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ON THE COHOMOLOGY OF SOLUBLE GROUPS OF FINITE RANK

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1. Introduction

An $\mathfrak{S}_0$-group, or soluble group with finite abelian section rank, is a soluble group in which elementary abelian p-sections* are finite for all primes p. Equivalently we could define these as the groups which possess a normal series of finite length whose factors are either torsion-free abelian groups of finite rank or direct products of abelian p-groups with Min (the minimal condition on subgroups) for different primes p.

Our attention will be directed primarily at two subclasses of $\mathfrak{S}_0$. The first is the class $\mathfrak{S}_1$ of all $\mathfrak{S}_0$-groups which have a series of finite length with abelian factors whose torsion-subgroups satisfy Min; secondly there is the narrower class of *soluble minimax groups.* Recall that a minimax group is group which has a series of finite length whose factors satisfy *either* Min *or* Max (the maximal condition on subgroups).

We shall prove theorems about the structure of cohomology groups of soluble minimax groups and $\mathfrak{S}_1$-groups in the case where the coefficient module, *qua* abelian group, is an $\mathfrak{S}_0$-group. The following is a consequence of one of these theorems, and the original motivation for this paper.

Theorem 1.1. *A finitely generated $\mathfrak{S}_0$-group is a minimax group.*

This settles an outstanding problem which was raised a number of years ago ([6]) and has been answered in special cases (see [7, p. 176]).

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* A *section* of a group is a quotient group of a subgroup.

2. The cohomology theorems

Let G be a finite extension of a soluble minimax group; then G has a series of finite length whose factors are finite or cyclic or quasicyclic. A simple application of the Schreier Refinement Theorem shows the set of primes p for which the series has a quasicyclic factor of type p^∞ to be an invariant of G; this set we call the *spectrum* of G and write as

$$\mathrm{sp}(G).$$

Clearly this is just the set of all p for which G has a p^∞-section. For example, G is polycyclic-by-finite if and only if $\mathrm{sp}(G)$ is empty.

Our first main result deals with modules which are periodic $\mathfrak{S}_0$-groups *qua* abelian groups, i.e., they are direct sums of abelian p-groups satisfying Min for different primes p.

Theorem 2.1. *Let G be a finite extension of a soluble minimax group and let A be a G-module. Assume that qua abelian group A is a periodic $\mathfrak{S}_0$-group without p^∞-subgroups for any p in* $\mathrm{sp}(G)$. *Then $H^n(G, A)$ is a periodic $\mathfrak{S}_0$-group for $n \geqslant 0$.*

In our application to prove Theorem 1.1 we need only the case $n = 2$. The significance of the periodicity of $H^2(G, A)$ was pointed out in [8, Lemma 10]. Let G be any group and A any G-module; suppose that the extension $A \rightarrowtail E \twoheadrightarrow G$ determines an element of finite order m in $H^2(G, A)$ and that θ denotes multiplication in A by m. Then there is a subgroup X of E such that

$$|E : XA| \leqslant |\mathrm{Coker}\,\theta|, \qquad |X \cap A| \leqslant |\mathrm{Ker}\,\theta|.$$

If A is an $\mathfrak{S}_0$-group *qua* additive group, Ker θ and Coker θ are finite, whence so are $|E: XA|$ and $|X \cap A|$. Under these circumstances E is said to *nearly split* over A.

Combining these remarks with Theorem 2.1, we can state the following.

Corollary 2.2. *Let G and A be as in Theorem* 2.1. *Then every extension of A by G is nearly split.*

Here we have used only the periodicity of $H^2(G, A)$. In fact $H^2(G, A)$ also belongs to $\mathfrak{S}_0$, which tells us that extensions of A by G that are associated with elements of $H^2(G, A)$ with bounded orders belong to finitely many equivalence classes.

