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Linear Algebra and its Applications 357 (2002) 35–44

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Normal matrices with a dominant eigenvalue and an eigenvector with no zero entries

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Received 30 October 2001; accepted 11 March 2002

Submitted by R.A. Brualdi

Abstract

We say that a square complex matrix is *dominant* if it has an algebraically simple eigenvalue whose modulus is strictly greater than the modulus of any other eigenvalue; such an eigenvalue and any associated eigenvector are also said to be dominant. We explore inequalities that are sufficient to ensure that a normal matrix is dominant and has a dominant eigenvector with no zero entries. For a real symmetric matrix, these inequalities force the entries of a dominant real eigenvector to have a prescribed sign pattern. In the cases of equality in our inequalities, we find that exceptional extremal matrices must have a very special form.
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Keywords: Dominant eigenvalue; Perron's Theorem; Normal matrices; Eigenvector with no zero entries

0. Introduction

A celebrated theorem of O. Perron says that a square matrix with positive entries has a positive eigenvalue that is algebraically simple and equal to the spectral radius, which is strictly greater than the modulus of any other eigenvalue. Moreover, this positive eigenvalue has an associated eigenvector with positive entries.

We say that a square complex matrix is *dominant* if it has an algebraically simple eigenvalue whose modulus is strictly greater than the modulus of any other eigenvalue; such an eigenvalue and any associated eigenvector are also said to be dominant.

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PII: S0024-3795(02)00366-X

It is easy to produce a rich variety of examples of complex or real normal matrices with a positive dominant eigenvalue and an associated positive eigenvector. Let u be any given positive unit vector, and let U be any n-by-n unitary matrix whose last column is u. Let $\lambda_1, \ldots, \lambda_{n-1}$ be any complex numbers such that $|\lambda_1| \leqslant \cdots \leqslant |\lambda_{n-1}| < 1$, and let $\Lambda \equiv \operatorname{diag}(\lambda_1, \ldots, \lambda_{n-1}, 1)$. Then $A = U\Lambda U^*$ is a normal matrix with the required properties. By choosing U and Λ to be real, one may obtain real symmetric (but not necessarily positive) matrices with the required property.

We explore inequalities that are sufficient to ensure that a normal matrix is dominant and has a dominant eigenvector with no zero entries. In the special case of a real symmetric matrix, these inequalities force the entries of a dominant real eigenvector to have a prescribed sign pattern. In the cases of equality in our inequalities, we find that exceptional extremal matrices must have a very special form.

Inequalities of the type that we discuss have been investigated in [2], but the methods employed there do not seem to extend to normal matrices that are not real and symmetric. Moreover, there appear to be gaps in the proof of the main theorem in [2], and it is not clear how they could be correctly filled.

1. Two basic lemmas

For a given positive integer $n \geqslant 2$, consider a set of complex numbers $\{\lambda_1, \ldots, \lambda_n\}$, indexed so that $|\lambda_n| \geqslant \cdots \geqslant |\lambda_1|$ and normalized so that $|\lambda_1|^2 + \cdots + |\lambda_n|^2 = 1$, and a given convex combination $\gamma = \alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n$. If $|\gamma| = 1$, then

$$\begin{aligned}1 = |\gamma|^2 = |\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n|^2 &\leqslant \left(\alpha_1^2 + \cdots + \alpha_n^2\right)\left(|\lambda_1|^2 + \cdots + |\lambda_n|^2\right)\\ &= \alpha_1^2 + \cdots + \alpha_n^2 \leqslant \alpha_1 + \cdots + \alpha_n = 1,\end{aligned}$$

which implies that $\alpha_n = |\lambda_n| = 1$ and all other $\lambda_i = 0$. Thus, if $|\gamma|$ is close to 1, then $|\lambda_n|$ must be strictly larger than all the other $|\lambda_i|$'s. On the other hand, if α_n is bounded away from 1, then so is $|\gamma|$. The following two lemmas quantify these simple observations.

