On a spectral property of doubly stochastic matrices and its application to their inverse eigenvalue problem

Bassam Mourad *

Department of Mathematics, Faculty of Science V, Lebanese University, Nabatieh, Lebanon

**Abstract**

In this note, we present a useful theorem concerning the spectral properties of doubly stochastic matrices. As applications, we use this result together with some known results that we recall, as a tool for extracting new sufficient conditions for the inverse eigenvalue problem for doubly stochastic matrices.

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1. Introduction

A doubly stochastic matrix is a nonnegative matrix such that each row and column sum is equal to 1. The theory of doubly stochastic matrices has been the object of research for such a long time. This particular interest in this theory as well as the theory of nonnegative matrices stems from the fact that it is endowed with a rich collection of applications that goes beyond mathematics to include many areas of physics and engineering as well as many other disciplines such as economics and operation research (see [1–3,7,8,6]).

The Perron–Frobenius theorem states that if $A$ is a nonnegative matrix, then it has a real eigenvalue $r$ (that is the Perron–Frobenius root) which is greater than or equal to the modulus of each of the other eigenvalues. Also, $A$ has an eigenvector $x$ corresponding to $r$ such that each of its entries are nonnegative. Furthermore, if $A$ is irreducible then $r$ is positive and the entries of $x$ are also positive.

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**Tel.**: +961 3 784363; fax: +961 7 768174.

**E-mail address**: bmourad@ul.edu.lb
(see [4,5,15,22]). In particular, it is well-known that if $A$ is an $n \times n$ doubly stochastic matrix then $r = 1$ and the corresponding eigenvector $x = e_n = \frac{1}{\sqrt{n}} (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ where \( \mathbb{R} \) denote the real line. Consequently, if $\lambda_i$ is any eigenvalue of a doubly stochastic matrix, then $|\lambda_i| \leq 1$. Now if $I$ is the imaginary unit and the complex plane is denoted by $\mathbb{C}$, then we have the following three inverse eigenvalue problems for doubly stochastic matrices that will refer to in this paper as the three major problems.

**Problem 1.1.** The doubly stochastic inverse eigenvalue problem denoted by (DIEP), is the problem of determining the necessary and sufficient conditions for a complex $n$-tuples to be the spectrum of an $n \times n$ doubly stochastic matrix. Equivalently, it is the problem of finding the region $\Theta_n$ (which is referred to as the region corresponding to this problem) of $\mathbb{C}^n$ where the spectra of all doubly stochastic matrices lie.

**Problem 1.2.** The real doubly stochastic inverse eigenvalue problem (RDIEP) asks which lists of $n$ real numbers occur as the spectrum of an $n \times n$ doubly stochastic matrix. This problem is essentially equivalent to describing the region $\Theta_n^R$ of $\mathbb{R}^n$ where the real spectra of doubly stochastic matrices lie. More precisely, the corresponding region to this problem is given by $\Theta_n^R = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n; \text{ such that there exists a doubly stochastic with spectrum } \lambda \}$. Moreover, $\Theta_n^s$ is the spectrum of the doubly stochastic matrix $B = \begin{pmatrix} 1/12 & 1/6 & 1/4 \\ 1/12 & 2/3 & 1/4 \\ 1/3 & 1/6 & 1/2 \end{pmatrix}$.

**Problem 1.3.** The symmetric doubly stochastic inverse eigenvalue problem (SDIEP) asks which sets of $n$ real numbers occur as the spectrum of an $n \times n$ symmetric doubly stochastic matrix. This problem corresponds the region $\Theta_n^s$ of $\mathbb{R}^n$ where the spectra of symmetric doubly stochastic matrices lie. More precisely, $\Theta_n^s = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n; \text{ such that there exists a symmetric doubly stochastic with spectrum } \lambda \}$. The following example appears in [19] and helps to see the difference among the three major problems.

**Example 1.** The point $\alpha = (1, -1/2 + i\sqrt{3}/2, -1/2 - i\sqrt{3}/2)$ is in $\Theta_3$ as $\alpha$ is the spectrum of the doubly stochastic matrix $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. On the other hand, the list $\lambda = (1, 1/2, 1/4)$ is in $\Theta_3^s$ since $\lambda$ is the spectrum of the doubly stochastic matrix $B = \begin{pmatrix} 7/12 & 1/6 & 1/4 \\ 1/12 & 2/3 & 1/4 \\ 1/3 & 1/6 & 1/2 \end{pmatrix}$. Moreover, $\lambda$ is in $\Theta_3^s$ as $\lambda$ is also the spectrum of the symmetric doubly stochastic $C = \begin{pmatrix} 13/24 & 7/24 & 1/6 \\ 7/24 & 13/24 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{pmatrix}$.