One cannot expect the cohomology groups of Theorem 2.1 to be finite. For example, if G is a finitely generated torsion-free nilpotent group and A is any trivial G-module,

$$H^r(G, A) \simeq A$$

where r is the Hirsch number* of G (Gruenberg [3, p. 152]). However it turns out that under somewhat more general circumstances the cohomology groups are eventually all finite, a fact which rests upon results of Gruenberg on the cohomological dimension of soluble groups and on detailed information about the structure of $\mathfrak{S}_1$-groups.

We shall prove the following.

Theorem 2.3. *Let G be a finite extension of an $\mathfrak{S}_1$-group and let A be a G-module which is an $\mathfrak{S}_0$-group qua abelian group. Assume that if G has a subgroup of type p^∞, then A has no such subgroups and A is divisible by p modulo its torsion subgroup. Denote by r the Hirsch number of G. Then $H^n(G, A)$ is finite for all $n > r+1$, and even for $n > r$ if G is an extension of a group with* Min *by one with* Max.

An essential tool in proving Theorems 2.1 and 2.3 is the spectral sequence of Lyndon–Hochschild–Serre; its consequence the exact sequence of Hochschild-Serre–Hattori is also useful (for these see, for example, [5,pp. 351, 355]). Theorems 2.1 and 2.3 are proved in §§4 and 5.

In Theorems 2.1 and 2.3 restrictions are placed upon the p^∞-sections or subgroups of group and module. Without these restrictions both theorems fail badly, as examples in §6 show. In the same section appear examples which show Theorem 2.1 to be false for $\mathfrak{S}_1$-groups and Theorem 2.3 for $\mathfrak{S}_0$-groups.

3. Deduction of Theorem 1.1 from Theorem 2.1

Let E be a finitely generated $\mathfrak{S}_0$-group. Then E has a *normal* series of finite length l whose factors are abelian and either periodic or torsion-free: we can assume that $l > 1$ and proceed by induction on l. Let A be the least non-trivial term of the series. Then $G = E/A$ is a minimax group and A is abelian. If A is torsion-free, E is certainly an $\mathfrak{S}_1$-group and by previous work it is a minimax group [7, Theorem (10.38)]. Thus we are left with the case where A is periodic.

Let $\pi = \mathrm{sp}(G)$, a finite set of primes. The π-component of A satisfies Min and can therefore be factored out. A is a G-module via conjugation and the conditions of Theorem 2.1 are satisfied. By Corollary 2.2 there is a subgroup X of E such that $|E : XA|$ and $|X \cap A|$ are finite. XA is finitely generated, say by $x_1a_1, \ldots, x_ra_r$. Set F equal to the smallest X-admissible subgroup of A containing $\{a_1, \ldots, a_r\}$ and $X \cap A$; then F is finite. Also $XA = XF$, so that $A = (XF) \cap A = F$. Hence E is a minimax group.

* A group has finite *Hirsch number* (or torsion-free rank) r if it has a series of finite length whose factors are infinite cyclic or periodic and r is the number of infinite cyclic factors. For locally polycyclic groups this is equivalent – although not immediately so – to the definition given in [3, p. 149].

4. Necessary lemmas

We shall now prove three lemmas which will be used in the proofs of Theorems 2.1 and 3. If one wants to prove Theorem 1.1 as economically as possible however, only Lemma 4.1 is needed, and that in a very special case.

Lemma 4.1. *Let G be a group which is the union of a well-ordered ascending chain of subgroups $\{G_\alpha : \alpha < \beta\}$ and let A be a G-module such that for some positive integer n and all α one has $H^{n-1}(G_\alpha, A) = 0 = H^n(G_\alpha, A)$. Then $H^n(G, A) = 0$.*