Lemma 1. *Let $n \geqslant 2$ and let complex numbers $\lambda_1, \ldots, \lambda_n$ and nonnegative real numbers $\alpha_1, \ldots, \alpha_n$ be given. Suppose that $|\lambda_1|^2 + \cdots + |\lambda_n|^2 = \alpha_1 + \cdots + \alpha_n = 1$.*

1. *Suppose that $|\lambda_n| \geqslant \cdots \geqslant |\lambda_1|$.*
   (a) *If $|\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n| > 1/\sqrt{2}$, then $|\lambda_n| > 1/\sqrt{2} > |\lambda_{n-1}|$.*
   (b) *If $|\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n| = 1/\sqrt{2}$ and $|\lambda_n| = |\lambda_{n-1}|$, then $|\lambda_n| = |\lambda_{n-1}| = 1/\sqrt{2}$ and $\lambda_1 = \cdots = \lambda_{n-2} = 0$.*
2. *Suppose that $\lambda_1, \ldots, \lambda_n$ are real and $\lambda_n \geqslant \cdots \geqslant \lambda_1$.*
   (a) *If $\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n > 1/\sqrt{2}$, then $\lambda_n > 1/\sqrt{2} > |\lambda_i|$ for all $i = 1, \ldots, n-1$.*

(b) *Suppose that* $\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n = 1/\sqrt{2}$ *and* $\lambda_n = |\lambda_k|$ *for some* $k \in \{1, \ldots, n-1\}$. *Then* $k \in \{1, n\}$. *If* $n = 2$, *then* $\lambda_1 = \pm 1/\sqrt{2}$. *If* $n \geqslant 3$, *then either* $k = n-1$ *and* $\lambda_{n-1} = 1/\sqrt{2}$ *or* $k = 1$ *and* $\lambda_1 = -1/\sqrt{2}$. *In all cases*, $\lambda_i = 0$ *for all* $i \notin \{k, n\}$.

Proof. For part 1(a), convexity and the triangle inequality ensure that

$$|\lambda_n| \geqslant \alpha_1 |\lambda_1| + \cdots + \alpha_n |\lambda_n| \geqslant |\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n| > \frac{1}{\sqrt{2}}. \tag{1}$$

Since

$$1 = |\lambda_1|^2 + \cdots + |\lambda_n|^2 \geqslant |\lambda_{n-1}|^2 + |\lambda_n|^2 > |\lambda_{n-1}|^2 + \frac{1}{2},$$

it follows that $|\lambda_{n-1}| < 1/\sqrt{2}$. The same manipulations with the first hypothesis of 1(b) show that $|\lambda_n| \geqslant 1/\sqrt{2} \geqslant |\lambda_{n-1}|$, and the second implies that both of these inequalities are equalities. The normalization $|\lambda_1|^2 + \cdots + |\lambda_n|^2 = 1$ then implies that $\lambda_1 = \cdots = \lambda_{n-2} = 0$.

Similar considerations establish the assertions in part 2. □

We denote the Euclidean norm of a complex n-vector $v = [v_i]$ by $\|v\| \equiv \sqrt{v^* v}$. If v has no zero entries and if k is an index such that $|v_k| \leqslant |v_i|$ for all $i = 1, \ldots, n$, we define

$$\kappa(v) \equiv \sqrt{|v_k|^4 + \left(\|v\|^2 - |v_k|^2\right)^2} \tag{2}$$

and observe that $\kappa(cv) = |c|^2 \kappa(v)$ for any scalar c.

Lemma 2. *Let* $n \geqslant 2$, *let* $u = [u_i]$ *and* $v = [v_i]$ *be given complex n-vectors*; *suppose that* u *is a unit vector and that no entry of* v *is zero. Let complex numbers* $\lambda_1, \ldots, \lambda_n$ *and nonnegative real numbers* $\alpha_1, \ldots, \alpha_{n-1}$ *be given*; *suppose that* $|\lambda_n| \geqslant \cdots \geqslant |\lambda_1|$, $|\lambda_1|^2 + \cdots + |\lambda_n|^2 = 1$, *and* $\alpha_1 + \cdots + \alpha_{n-1} + |v^* u|^2 = \|v\|^2$. *If at least one entry of* u *is zero*, *then*

$$\left|\alpha_1\lambda_1 + \cdots + \alpha_{n-1}\lambda_{n-1} + \left|v^* u\right|^2 \lambda_n\right| \leqslant \kappa(v)\,.$$