When $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a solution of any of these problems, i.e., $\lambda$ is the spectrum of an $n \times n$ doubly stochastic matrix $A$, then we will say that $A$ realizes $\lambda$ or that $\lambda$ is realized by $A$. Moreover it is clear that $\Theta_n^s \subseteq \Theta_n^R \subset \Theta_n$. However, the question whether $\Theta_n^s$ is strictly contained in $\Theta_n^R$ or not remains open. This is also essentially equivalent to answering the question of whether (RDIEP) and (SDIEP) are generally equivalent. For $n = 3$, the two problems are equivalent, i.e., $\Theta_3^s = \Theta_3^R$ (see [19]). While for $n \geq 4$, it remains a very interesting open problem. For more on this subject see [11, 16, 20, 23, 24] and the references therein.

The (SDIEP) was studied in [10, 12, 17, 18, 21], and earlier work can be found in [11, 20, 24] and all the results obtained are partial. In addition, all three problems have been completely solved for $n = 3$.
in [19, 20]. For general \( n \geq 4 \) all three problems remain open. By analogy to what was explained in [18] for (SDIEP), complete solutions to any of the three major problems require complete characterizations of all the boundary sets of its corresponding region. These boundary sets contain in particular the spectra of reducible doubly stochastic matrices in one hand and those of trace-zero on the other hand as well as those spectra that have \(-1\) as any of its components. Therefore, solving the trace-zero inverse eigenvalue problem for any of the three major problems is perhaps the first step in order to offer complete solutions.

Let \( I_n \) be the \( n \times n \) identity matrix and \( J_n \) the \( n \times n \) matrix whose all entries are \( \frac{1}{n} \) and denote by \( K_n \) to be the \( n \times n \) matrix whose diagonal entries are zeros and all whose off-diagonal entries are equal to \( \frac{1}{n-1} \). Note that \((1, \ldots, 1), (1, 0, \ldots, 0)\) and \((1, -\frac{1}{n-1}, \ldots, -\frac{1}{n-1})\) are, respectively, realized by \( I_n, J_n \) and \( K_n \). Now if \((1, \lambda_2, \ldots, \lambda_m)\) and \((1, \mu_2, \ldots, \mu_n)\) are realized by the two \( m \times m \) and \( n \times n \) doubly stochastic matrices \( A \) and \( B \), respectively, then clearly \((1, 1, \lambda_2, \ldots, \lambda_m, \mu_2, \ldots, \mu_n)\) is realized by the \((m+n) \times (m+n)\) reducible doubly stochastic matrix \( A \oplus B \). As a generalization of this construction to irreducible doubly stochastic matrices, we present in the second section, a result concerning spectral properties of doubly stochastic matrices which in turn leads to a method that serves as a technique for finding sufficient conditions for the above three problems. The proofs are constructive in the sense that one can easily construct the realizing matrix. To illustrate the efficiency of this method, we study some of its consequences on the above three major problems. In particular, we use it in Section 3, to identify new subregions of \( \Theta_n^s, \Theta_n^r \) and \( \Theta_n \) some of which are boundary sets.

2. Main observations

Our main goal here is to first present some auxiliary results that we are going to exploit their advantages later. The first one is due to Fiedler [9].

**Lemma 2.1** [9]. Let \( A \) be an \( m \times m \) symmetric matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_m \), and let \( u \) be the unit eigenvector corresponding to \( \lambda_1 \). Let \( B \) be an \( n \times n \) symmetric matrix with eigenvalues \( \mu_1, \mu_2, \ldots, \mu_n \), and let \( v \) be the unit eigenvector corresponding to \( \mu_1 \). Then for any \( \rho \), the matrix \( C = \begin{pmatrix} A & \rho uv^T \\ \rho vu^T & B \end{pmatrix} \) has eigenvalues \( \lambda_2, \ldots, \lambda_m, \mu_2, \ldots, \mu_n \) and \( \gamma_1, \gamma_2 \) where \( \gamma_1, \gamma_2 \) are the eigenvalues of the matrix

\[
\begin{pmatrix}
\lambda_1 & \rho \\
\rho & \mu_1
\end{pmatrix}.
\]

