1)$-connected. On the other hand, the algorithm may start by adding $k-1$ saturating edges pairing $2k-2$ new vertices and then adding saturating edges connecting old vertices of degree k. In this case the solution it produces may contain $2k-1+\lceil (k-1)/2 \rceil$ edges.

Recently Cheriyan and Thurimella [1] gave a more efficient algorithm with running time $O(\min(k, n^{1/2})\, k^2 n^2 + (\log n)\, kn^2)$ for computing an augmentation of size at most $m(G)+k-2$. Using their method, the smaller gap described here can also be achieved in a more efficient way.

Another idea is to define an even stronger lower bound as follows. Replace every edge of G by two oppositely directed edges and let $m(G_d)$ be the minimum number of new directed edges which make the new graph G_d $(k+1)$-connected. Clearly, $\lceil m(G_d)/2 \rceil$ is a lower bound for $m(G)$. However, our previous example graph H_k^r shows that we cannot achieve a better gap using this bound instead of $\lceil t^*(G)/2 \rceil$. (The parameter $m(G_d)$ can be computed in polynomial time, see [3]. It is easy to see, as in Lemma 2.1, that $m(G_d) \geqslant t^*(G)$.)

Finally note that for all the graphs we have showing that the gap in Theorem 2.3 is almost sharp (like in the case of H_k^r) the size of an optimal augmentation is small, that is, it can be bounded by a function of k. This suggest the following conjecture: there exists a function $f(k)$ such that for a k-connected graph G with $m(G) \geqslant f(k)$ the equality $m(G) = M(G)$ holds.

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