Proof. Let j and k be indices such that $u_j = 0$ and $|v_k| \leqslant |v_i|$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned} \left|v^* u\right|^2 &= \left|\sum_{i=1}^{n} \bar{v}_i u_i\right|^2 = \left|\sum_{i \neq j}^{n} \bar{v}_i u_i\right|^2 \leqslant \left(\sum_{i \neq j}^{n} |v_i|^2\right)\left(\sum_{i \neq j}^{n} |u_i|^2\right) \\ &= \left(\sum_{i \neq j}^{n} |v_i|^2\right) \leqslant \left(\sum_{i \neq k}^{n} |v_i|^2\right) = \|v\|^2 - |v_k|^2 \end{aligned}$$

and

$$
\begin{aligned}
&\left|\alpha_1\lambda_1+\cdots+\alpha_{n-1}\lambda_{n-1}+\left|v^*u\right|^2\lambda_n\right| \\
&\quad\leqslant \alpha_1\left|\lambda_1\right|+\cdots+\alpha_{n-1}\left|\lambda_{n-1}\right|+\left|v^*u\right|^2\left|\lambda_n\right| \\
&\quad\leqslant \left|v_k\right|^2\left|\lambda_{n-1}\right|+\left(\|v\|^2-\left|v_k\right|^2\right)\left|\lambda_n\right| \\
&\quad\leqslant \left(\sqrt{\left|v_k\right|^4+\left(\|v\|^2-\left|v_k\right|^2\right)^2}\right)\left(\sqrt{\left|\lambda_{n-1}\right|^2+\left|\lambda_n\right|^2}\right) \\
&\quad\leqslant \left(\sqrt{\left|v_k\right|^4+\left(\|v\|^2-\left|v_k\right|^2\right)^2}\right)=\kappa(v). \qquad \square
\end{aligned}
$$

For any n-vector v with no zero entries and $n \geqslant 2$, a computation reveals that

$$
\begin{aligned}
1 > \frac{\kappa(v)}{\|v\|^2} &= \kappa\left(\frac{v}{\|v\|}\right)=\sqrt{\left|\frac{v_k}{\|v\|}\right|^4+\left(1-\left|\frac{v_k}{\|v\|}\right|^2\right)^2} \\
&\overset{\star}{\geqslant} \sqrt{\frac{1}{n^2}+\left(1-\frac{1}{n^2}\right)^2} \\
&\geqslant \begin{cases} \dfrac{1}{\sqrt{2}} & \text{if } n=2 \\ \sqrt{\dfrac{5}{9}}>\dfrac{1}{\sqrt{2}} & \text{if } n>2 \end{cases}
\end{aligned}
\tag{3}
$$

with equality at $\star$ if and only if $|v_i| = \|v\| / \sqrt{n}$ for all $i = 1, \ldots, n$.

2. Dominant normal matrices

Our first theorem gives a sufficient condition for a normal matrix to be dominant. The Frobenius norm of a complex matrix A is denoted by $\|A\| \equiv (\mathrm{tr} A^*A)^{1/2}$; if A is normal, then $\mathrm{tr} A^*A$ is equal to the sum of the squares of the moduli of the eigenvalues of A.

Theorem 3. *Suppose that $n \geqslant 2$, let A be a given n-by-n complex normal matrix with eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$, and let v be a given complex n-vector.*

1. *If $|\lambda_n| \geqslant \cdots \geqslant |\lambda_1|$ and*

$$\left|v^*Av\right| > \frac{\|A\|\ \|v\|^2}{\sqrt{2}}, \tag{4}$$

then $|\lambda_n| > \|A\| / \sqrt{2} > |\lambda_{n-1}|$, so A is dominant.

2. *If A is Hermitian, $\lambda_n \geqslant \cdots \geqslant \lambda_1$, and*

$$v^*Av > \frac{\|A\|\ \|v\|^2}{\sqrt{2}}, \tag{5}$$

then $\lambda_n > \|A\| / \sqrt{2} > |\lambda_i|$ *for all* $i = 1, \ldots, n-1$, *so* A *has a positive dominant eigenvalue.*

Proof. The hypotheses ensure that A and v are both nonzero, so we may assume that $\|A\| = \|v\| = 1$. Let $A = U \Lambda U^*$ be a spectral decomposition of A, with a unitary $U = [u_1 \ \ldots \ u_n]$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then

$$v^* A v = |v^* u_1|^2 \lambda_1 + \cdots + |v^* u_{n-1}|^2 \lambda_{n-1} + |v^* u_n|^2 \lambda_n \tag{6}$$

expresses $v^* A v$ as a convex combination of the eigenvalues of A. For the first assertion, we know that $|v^* A v| > 1/\sqrt{2}$, so Lemma 1(1) ensures that $|\lambda_n| > 1/\sqrt{2} > |\lambda_{n-1}|$.