In the proof of the above lemma, the author uses the fact that each of the symmetric matrices \( A \) and \( B \) has a complete set of orthogonal eigenvectors (see [9]). However, a closer look at the proof shows that a weaker hypothesis is needed. More precisely, the fact that the matrices \( A \) and \( B \) have complete set of orthogonal eigenvectors can be replaced by the hypothesis that \( A \) and \( B \) have each a complete set of eigenvectors such that the unit eigenvector \( u \) of \( A \) is orthogonal to all other eigenvectors of \( A \) and the same is true for the unit eigenvector \( v \) of \( B \). For the record, we state this observation in here.

**Lemma 2.2.** Let \( A \) be an \( m \times m \) diagonalizable matrix with eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_m \), and let \( u \) be the unit eigenvector corresponding to \( \lambda_1 \) such that \( u \) is orthogonal to all other eigenvectors of \( A \). Let \( B \) be an \( n \times n \) diagonalizable matrix with eigenvalues \( \mu_1, \mu_2, \ldots, \mu_n \), and let \( v \) be the unit eigenvector corresponding to \( \mu_1 \) such that \( v \) is orthogonal to all other eigenvectors of \( B \). Then for any \( \rho \), the matrix \( C = \begin{pmatrix} A & \rho uv^T \\ \rho vu^T & B \end{pmatrix} \) has eigenvalues \( \lambda_2, \ldots, \lambda_m, \mu_2, \ldots, \mu_n \) and \( \gamma_1, \gamma_2 \) where \( \gamma_1, \gamma_2 \) are the eigenvalues of the matrix

\[
\begin{pmatrix}
\lambda_1 & \rho \\
\rho & \mu_1
\end{pmatrix}.
\]
Now we identify a potential class of matrices for which the conditions of the preceding lemma are satisfied.

**Lemma 2.3.** Let $D = (d_{ij})$ be an $n \times n$ matrix whose each row and column sum is equal to $s$. Then clearly $e_n$ is an eigenvector of $D$ corresponding to the eigenvalue $s$ and any other eigenvector $x = (x_1, x_2, \ldots, x_n)^T$ that corresponds to an eigenvalue $\lambda \neq s$ is orthogonal to $e_n$.

**Proof.** For any vector $x$, let $\sigma(x)$ denote in general the sum of the components of $x$. Hence in order to prove the lemma, it suffices to prove that $\sigma((x_1, x_2, \ldots, x_n)^T) = 0$. Using the fact that $x = (x_1, x_2, \ldots, x_n)^T$ is a eigenvector of $D$ with $Dx = \lambda x$, we obtain

$$Dx = \begin{pmatrix} d_{11}x_1 + d_{12}x_2 + \cdots + d_{1n}x_n \\ d_{21}x_1 + d_{22}x_2 + \cdots + d_{2n}x_n \\ \vdots \\ d_{(n-1)1}x_1 + d_{(n-1)2}x_2 + \cdots + d_{(n-1)n}x_{n-1} \\ d_{n1}x_1 + d_{n2}x_2 + \cdots + d_{nn}x_n \end{pmatrix},$$

and then

$$\sigma(Dx) = x_1\sum_{i=1}^n d_{1i} + \cdots + x_n\sum_{i=1}^n d_{ni} = x_1(s) + \cdots + x_n(s) = s\sigma(x) = \sigma(\lambda x) = \lambda\sigma(x).$$

Thus $\sigma(x) = \lambda\sigma(x)$ and since $\lambda \neq s$ then $\sigma(x) = 0$. □

For the class of doubly stochastic matrices, we have the following.

**Lemma 2.4.** Let $D = (d_{ij})$ be a $d \times d$ doubly stochastic matrix. Then for each eigenvalue $\lambda$ of $D$, there exists an associated eigenvector $x$ such that if $x \neq e_d$ then $x$ is orthogonal to $e_d$.