Proof. Let f be an n-cocycle of G with coefficients in A and write $f^{(\alpha)}$ for the restriction of f to G_α. Since $H^n(G_\alpha, A) = 0$, there is an $(n-1)$-cochain g_α of G_α such that $f^{(\alpha)} = \delta g_\alpha$. Now $\delta(g^{(\alpha)}_{\alpha+1} - g_\alpha) = 0$ and, since $H^{n-1}(G_\alpha, A) = 0$, it follows that $g^{(\alpha)}_{\alpha+1} - g_\alpha = \delta h_\alpha$ for some $(n-2)$-cochain h_α of G_α. Extend h_α to $G_{\alpha+1}$ in any way and define $\bar{g}_{\alpha+1} = g_{\alpha+1} - \delta h_\alpha$. This is a new $(n-1)$-cochain of $G_{\alpha+1}$ which extends g_α and satisfies $\delta \bar{g}_{\alpha+1} = f^{(\alpha+1)}$. In this way we may extend g_0 to an $(n-1)$-cochain of G such that $f = \delta g$.

Corollary 4.2. *Let G be a countable locally finite group and let A be a G-module. Assume that as an abelian group A is uniquely divisible by all primes dividing orders of elements of G. Then $H^n(G, A) = 0$ for all $n > 1$. If A is a trivial G-module, then $H^1(G, A) = 0$ as well.*

To prove the corollary form a chain of finite subgroups $G_1 < G_2 < ...$ with union G; then $H^n(G_i, A) = 0$ for $n > 0$ by a classical theorem. Of course, if A is a trivial module, $H^1(G, A) = \mathrm{Hom}(G, A) = 0$.

Note, however, that if A is non-trivial, $H^1(G, A)$ may well be non-zero, a fact which is reflected in the non-conjugacy of the Sylow 2-subgroups in the restricted direct product of a countable infinity of copies of the symmetric group of degree 3 (see also the structure of $H^1(Q, A)$ in §6, Example (B)).

Lemma 4.3. *Let G be a finite group of order m and let A be a G-module. Assume that the endomorphism $\theta : a \mapsto ma$ of A has finite kernel and cokernel. Then $H^n(G, A)$ is finite for all $n > 0$: moreover primes dividing the orders of these groups must divide $|\mathrm{Ker}\,\theta| \cdot |\mathrm{Coker}\,\theta|$.*

Proof. Let M denote the sum of the p-components of A for p dividing m, and write $A_1 = A/M$. Then θ induces a monomorphism in A_1 and $A_1 \overset{\theta}{\rightarrowtail} A_1 \twoheadrightarrow A_1/A_1\theta$ is exact. Applying the cohomology sequence we get for $n > 0$

$$... \to H^n(G, A_1) \xrightarrow{\theta^*} H^n(G, A_1) \to H^n(G, A_1/A_1\theta) \to ...$$

Obviously θ^* is multiplication by m; hence $\theta^* = 0$ since $|G| = m$. Thus $H^n(G, A_1) \rightarrowtail H^n(G, A_1/A_1\theta)$, and the latter is finite since $|A_1 : A_1\theta| \leq |\mathrm{Coker}\,\theta|$.

Let D be the maximal divisible subgroup of M. Then, since $\operatorname{Ker}\theta$ is finite, M satisfies Min and M/D is finite with order dividing a power of $|\operatorname{Ker}\theta|$. Hence $H^n(G, M/D)$ is finite. Finally, θ induces an epimorphism in D and application of the cohomology sequence to $K \rightarrowtail D \xrightarrow{\theta} D$ (where $K = \operatorname{Ker}(\theta|_D)$) yields $H^n(G, K) \twoheadrightarrow H^n(G, D)$, whence $H^n(G, A)$ is finite, and the proof is completed by a count of primes. (In the case $\operatorname{Ker}\theta = 0 = \operatorname{Coker}\theta$ the conclusion is that $H^n(G, A) = 0$ for $n > 0$, a well known fact.)