The second assertion follows in the same way from Lemma 1(2). □

If A is dominant, then the eigenspace associated with its dominant eigenvalue is one-dimensional, so every dominant eigenvector has no zero entries if and only if some dominant eigenvector has no zero entries. We are interested in conditions that prevent an eigenvector associated with a dominant (or any algebraically simple) eigenvalue λ from having any zero entries. A necessary and sufficient condition is, of course, that every set of $n-1$ columns of $A - \lambda I$ has rank $n-1$, but we are interested in inequalities similar to those in (4), which do not involve the value of the eigenvalue.

The 2-by-2 case is special and easy to analyze. Let u_1 and u_2 be orthonormal eigenvectors of a 2-by-2 normal matrix that is not a scalar multiple of the identity. Then both u_1 and u_2 are uniquely determined up to a unit scalar factor. If u_2 has a zero entry, then u_1 has a zero in the other entry and the matrix is diagonal. Thus, dominant or not, an eigenvector of a 2-by-2 normal non-scalar matrix has no zero entry if and only if the matrix is not diagonal.

Theorem 4. *Suppose* $n \geqslant 2$, *let* A *be a given* n*-by-*n *complex normal matrix*, *and let* $v = [v_i]$ *be a given complex* n*-vector with no zero entries.*

1. *If* $|v^* A v| > \kappa(v) \|A\|$, *then* A *is dominant and no entry of a dominant eigenvector of* A *is zero.*
2. *If* A *is real and symmetric*, *if* v *is real*, *and if* $v^{\mathrm{T}} A v > \kappa(v) \|A\|$, *then* A *has a positive dominant eigenvalue and an associated real dominant eigenvector whose entries are nonzero and have the same signs as the corresponding entries of* v.

Proof. The hypothesis for the first assertion and the bounds (3) ensure that

$$\left|v^* A v\right| > \kappa(v) \|A\| \geqslant \frac{\|A\| \|v\|^2}{\sqrt{2}}, \tag{7}$$

so it follows from Theorem 3 that A is dominant.

Let $A = U\Lambda U^*$ be a spectral decomposition of A, with a unitary $U = [u_1 \ \cdots \ u_n]$ and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, in which λ_n is the dominant eigenvalue; the last column of U is then a dominant unit eigenvector. Since

$$\begin{aligned} \left|v^* A v\right| &= \left| \left|v^* u_1\right|^2 \lambda_1 + \cdots + \left|v^* u_{n-1}\right|^2 \lambda_{n-1} + \left|v^* u_n\right|^2 \lambda_n \right| \\ &> \kappa(v) \, \|A\| \,, \end{aligned} \tag{8}$$

Lemma 2 ensures that u_n has no zero entry.

Now suppose that A is real and symmetric and that v is real. Consider the real symmetric analytic family $A(t) \equiv tA + (1-t)\, vv^{\mathrm{T}}$ for $0 \leqslant t \leqslant 1$ and use (3) and the triangle inequality to verify that

$$\begin{aligned} v^{\mathrm{T}} A(t) v = t v^{\mathrm{T}} A v + (1-t) v^{\mathrm{T}} v v^{\mathrm{T}} v &= t v^{\mathrm{T}} A v + (1-t) \left\| v v^{\mathrm{T}} \right\| \|v\|^2 \\ &> t\kappa(v) \, \|A\| + (1-t)\kappa(v) \left\| v v^{\mathrm{T}} \right\| \\ &\geqslant \kappa(v) \left\| tA + (1-t) v v^{\mathrm{T}} \right\| = \kappa(v) \, \|A(t)\| \,, \end{aligned} \tag{9}$$