**Proof.** If $D$ is irreducible then by the preceding lemma, the result is true. On the other hand, if $D$ is reducible then it suffices to prove the statement for the multiple eigenvalue $1$. As $D$ is reducible then it is cogredient to a direct sum of irreducible doubly stochastic matrices (see [15, p. 109]), i.e., there exist irreducible doubly stochastic matrices $A_1, A_2, \ldots, A_k$, and a permutation matrix $P$ such that $D = P^T(A_1 \oplus A_2 \oplus \cdots \oplus A_k)P$. For simplicity we assume that $D$ is cogredient to a direct sum of two irreducible doubly stochastic matrices, i.e., $D = P^T(A \oplus B)P$ where $A$ and $B$ are, respectively, $m \times m$ and $n \times n$ irreducible doubly stochastic matrices with $d = m + n$ (the case for more can be done in a similar fashion by induction). Clearly $e_m$ and $e_n$ are, respectively, the eigenvectors of $A$ and $B$ corresponding to the eigenvalue 1. Now a direct verification shows that the two vectors $x_1 = \begin{pmatrix} e_m \\ 0 \end{pmatrix}$ and $x_2 = \begin{pmatrix} 0 \\ e_n \end{pmatrix}$ of $\mathbb{R}^{m+n}$ are eigenvectors of $A \oplus B$ corresponding to the double eigenvalue 1. In addition, the two linearly independent vectors $w_1 = \frac{1}{\sqrt{m}}x_1 + \frac{1}{\sqrt{n}}x_2 = e_{m+n}$ and $w_2 = \frac{1}{\sqrt{m}}x_1 - \frac{1}{\sqrt{n}}x_2$ are also eigenvectors of $A \oplus B$ corresponding to the double eigenvalue 1 such that the sum of the entries of $w_2$ is zero. However, $A \oplus B = PDP^T$ and then $(A \oplus B)w_2 = w_2 = PDP^Tw_2$ or $DP^Tw_2 = P^Tw_2$. Hence $P^Tw_2$ is the second eigenvector of $D$ corresponding to the double eigenvalue 1 with the sum of the entries of $P^Tw_2$ is zero. This completes the proof. □

Thus without loss of generality, we can assume that any eigenvector $x \neq e_d$ of a $d \times d$ doubly stochastic matrix is orthogonal to $e_d$. Hence we have the following lemma which can be thought as a generalization of Lemma 2.1 in the case of doubly stochastic matrices.

**Lemma 2.5.** Let $A$ be an $m \times m$ diagonalizable doubly stochastic matrix with eigenvalues $1, \lambda_2, \ldots, \lambda_m$, and let $B$ be an $n \times n$ diagonalizable doubly stochastic matrix with eigenvalues $1, \mu_2, \ldots, \mu_n$. Then for
any $\rho$, the matrix $C = \begin{pmatrix} A & \rho e_m e_n^T \\ \rho e_n e_m^T & B \end{pmatrix}$ has eigenvalues $\lambda_2, \ldots, \lambda_m, \mu_2, \ldots, \mu_n$, and $\gamma_1, \gamma_2$ where
\[
\gamma_1, \gamma_2 \in \{1 + \rho, 1 - \rho\} \text{ which are the eigenvalue of } \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.
\]

**Proof.** Let \{e_m, u_2, \ldots, u_m\} and \{e_n, v_2, \ldots, v_n\} be the complete sets of eigenvectors of $A$ and $B$, respectively, with $u_i$ corresponds to $\lambda_i$ and $v_i$ corresponds to $\mu_i$. Using the preceding lemma with a direct verification shows that for $i = 2, \ldots, m$ and $j = 2, \ldots, n$, the $m + n - 2$ vectors \( \begin{pmatrix} u_i \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ v_j \end{pmatrix} \) of $\mathbb{R}^{m+n}$ are eigenvectors of $C$ corresponding to the eigenvalues $\lambda_i$ and $\mu_i$, respectively. In addition, the other two eigenvectors of $C$ are given by: \( \begin{pmatrix} e_m \\ e_n \end{pmatrix} \) and \( \begin{pmatrix} e_m \\ -e_n \end{pmatrix} \) which corresponds to the eigenvalues $1 + \rho$ and $1 - \rho$, respectively, and the proof is complete. $\square$

As a consequence, we have the following useful theorem.