Lemma 4.4. *Let G be a finite extension of an $\mathfrak{S}_0$-group and let A be a finite G-module. Then $H^n(G, A)$ is finite for all $n \geqslant 0$.*

Proof. Form in G a series of finite length whose factors are of the following types: (i) finite, (ii) periodic $\mathfrak{S}_0$ with elements of orders coprime to $|A|$, (iii) torsion-free abelian of rank 1, (iv) p^∞ for p dividing $|A|$. Let N be the largest proper term of this series. Then $N \lhd G$ and by induction on the length of the series $H^q(N, A)$ is finite. Let

$$E^{p,q} = H^p(G/N, H^q(N, A))$$

where $p + q = n$. By the spectral sequence of Lyndon–Hochschild–Serre there is a series in $H^n(G, A)$ of length $n + 2$ whose factors are isomorphic with sections of the $E^{p,q}$ for $p + q = n$. It is therefore sufficient to prove each $E^{p,q}$ finite. Hence we can assume that the series has length 1 and that G is of one of the types (i)–(iv). If G is finite the result is clear; thus we are left with types (ii)–(iv). If C is the centraliser of A in G, then G/C is finite and the spectral sequence argument just given shows that it is enough to prove $H^n(C, A)$ finite. Hence we can assume that A is a trivial G-module.

If G is of type (ii), then $H^n(G, A) = 0$ for $n > 0$ by Corollary 4.2. If G is of type (iii), $H^1(G, A) = \operatorname{Hom}(G, A)$, which is finite since G/G^m is finite for every $m > 0$. An extension of A by G is necessarily abelian, since G has rank 1, and also splits over A since the latter is finite: hence $H^2(G, A) = 0$. For $n > 2$ we have $H^n(G, A) = 0$ because G has cohomological dimension $\leqslant 2$.

Finally, suppose that G is of type p^∞. Writing $\mathbf{Q}_p$ for the additive group of p-adic rationals mp^n, $(m, n \in \mathbf{Z})$, we have an exact sequence

$$\mathbf{Z} \rightarrowtail \mathbf{Q}_p \twoheadrightarrow G.$$

Thus A becomes a trivial $\mathbf{Q}_p$-module. Since $H^n(\mathbf{Z}, A) = 0$ for $n > 1$ and $H^1(\mathbf{Z}, A) \overset{G}{\simeq} A$, the Hochschild–Serre–Hattori exact sequence yields

$$\ldots \to H^n(G, A) \to H^n(\mathbf{Q}_p, A) \to H^{n-1}(G, A) \to H^{n+1}(G, A) \to H^{n+1}(\mathbf{Q}_p, A) \to \ldots$$

for $n > 0$. Now $H^n(\mathbf{Q}_p, A) = 0$ for $n > 1$ by the last paragraph, so one obtains

$$H^{n-1}(G, A) \simeq H^{n+1}(G, A) \quad \text{if } n > 1,$$

which shows that $H^{2n+1}(G, A) \simeq H^1(G, A) = \mathrm{Hom}(G, A) = 0$. For $n = 1$ the exact sequence yields $A \to H^2(G, A) \to 0$. Hence $H^2(G, A)$ is finite, and $H^{2n}(G, A) \simeq H^2(G, A)$, $(n > 0)$, gives the result. (This method can be used to compute the cohomology of quasicyclic groups with arbitrary coefficient modules.)

5. Proofs of the main results

Proof of Theorem 2.1. Let π denote the spectrum of G. By hypothesis the π-component P of A is finite and Lemma 4.4 shows that $H^n(G, P)$ is finite for all $n \geqslant 0$. Henceforth we assume that $P = 0$ and A has no elements of order p for any p in π.

Form a series in G with finite length whose factors are finite, cyclic or quasicyclic, and denote by N the largest proper term. Then G/N is finite, cyclic or quasicyclic. By induction on the length of the series and the Lyndon–Hochschild–Serre spectral sequence we can assume that $N = 1$: notice here that $H^q(N, A)$, like A, can have no elements of order p in π; the reason is that $a \mapsto pa$, being an N-automorphism of A, induces an automorphism in $H^q(N, A)$.