so $v^{\mathrm{T}} A(t) v > \kappa(v) \, \|A(t)\|$ for each $t \in [0, 1]$. Theorem 3 ensures that each $A(t)$ has a positive dominant eigenvalue and part 1 of the present theorem says that a dominant eigenvector has no zero entries. Standard results ensure that there is a unit vector function $x(t)$ that is continuous on $[0, 1]$ and is such that $x(t)$ is a dominant eigenvector of $A(t)$ for each $t \in [0, 1]$ [1, Theorem 18.2.1]. Since v, a dominant eigenvector of $A(0)$, is a positive or negative multiple of $x(0)$, by considering $-x(t)$ instead of $x(t)$ we may assume that each entry of $x(0)$ has the same sign as the corresponding entry of v. Since no entry of $x(t)$ becomes zero as t evolves from 0 to 1, each entry of the dominant eigenvector $x(1)$ must have the same sign as the corresponding entry of v. □

For the n-vector $v = e$ whose entries are all equal to +1, $\kappa(e) = \sqrt{(n-1)^2 + 1}$. The condition

$$e^{\mathrm{T}} A e > \sqrt{(n-1)^2 + 1} \, \|A\|$$

is then sufficient to ensure that a real n-by-n symmetric matrix A has a positive dominant eigenvalue and an associated eigenvector with positive entries. This is the essential assertion in Theorem 4.1 in [2].

3. The cases of equality

Suppose that A is normal, $A \neq 0$, $v \neq 0$, and inequality (4) is an equality; for convenience assume that A and v are normalized so that $\|A\| = \|v\| = 1$.

If A is not dominant then Lemma 1(1b) ensures that $|\lambda_n| = |\lambda_{n-1}| = 1/\sqrt{2}$ and $\lambda_i = 0$ if $i \notin \{n-1, n\}$, so rank $A = 2$. Moreover, all of the following inequalities in (1) are equalities:

$$|\lambda_n| = \frac{1}{\sqrt{2}} \geqslant \left|v^* u_{n-1}\right|^2 \frac{1}{\sqrt{2}} + \left|v^* u_n\right|^2 \frac{1}{\sqrt{2}} = \left|v^* u_{n-1}\right|^2 |\lambda_{n-1}| + \left|v^* u_n\right|^2 |\lambda_n|$$
$$\geqslant \left| \left|v^* u_{n-1}\right|^2 \lambda_{n-1} + \left|v^* u_n\right|^2 \lambda_n \right| = 1 \Big/ \sqrt{2}. \tag{10}$$

The first inference we make from (10) is that $|v^* u_{n-1}|^2 + |v^* u_n|^2 = 1$, so $v \in \operatorname{span}\{u_{n-1}, u_n\}$; the second is that $|v^* u_{n-1}|^2 \lambda_{n-1}$ and $|v^* u_n|^2 \lambda_n$ both lie on the same ray from the origin. There are two possibilities: either (a) $|v^* u_{n-1}|^2$ and $|v^* u_n|^2$ are both positive, which forces $\lambda_{n-1} = \lambda_n$ and ensures that span $\{u_{n-1}, u_n\}$ is an eigenspace of A; or (b) one of $|v^* u_{n-1}|^2$ or $|v^* u_n|^2$ is equal to zero and the other is equal to one, which means that either $v \in$ span $\{u_{n-1}\}$ or $v \in$ span $\{u_n\}$. In either event (a) or (b), v is an eigenvector of A.

Conversely, if rank $A = 2$ then $A = \lambda_{n-1} u_{n-1} u_{n-1}^* + \lambda_n u_n u_n^*$ and $\lambda_n \neq 0 \neq \lambda_{n-1}$. Suppose v is an eigenvector of A, so that $Av = \lambda v$, $|\lambda| = |v^* A v| = 1/\sqrt{2}$, and $\lambda = \lambda_{n-1}$ or $\lambda = \lambda_n$. In either case, the normalization $|\lambda_{n-1}|^2 + |\lambda_n|^2 = 1$ ensures that $|\lambda_{n-1}| = |\lambda_n| = 1/\sqrt{2}$, so A is not dominant.

Similar arguments can be made for the case of equality in (5), but we can say a little more: since $\lambda_n \geqslant v^* A v = 1/\sqrt{2}$, we know that the spectral radius of A is positive and equals λ_n.