**Theorem 2.6.** Let $A$ be an $m \times m$ diagonalizable doubly stochastic matrix with eigenvalues $1, \lambda_2, \ldots, \lambda_m$, and let $B$ be an $n \times n$ diagonalizable doubly stochastic matrix with eigenvalues $1, \mu_2, \ldots, \mu_n$. For any $\alpha \geq 0$ and for any $\rho \geq 0$ such that $\alpha$ and $\rho$ do not vanish simultaneously, then the $(m+n) \times (m+n)$ matrix $C$ given by:

- For $n \geq m$, $C = \frac{1}{\alpha + \frac{\rho n}{\sqrt{mn}}} \begin{pmatrix} \alpha A & \rho e_m e_n^T \\ \rho e_n e_m^T & (\alpha + \rho \frac{n-m}{\sqrt{mn}}) B \end{pmatrix}$ is doubly stochastic with eigenvalues given by:
  
  $1, \frac{\alpha \sqrt{mn} - \rho m}{\alpha \sqrt{mn} + \rho n}, \frac{\alpha \sqrt{mn} - \rho m}{\alpha \sqrt{mn} + \rho n} \lambda_2, \ldots, \frac{\alpha \sqrt{mn} - \rho m}{\alpha \sqrt{mn} + \rho n} \lambda_m, \frac{\alpha \sqrt{mn} + \rho (n-m)}{\alpha \sqrt{mn} + \rho n} \mu_2, \ldots, \frac{\alpha \sqrt{mn} + \rho (n-m)}{\alpha \sqrt{mn} + \rho n} \mu_n.$
- For $m \geq n$, $C = \frac{1}{\alpha + \frac{\rho n}{\sqrt{mn}}} \begin{pmatrix} \alpha A & \rho e_m e_n^T \\ \rho e_n e_m^T & (\alpha + \rho \frac{n-m}{\sqrt{mn}}) B \end{pmatrix}$ is doubly stochastic with eigenvalues given by:
  
  $1, \frac{\alpha \sqrt{mn} - \rho m}{\alpha \sqrt{mn} + \rho n}, \frac{\alpha \sqrt{mn} - \rho m}{\alpha \sqrt{mn} + \rho n} \lambda_2, \ldots, \frac{\alpha \sqrt{mn} - \rho m}{\alpha \sqrt{mn} + \rho n} \lambda_m, \frac{\alpha \sqrt{mn} + \rho (n-m)}{\alpha \sqrt{mn} + \rho n} \mu_2, \ldots, \frac{\alpha \sqrt{mn} + \rho (n-m)}{\alpha \sqrt{mn} + \rho n} \mu_n.$

**Proof.** Let $n \geq m$. First note that $\rho e_m e_n^T$ is an $m \times n$ matrix whose all entries are equal to $\frac{\rho}{\sqrt{mn}}$ where each row sum is $\frac{n \rho}{\sqrt{mn}}$ and each column sum is $\frac{m \rho}{\sqrt{mn}}$. Also as $\rho e_m e_n^T$ is the transpose of $\rho e_n e_m^T$ and $A$ and $B$ are doubly stochastic, then $C$ is doubly stochastic. In addition, a virtually identical proof to that of the preceding lemma shows that the eigenvalues of $C$ are given by:

$$
\begin{pmatrix}
\frac{\alpha}{\alpha + \frac{\rho n}{\sqrt{mn}}} \lambda_2, \ldots, \frac{\alpha}{\alpha + \frac{\rho n}{\sqrt{mn}}} \lambda_m, \frac{\alpha \sqrt{mn} - \rho m}{\alpha \sqrt{mn} + \rho n} \mu_2, \ldots, \frac{\alpha \sqrt{mn} + \rho (n-m)}{\alpha \sqrt{mn} + \rho n} \mu_n, \gamma_1, \gamma_2
\end{pmatrix}
$$

where $\gamma_1, \gamma_2$ are the eigenvalues of the matrix $X = \frac{1}{\alpha + \frac{\rho n}{\sqrt{mn}}} \begin{pmatrix} \alpha & \rho \\ \rho & \alpha + \rho \frac{n-m}{\sqrt{mn}} \end{pmatrix}$ which are given by $\gamma_1, \gamma_2 \in \left\{1, \frac{\alpha \sqrt{mn} - \rho m}{\alpha \sqrt{mn} + \rho n} \right\}$. The case $m \geq n$ is done obviously by exchanging the roles of $m$ and $n$. $\square$
Suppose that

Proof. It suffices to note that for

Lemma 3.1. Theorem 3.2 mentioned earlier that the three major problems are completely solved only for $n = 2$ and $n = 3$. Indeed, the three problems are equivalent for the case $n = 2$ since obviously any $2 \times 2$ doubly stochastic matrix is necessarily symmetric. In addition, the following lemma solves the problem for the case $n = 2$.