If G is finite, so is $H^n(G, A)$ for $n > 0$ by Lemma 4.3. If G is cyclic, well-known formulae show $H^n(G, A)$ to be isomorphic with a section of A and hence a periodic $\mathfrak{S}_0$-group. Finally, suppose G is of type p^∞. Now the automorphism group of an abelian group with Min – and hence that of A – is residually finite. Therefore A is in fact a trivial G-module and $H^n(G. A) = 0$ for $n > 0$ by Corollary 4.2.

Proof of Theorem 2.3. By the structure of $\mathfrak{S}_1$-groups [7, Theorems 10.33 and 9.39.3] there exist normal subgroups R and N of G such that

(i) R is a divisible abelian group with Min,

(ii) $R \leqslant N$,

(iii) N/R is an extension of a torsion-free nilpotent group of a finite rank by a free abelian group of finite rank,

(iv) G/N is finite, of order m say.

Step 1. $H^n(R, A)$ is finite for all $n > 0$.

Let π be the set of primes dividing orders of elements of R; then π is finite and by hypothesis the π-component A_0 of A is finite. Lemma 4.4 now shows that $H^n(R, A_0)$ is finite for all n. Hence we may assume in this part of the proof that $A_0 = 0$. Denote by T the torsion-subgroup of A; then $H^n(R, T) = 0$ for all $n > 0$ by Corollary 4.2, since R must act trivially on T. Consider next $A_1 = A/T$: by hypothesis A_1 is (uniquely) divisible by primes in π. Since a p^∞-group can have no non-trivial representations of finite degree over the rational field*, A_1 is a trivial R-module. Corollary 4.2 now implies that $H^n(R, A_1) = 0$ for all $n > 0$. Hence $H^n(R, A) = 0$ if $n > 0$.

* More generally, periodic subgroups of $GL(n, Q)$ are finite, a result due essentially to Schur: see for example [7, Vol. 2, p. 85].

Step 2. $H^n(N, A)$ is finite if $n > r + 1$ (and if $n > r$ in case G is Min-by-Max).

Let $n = p + q > r + 1$ (or r) and consider

$$E^{p,q} = H^p(N/R, H^q(R, A)).$$

It is sufficient to prove that $E^{p,q}$ is finite. Now if $q > 0$, Step 1 shows that $H^q(R, A)$ is finite, so that $E^{p,q}$ is finite by Lemma 4.4. If $q = 0$, then $n = p > r + 1$ (or r) and $E^{p,0} = 0$ because N/R has cohomological dimension $\leqslant r + 1$ (or $\leqslant r$ if N/R has Max); this follows from results of Gruenberg [3, §8, Theorem 5 and Proposition 9].*

Step 3. Case A periodic.

By the structure of A there is a finite submodule B such that A/B is divisible by m. From Lemma 4.4 we know that $H^n(G, B)$ is finite, so we can assume that $B = 0$ and $A = mA$. Then multiplication in A by m is an epimorphism which we call θ. Applying the cohomology sequence for N to $K = \operatorname{Ker}\theta \rightarrowtail A \xrightarrow{\theta} A$ we obtain

$$\ldots \to H^n(N, K) \to H^n(N, A) \xrightarrow{\theta^*} H^n(N, A) \to H^{n+1}(N, K) \to \ldots$$

Here $H^n(N, K)$ and $H^{n+1}(N, K)$ are finite ($n \geqslant 0$) since K is. Consequently θ^*, which is just multiplication by m, has finite kernel and cokernel. Lemma 4.3 now shows that

$$E^{p,q} = H^p(G/N, H^q(N, A))$$

is finite provided $p > 0$.

Let $n = p + q > r + 1$ (or r). We have only to prove that $E^{p,q}$ is finite and only the case $p = 0$ is questionable; but then $n = q > r + 1$ (or r) and $H^q(N, A)$ is finite by Step 2; hence $E^{0,q}$ is finite.

Step 4. Case A torsion-free.

Here $\theta: a \mapsto ma$ is monomorphic and $A \xrightarrow{\theta} A \twoheadrightarrow \operatorname{Coker}\theta$ is exact. The cohomology sequence for N shows that θ^* has finite kernel and cokernel. One then argues that $E^{p,q}$ is finite for $n = p + q > r + 1$ (or r) as in Step 3.