We summarize what we have just learned as follows:

Theorem 5. *Suppose that $n \geqslant 2$, let A be a given nonzero n-by-n complex matrix, and let v be a given nonzero complex n-vector.*

1. *Suppose that A is normal and $|v^* A v| = \|A\| \; \|v\|^2 / \sqrt{2}$. If* rank $A > 2$, *then A is dominant. If* rank $A = 2$, *then A is dominant if and only if v is not an eigenvector of A.*
2. *Suppose that A is Hermitian and $v^* A v = \|A\| \; \|v\|^2 / \sqrt{2}$. Then the spectral radius of A is positive and is an eigenvalue of A. If* rank $A > 2$, *then A is dominant. If* rank $A = 2$, *then A is dominant if and only if v is not an eigenvector of A.*

Of course, every rank one normal matrix is dominant and is a scalar multiple of a Hermitian matrix.

Now consider the cases of equality in Theorem 4, whose notation we adopt. For convenience, we again assume that $\|A\| = \|v\| = 1$ and that $|v_k| = \min_{1 \leqslant i \leqslant n} |v_i|$. Assume that $n \geqslant 3$ and that equality holds in (8), so A is dominant: $|\lambda_n| > |\lambda_{n-1}|$. We assume that the jth entry of u_n is zero in order to discover the remarkably special form that A must then have. The inequalities that must all be equalities are

$$\begin{aligned}\kappa(v) = \left|v^* A v\right| &= \left|\sum_{i=1}^{n} \left|v^* u_i\right|^2 \lambda_i\right| \overset{\blacktriangle}{\leqslant} \sum_{i=1}^{n} \left|v^* u_i\right|^2 |\lambda_i| \\ &\overset{\blacklozenge}{\leqslant} |v_k|^2 |\lambda_{n-1}| + \left(1 - |v_k|^2\right) |\lambda_n| \overset{\spadesuit}{\leqslant} \kappa(v) \sqrt{|\lambda_{n-1}|^2 + |\lambda_n|^2} \overset{\bigstar}{\leqslant} \kappa(v).\end{aligned}$$

Equality at $\bigstar$ implies that $|\lambda_{n-1}|^2 + |\lambda_n|^2 = 1$, so $\lambda_1 = \cdots = \lambda_{n-2} = 0$ and rank $A \leqslant 2$.

Equality in the Cauchy–Schwarz inequality at $\spadesuit$ implies that the vectors

$$\begin{bmatrix} |\lambda_n| \\ |\lambda_{n-1}| \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 - |v_k|^2 \\ |v_k|^2 \end{bmatrix}$$

are proportional. Since the vector on the left is a unit vector and the norm of the vector on the right is $\kappa(v)$, this means that $|\lambda_n| = \left(1 - |v_k|^2\right)/\kappa(v)$ and $|\lambda_{n-1}| = |v_k|^2 / k(v) > 0$, so rank $A = 2$.

Equality at $\blacklozenge$ and the bounds $|v^* u_n|^2 \leqslant 1 - |v_k|^2$ and $|\lambda_{n-1}| < |\lambda_n|$ imply that $|v^* u_n|^2 = 1 - |v_k|^2$ and $|v^* u_{n-1}|^2 = |v_k|^2$, so $|v^* u_{n-1}|^2 + |v^* u_n|^2 = 1$, both terms are nonzero, $v \in \operatorname{span}\{u_{n-1}, u_n\}$, and we have equalities in the inequalities

$$1 - |v_k|^2 = \left|v^* u_n\right|^2 = \left|\sum_{i \neq j}^{n} \bar{v}_i u_i^{(n)}\right|^2 \leqslant \sum_{i \neq j}^{n} |v_i|^2 \leqslant \sum_{i \neq k}^{n} |v_i|^2 = 1 - |v_k|^2 .$$