Lemma 3.1. There exists a $2 \times 2$ symmetric doubly stochastic matrix with spectrum $(1, \lambda_1)$ if and only if $-1 \leq \lambda_1 \leq 1$.

Proof. It suffices to note that for $-1 \leq \lambda_1 \leq 1$, the matrix $X = \begin{pmatrix} \frac{1+\lambda_1}{2} & \frac{1-\lambda_1}{2} \\ \frac{1-\lambda_1}{2} & \frac{1+\lambda_1}{2} \end{pmatrix}$ is doubly stochastic and has spectrum $(1, \lambda_1)$. □

For the case $n = 3$, $\Theta_3^{\mathbb{C}} = \Theta_3^{\mathbb{R}}$ (see [19]) and the following theorem solves this case.

Theorem 3.2 [20]. There exists a symmetric $3 \times 3$ doubly stochastic matrix with spectrum $(1, \lambda, \mu)$ if and only if $-1 \leq \lambda \leq 1$, $-1 \leq \mu \leq 1$, $\lambda + 3\mu + 2 \geq 0$ and $3\lambda + \mu + 2 \geq 0$.

Proof. Suppose that $\lambda \geq \mu$, then the matrix $X = \frac{1}{6} \begin{pmatrix} 2 + 4\lambda & 2 - 2\lambda & 2 - 2\lambda \\ 2 - 2\lambda & 2 + \lambda + 3\mu & 2 + \lambda - 3\mu \\ 2 - 2\lambda & 2 + \lambda - 3\mu & 2 + \lambda + 3\mu \end{pmatrix}$ is doubly stochastic and has spectrum $(1, \lambda, \mu)$. Now if $\mu \geq \lambda$, then the proof can be completed by exchanging the roles of $\lambda$ and $\mu$. □

Now if we let $\Pi_3$ denote the region of the complex plane specified as the convex hull of the cubic roots of unity. Then in the complex case and for $n = 3$, we have the following theorem which is due to [20] and the proof which we include here appears in [14].

Theorem 3.3 [20]. Let $z$ be a complex number and $\bar{z}$ be its complex conjugate. Then $(1, z, \bar{z})$ is the spectrum of a $3 \times 3$ doubly stochastic matrix if and only if $z \in \Pi_3$. 
Proof. It suffices to check that the circulant matrix
\[
X = \frac{1}{3} \begin{pmatrix} 1 + 2r \cos \theta & 1 - 2r \cos(\frac{\pi}{3} + \theta) & 1 - 2r \cos(\frac{\pi}{3} - \theta) \\ 1 - 2r \cos(\frac{\pi}{3} - \theta) & 1 + 2r \cos \theta & 1 - 2r \cos(\frac{\pi}{3} + \theta) \\ 1 - 2r \cos(\frac{\pi}{3} + \theta) & 1 - 2r \cos(\frac{\pi}{3} - \theta) & 1 + 2r \cos \theta \end{pmatrix}
\]
has spectrum \((1, z = re^{i\theta}, \bar{z} = re^{-i\theta})\), and \(X\) is doubly stochastic if and only if \(z = re^{i\theta} \in \Pi_3\). \(\square\)