The theorem now follows from Steps 3 and 4.

Remark. Theorem 2.3 is still true if A/T has infinite rank but one must now write $r + 1$ for r in the conclusion. The reason is that in Step 1 A_1 need not be a trivial module and consequently $H^1(R, A_1)$ may not vanish.

6. Counterexamples

In this final section we shall describe examples which block various plausible generalisations of Theorems 2.1 and 2.3.

(A) *Theorem 2.1 and Corollary 2.2 are false without the restriction on p^∞-subgroups.*

* For further results in this direction see Bieri [1].

Let G be the direct product of two copies of the additive group of p-adic rationals mp^n, $(m, n \in \mathbf{Z})$, and let G_1 be one of the direct factors. Then G is a torsion-free abelian minimax group with spectrum $\{p\}$. Let A be a group of type p^∞ regarded as a trivial G-module. Then $H^1(G_1, A) \simeq \mathrm{Hom}(G_1, A)$ and $H^2(G_1, A) = 0$ because any extension of A by G_1 is abelian. Now apply the Hochschild–Serre–Hattori exact sequence to $G_1 \rightarrowtail G \twoheadrightarrow G/G_1 \simeq G_1$. Since $H^1(G_1, A) \overset{G}{\simeq} \mathrm{Hom}(G_1, A)$, one obtains

$$H^2(G, A) \simeq \mathrm{Hom}(G_1, \mathrm{Hom}(G_1, A)) \simeq \mathrm{Hom}(G_1 \otimes G_1, A) \simeq \mathrm{Hom}(G_1, A).$$

Observe that $\mathrm{Hom}(G_1, A)$ is a rational vector space V of dimension the continuum. Our conclusion is, then, that

$$H^1(G, A) \simeq V \simeq H^2(G, A).$$

(Alternatively this can be deduced from the Universal Coefficient Theorem for cohomology.)

It follows that there are continuously many inequivalent extensions of A by G none of which is nearly split.

(B) *Theorem 2.1 is false if G is merely an $\mathfrak{S}_1$-group.*

Strictly speaking this statement requires qualification since the spectrum of an $\mathfrak{S}_1$-group has not been defined. Our example, however, seems to exclude any reasonable formulation.

Choose a sequence of distinct primes $p_1, q_1, p_2, q_2, \ldots$, with the property $p_i \equiv 1 \bmod q_i$; that this is possible is a consequence of the infinity of primes in an arithmetic progression. Let G be the additive group generated by the rational numbers $1/q_1, 1/q_2, \ldots$, and let A be the direct sum of cyclic groups $A_1, A_2, \ldots$, of orders $p_1, p_2, \ldots$. Since $p_i \equiv 1 \bmod q_i$, there is a non-zero homomorphism $G \to \mathrm{Aut}\, A_i$ under which $\mathbf{Z}$ maps to the identity subgroup; these homomorphisms combine to yield $G \to \mathrm{Aut}\, A$ and this turns A into a G-module. Let Q be a direct sum of cyclic groups of orders $q_1, q_2, \ldots$; apply the Hochschild–Serre–Hattori exact sequence to $\mathbf{Z} \rightarrowtail G \twoheadrightarrow Q$, noting that $A^{\mathbf{Z}} = A$, $A^G = 0$ and $H^n(Q, A) = 0$ for $n > 1$ (by Lemma 4.1). The conclusion is that

$$H^1(G, A) \simeq H^1(Q, A) \simeq H^2(G, A).$$

$H^1(Q, A)$ can be computed directly by constructing all derivations of Q to A and identifying the inner ones. One finds that

$$H^1(Q, A) \simeq (\mathop{\mathrm{Cr}}_i A_i)/(\mathop{\mathrm{Dr}}_i A_i),$$

which is a rational vector space with the dimension of the continuum.