This means that $|v_k| = \left|v_j\right|$ and that $v - v_j e_j = \gamma u_n$ for some scalar γ; it follows that only the jth entry of u_n is zero. Since u_n and e_j are orthogonal unit vectors, we have $1 = \|v\|^2 = \left\|\gamma u_n + v_j e_j\right\|^2 = |\gamma|^2 + \left|v_j\right|^2$, or $|\gamma|^2 = 1 - |v_k|^2$. This tells us that u_n is a unit scalar multiple of $\left(1 - |v_k|^2\right)^{-1/2} \left(v - v_j e_j\right)$. Since $v = \gamma u_n + v_j e_j$, u_{n-1} and e_j are orthogonal to u_n, and $v \in \operatorname{span}\{u_{n-1}, u_n\}$, we can conclude that u_{n-1} is a unit scalar multiple of e_j, so e_j is an eigenvector of A associated with the eigenvalue λ_{n-1}. Since A is normal, e_j is also a left eigenvector of A associated with the eigenvalue λ_{n-1}. This means that the jth column of A is $\lambda_{n-1} e_j$, and the jth row of A is $\lambda_{n-1} e_j^{\mathrm{T}}$.

Equality in the triangle inequality at $\blacktriangle$ means that $|v^* u_{n-1}|^2 \lambda_{n-1}$ and $|v^* u_n|^2 \lambda_n$ lie on the same ray from the origin. Since $|v^* u_{n-1}|$ and $|v^* u_n|$ are both nonzero, we conclude that λ_{n-1} and λ_n both lie on the same ray from the origin, i.e., for some real θ, $\lambda_{n-1} = \mathrm{e}^{\mathrm{i}\theta} |\lambda_{n-1}| = \mathrm{e}^{\mathrm{i}\theta} |v_k|^2 / \kappa(v)$ and $\lambda_n = \mathrm{e}^{\mathrm{i}\theta} |\lambda_n| = \mathrm{e}^{\mathrm{i}\theta} \left(1 - |v_k|^2\right)/\kappa(v)$. This means that A is *essentially positive semidefinite*, i.e., $\mathrm{e}^{-\mathrm{i}\theta} A$ is Hermitian and positive semidefinite.

If P is a given permutation matrix such that $P e_j = e_1$, define the $(n-1)$-vector w by

$$P v = \begin{bmatrix} v_j \\ w \end{bmatrix} = v_j P e_j + \gamma P u_n = v_j e_1 + \gamma P u_n .$$

Then

$$
\begin{aligned}
PAP^{\mathrm{T}} &= P\left(\lambda_{n-1}u_{n-1}u_{n-1}^{*} + \lambda_n u_n u_n^{*}\right)P^{\mathrm{T}} = P\left(\lambda_{n-1}e_j e_j^{\mathrm{T}} + \lambda_n u_n u_n^{*}\right)P^{\mathrm{T}} \\
&= \lambda_{n-1}\left(Pe_j\right)\left(Pe_j\right)^{\mathrm{T}} + \lambda_n\left(Pu_n\right)\left(Pu_n\right)^{*} \\
&= \lambda_{n-1}e_1e_1^{\mathrm{T}} + \frac{\lambda_n}{|\gamma|^2}\begin{bmatrix}0\\ w\end{bmatrix}\begin{bmatrix}0 & w^{*}\end{bmatrix} = \left[\frac{\mathrm{e}^{\mathrm{i}\theta}\,|v_k|^2}{\kappa(v)}\right] \oplus \left[\frac{\mathrm{e}^{\mathrm{i}\theta}}{\kappa(v)}ww^{*}\right].
\end{aligned}
$$

Theorem 6. *Suppose $n \geqslant 3$, let A be a given nonzero n-by-n complex normal matrix, let $v = [v_i]$ be a given nonzero complex n-vector with no zero entries, let k be an index such that $|v_k| \equiv \min_{1\leqslant i\leqslant n}|v_i|$, and suppose that $|v^{*}Av| = \kappa(v)\,\|A\|$. Then*
1. *A is dominant;*
2. *A dominant eigenvector of A has at most one zero entry; and*
3. *A dominant eigenvector of A has a zero entry in position j if and only if $\left|v_j\right| = |v_k|$ and there is a real number θ and a permutation matrix P such that $Pe_j = e_1$,*

$$
Pv = \begin{bmatrix} v_j \\ w \end{bmatrix}, \quad PAP^{\mathrm{T}} = \frac{\mathrm{e}^{\mathrm{i}\theta}\,\|A\|}{\kappa(v)}\begin{bmatrix} |v_k|^2 & 0^{\mathrm{T}} \\ 0 & ww^{*} \end{bmatrix}.
$$

In this event,

$$
P^{T}\begin{bmatrix}0\\ w\end{bmatrix} = v - v_j e_j
$$

is a dominant eigenvector of A, and except for the one zero entry in position j, its entries have the same arguments as those of v. The associated dominant eigenvalue is $\mathrm{e}^{\mathrm{i}\theta}\,\|A\|\left(\|v\|^2 - |v_k|^2\right)/\kappa(v)$.