Next we shall show that the above three theorems together with Theorem 2.6 can serve as building blocks for obtaining many new results concerning the three major problems. This can be illustrated as follows. For all \(i = 2, 3, \ldots\), and for all \(\alpha_i \geq 0\) and \(\rho_i \geq 0\) such that for each \(i, \alpha_i\) and \(\rho_i\) are not zeroes at the same time, define 
\[
X_{m,n}^{(i)} = \frac{\alpha_i \sqrt{mn} + \rho_i (n-m)}{n \sqrt{mn} + \rho_i n}, \quad Y_{m,n}^{(i)} = \frac{\alpha_i \sqrt{mn} + \rho_i (n-m)}{n \sqrt{mn} + \rho_i n}, \quad Z_{m,n}^{(i)} = \frac{\alpha_i \sqrt{mn} + \rho_i (n-m)}{n \sqrt{mn} + \rho_i n}
\]
and \(W_{m,n}^{(i)} = Y_{m,n}^{(i)} - X_{m,n}^{(i)}\) where \(m\) and \(n\) are positive integers. Also, for convenience, let 
\[
Y_{2,n}^{(i)} = y_{2,n}^{(i)}, \quad Z_{2,n}^{(i)} = z_{2,n}^{(i)} \quad \text{and} \quad W_{2,n}^{(i)} = w_{2,n}^{(i)}.
\]
In addition, for any two lists \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)\) and \(\mu = (\mu_1, \mu_2, \ldots, \mu_m)\), and for any real numbers \(a\) and \(b\), we will write \((a\lambda, b\mu)\) to denote the list \((a\lambda_1, a\lambda_2, \ldots, a\lambda_n, b\mu_1, b\mu_2, \ldots, b\mu_m)\).

Using these notations, we have the following theorem which identifies a subregion of \(\Theta_n\).

**Theorem 3.4.** Let \(n = 2k\) for \(n\) even and \(n = 2k+1\) for \(n\) odd, and let \(1 \leq \lambda_j \leq 1\) for all \(j = 1, 2, \ldots, k\). Also let \(\lambda\) be the \((2k-1)\)-list defined by:
\[
\lambda = \left( (w_{2k-2}^{(k)}), (z_{2k-2}^{(k)})_{2k-4}, (z_{2k-2}^{(k)})_{k-1}, (z_{2k-2}^{(k)})_{k-2}, \ldots, (z_{2k-2}^{(k)})_{2}, (y_{2k-2}^{(k)})_{k} \right)\]

Then the \(n\)-list \((1, \lambda)\) defines a subregion of \(\Theta_n\) for \(n = 2k\). In addition, for the case \(n = 2k + 1\) the \(n\)-list defined by \((1, W_{1,2k}^{(k+1)}, Z_{1,2k}^{(k+1)} \lambda)\) forms also a subregion of \(\Theta_n\).

**Proof.** For \(n = 2k\), the proof can be completed by performing the following \(k\) steps.

- **Step1:** Define the doubly stochastic matrix \(C_2 = \begin{pmatrix} 1+\lambda_1 & 1-\lambda_1 \\ 1-\lambda_1 & 1+\lambda_1 \end{pmatrix}\) whose eigenvalues are given by \((1, 1)\).

- **Step2:** In Theorem 2.6, taking \(A = \begin{pmatrix} 1+\lambda_2 & 1-\lambda_2 \\ 1-\lambda_2 & 1+\lambda_2 \end{pmatrix}\) and \(B = C_2\) of Step1, we obtain the \(4 \times 4\) doubly stochastic matrix \(C_4\) whose eigenvalues are given by \((1, (w_2^{(2)}), (z_2^{(2)})_{2}, (y_2^{(2)})_{2}, (z_2^{(2)})_{1})\).

- **Step3:** Repeating the same process by taking this time \(A = \begin{pmatrix} 1+\lambda_3 & 1-\lambda_3 \\ 1-\lambda_3 & 1+\lambda_3 \end{pmatrix}\) and \(B = C_4\) of Step2.

We then obtain the \(6 \times 6\) doubly stochastic matrix \(C_6\) whose eigenvalues are
\[
(1, (w_4^{(3)}), (z_4^{(3)})_{4}, (y_4^{(3)})_{4}, (z_4^{(3)})_{2}, (z_4^{(3)})_{1}).
\]

- **Step4:** Next the same process is repeated with \(A = \begin{pmatrix} 1+\lambda_4 & 1-\lambda_4 \\ 1-\lambda_4 & 1+\lambda_4 \end{pmatrix}\) and \(B = C_6\) of Step3 to obtain
the \(8 \times 8\) doubly stochastic matrix \(C_8\) whose eigenvalues are
\[
(1, (w_6^{(4)}), (z_6^{(4)})_{4}, (z_6^{(4)})_{2}, (z_6^{(4)})_{1}).
\]
Continuing this way and after \( k - 1 \) steps, we obtain the \((2k - 2) \times (2k - 2)\) doubly stochastic matrix \( C_{2k-2} \). So that the \( k \)th step is given by:

- **Stepk**: Let \( A = \left( \begin{array}{cc} \frac{1+\lambda_k}{2} & \frac{1-\lambda_k}{2} \\ \frac{1-\lambda_k}{2} & \frac{1+\lambda_k}{2} \end{array} \right) \) and \( B = C_{2k-2} \) which is found in the \((k-1)\)th Step. Then another application of Theorem 2.6 with these data results in obtaining the \( 2k \times 2k \) doubly stochastic matrix \( C_{2k} \) whose eigenvalues are given by the theorem.