Notice that G is a torsion-free abelian $\mathfrak{S}_1$-group: moreover G has no p^∞-sections for any p and corresponding to primes which are the orders of elements of A there are zero entries in the *type* of G (in the sense of Fuch [2, p. 147]).

(C) *Theorem 2.3 is false without the restriction on p^∞-subgroups.*

Let G be the direct product of a p^∞-group G_1 and an infinite cyclic group; let A be a p^∞-group regarded as a trivial G-module. Then

$$H^1(G,A) \simeq R_p \oplus A, \qquad H^n(G,A) \simeq R_p, \quad n > 1,$$

where R_p is the additive group of p-adic integers.

To obtain this, first compute the cohomology of G_1 using the exact sequence $\mathbf{Z} \rightarrowtail \mathbf{Q}_p \twoheadrightarrow G_1$, just as in the proof of Lemma 4.4 (last part). The result is that

$$H^{2n}(G_1,A) = 0, \quad (n > 0), \qquad H^{2n+1}(G_1,A) \simeq R_p.$$

We may now use a formula of Lyndon [4, Lemma 9.1 or Theorem 7]

$$H^n(G,A) \simeq H^n(G_1,A) \oplus H^{n-1}(G_1,A)$$

to give the result in question.

If we take A to be $\mathbf{Z}$ instead of p^∞, again regarded as a trivial module, then

$$H^1(G,A) \simeq \mathbf{Z}, \qquad H^n(G,A) \simeq R_p, \quad n > 1.$$

Hence both parts of the restriction on p^∞-subgroups are relevant.

(D) *Theorem 2.3 is false if G is merely an $\mathfrak{S}_0$-group.*

Let G and A each be direct sums of cyclic groups of all possible prime orders and regard A as a trivial G-module. Then

$$H^n(G,A) \simeq \operatorname*{Cr}_p \mathbf{Z}_p, \quad n > 0,$$

where $\mathbf{Z}_p$ is cyclic of prime order p.

This may be seen as follows:

Let P be the additive group generated by all $1/p$ for p a prime. Then there is an exact sequence $\mathbf{Z} \rightarrowtail P \twoheadrightarrow G$. With A a trivial P-module one has $H^n(P,A) = 0$ for $n > 2$ while $H^2(P,A) \simeq \operatorname{Ext}(P,A)$ is divisible since P is torsion-free, and $H^1(G,A) \simeq \operatorname{Cr}_p \mathbf{Z}_p$ is residually finite. Hence any homomorphism $H^2(P,A) \to H^1(G,A)$ must be zero. One putting this information into the Hochschild–Serre–Hattori exact sequence one finds that $H^{n+1}(G,A) \simeq H^{n-1}(G,A)$ for $n \geqslant 2$. Moreover

$$H^2(G,A) \simeq \operatorname{Ext}(G,A) \simeq \operatorname*{Cr}_p(\operatorname{Ext}(\mathbf{Z}_p,A)) \simeq \operatorname*{Cr}_p \mathbf{Z}_p.$$

The result now follows.

References

[1] R. Bieri, Ueber die cohomologische Dimension der auflösbaren Gruppen. Math. Z. 128 (1972) 235–243.
[2] L. Fuchs, Abelian groups (Pergamon Press, Oxford, 1960).
[3] K.W. Gruenberg, Cohomological topics in group theory, Lecture Notes in Mathematics 143 (Springer, Berlin, 1970).
[4] R.C. Lyndon, The cohomology theory of group extensions. Duke Math. J. 15 (1948) 271–292.
[5] S. MacLane, Homology (Springer, Berlin, 1963).
[6] D.J.S. Robinson, Residual properties of some classes of infinite soluble groups. Proc. London Math. Soc. (3) 18 (1968) 495–520.
[7] D.J.S. Robinson, Finiteness conditions and generalized soluble groups (Springer, Berlin, 1972).
[8] D.J.S. Robinson, Splitting theorems for infinite groups, Symposia Math. (to appear).