Suppose that A is nonzero and normal and that $|v^{*}Av| = \kappa(v)\,\|A\|$. Theorem 6(3) ensures that a dominant eigenvector of A has no zero entries if any of the following conditions holds:
(a) A has rank greater than two; or
(b) A does not have the prescribed pattern of zero entries; or
(c) Not all the diagonal entries of A are nonzero and lie on the same ray from the origin.

However, if an entry (and only one) of a dominant eigenvector of A is zero, and if all the entries of v are positive, then there is a dominant eigenvector with $n-1$ positive entries and one zero entry.

Our final result concerns the signs of the entries of a dominant eigenvector in the equality case of Theorem 4.

Theorem 7. *Suppose $n \geqslant 3$, let A be a given nonzero n-by-n real symmetric matrix, let $v = [v_i]$ be a real n-vector with no zero entries, let k be an index such that $|v_k| \equiv \min_{1\leqslant i\leqslant n}|v_i|$, and suppose that $v^{\mathrm{T}}Av = \kappa(v)\,\|A\|$. Then A has a positive dominant eigenvalue and an associated real eigenvector with at most one zero entry.*

1. *A dominant eigenvector has a zero entry in position j if and only if $\left|v_j\right| = |v_k|$ and there is a permutation matrix P such that $Pe_j = e_1$,*
$$Pv = \begin{bmatrix} v_j \\ w \end{bmatrix}, \quad PAP^{\mathrm{T}} = \frac{\|A\|}{\kappa(v)} \begin{bmatrix} v_k^2 & 0^{\mathrm{T}} \\ 0 & ww^{\mathrm{T}} \end{bmatrix}. \tag{11}$$
In this event,
$$P^{\mathrm{T}} \begin{bmatrix} 0 \\ w \end{bmatrix} = v - v_j e_j$$
is a real dominant eigenvector of A, and except for the one zero entry in position j its entries have the same signs as those of v. The associated dominant eigenvalue is $\|A\| \left(\|v\|^2 - |v_k|^2\right)/\kappa(v)$.
2. *If there is no permutation matrix P that achieves the representations in* (11), *then A has a real dominant eigenvector whose entries are nonzero and have the same signs as the corresponding entries of v.*

Proof. The first assertion follows immediately from Theorem 6. For the second, we proceed as in the proof of Theorem 4. Consider the analytic real symmetric family $A(t) \equiv tA + (1-t)\,vv^{\mathrm{T}}$ for $0 \leqslant t \leqslant 1$. Since $\kappa(v) < \|v\|^2$, the computation (9) shows that $v^{\mathrm{T}}A(t)v > \kappa(v)\,\|A(t)\|$ for all $t \in [0, 1)$. The argument in the proof of Theorem 4 shows that there is a real vector family $x(t)$ such that: $x(t)$ is continuous on the whole interval $[0, 1]$, $x(t)$ is a dominant eigenvector of $A(t)$ for each $t \in [0, 1]$, and $x(0)$ is a positive scalar multiple of v. Theorem 4 ensures that no entry of $x(t)$ is zero for $0 \leqslant t < 1$, and Theorem 6 tells us that no entry of $x(1)$ is zero. We conclude that no entry of $x(t)$ can become zero as t moves from 0 to 1, so each entry of $x(1)$ has the same sign as the corresponding entry of v. □

One consequence of this final theorem is a result of Perron–Frobenius type: if A is a nonzero real symmetric matrix of size at least three, and if there is a vector v with positive entries such that $v^{\mathrm{T}}Av \geqslant \kappa(v)\,\|A\|$, then, regardless of the signs of its entries, A has a positive dominant eigenvalue and an associated eigenvector with nonnegative entries and at most one zero entry; if A does not have the very special form (11), then A has a dominant eigenvector with positive entries.

Acknowledgements

It is a pleasure to acknowledge valuable conversations with Raphael Loewy and Leiba Rodman about this article.

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