For \( n = 2k + 1 \), it suffices to again apply Theorem 2.6 for \( A = 1 \) and \( B = C_{2k} \) which is the \( 2k \times 2k \) matrix obtained in Stepk. \( \square \)

Next to simplify notations, let \( W^{(i)}_{3,n} = a_n^{(i)}, Y^{(i)}_{3,n} = b_n^{(i)} \) and \( Z^{(i)}_{3,n} = c_n^{(i)} \). Then making use of Theorem 2.6 and Theorem 3.3, we obtain the following result which identifies a subregion of \( \Theta_n \).

**Theorem 3.5.** For all \( j = 1, 2, \ldots, k \), let \( \beta_j \) be any point in \( \Pi_3 \) and let \( \mu \) be the \((3k - 1)\)-list defined by:

\[
\beta = \left( (a_{3k-3}^{(k)}), (c_{3k-3}^{(k-1)} a_{3k-6}^{(k-1)}), (c_{3k-3}^{(k-1)} a_{3k-9}^{(k-2)}), \ldots, (c_{3k-3}^{(k)} \cdots c_{6}^{(2)} a_{3}^{(2)}), (b_{3k-3}^{(k)} \beta_k),\right. \\
\left. (b_{3k-3}^{(k)}) \bar{\beta}_k, (c_{3k-3}^{(k-1)} b_{3k-6}^{(k-1)} \beta_{k-1}), (c_{3k-3}^{(k-1)} b_{3k-6}^{(k-1)} \bar{\beta}_{k-1}), (c_{3k-3}^{(k-1)} b_{3k-6}^{(k-2)} \beta_{k-2}),\right. \\
\left. (c_{3k-3}^{(k-1)} b_{3k-6}^{(k-2)} \bar{\beta}_{k-2}), \ldots, (c_{3k-3}^{(k)} \cdots c_{6}^{(3)} b_{3}^{(2)} \beta_2), (c_{3k-3}^{(k)} \cdots c_{6}^{(3)} b_{3}^{(2)} \bar{\beta}_2),\right. \\
\left. (c_{3k-3}^{(k)} \cdots c_{6}^{(3)} b_{3}^{(2)} \beta_1), (c_{3k-3}^{(k)} \cdots c_{6}^{(3)} b_{3}^{(2)} \bar{\beta}_1). \right)
\]

Then the following two \( n \)-lists \((1, \beta)\) and \((1, W^{(k+1)}_{1,3k}, Z^{(k+1)}_{1,3k} \beta)\) define subregions of \( \Theta_n \) for the cases \( n = 3k \) and \( n = 3k + 1 \), respectively. Moreover for \( n = 3k + 2 \), the following two \( n \)-lists

- \( (1, W^{(k+2)}_{1,3k+3}, Z^{(k+2)}_{1,3k+3} \beta) \)
- \( (1, W^{(k+1)}_{2,3k}, Y^{(k+1)}_{3k} t, Z^{(k+1)}_{3k} \beta) \) for all \(-1 \leq t \leq 1\)

define also two subregions of \( \Theta_n \).

**Proof.** We only need to employ a similar strategy to that of the previous proof so that the proof of the first part can be completed by performing the following \( k \) steps.

- **Step1**: Define the \( 3 \times 3 \) matrix doubly stochastic matrix \( C_3 \) given by Theorem 3.3 and whose eigenvalues are given by \((1, \beta_1, \bar{\beta}_1)\).
- **Step2**: In Theorem 2.6, taking \( A \) to be the \( 3 \times 3 \) doubly stochastic matrix whose eigenvalues are \((1, \beta_2, \bar{\beta}_2)\) and \( B = C_3 \) of Step1, we then obtain the \( 6 \times 6 \) doubly stochastic matrix \( C_6 \) whose eigenvalues are given by \((1, (a_2^{(2)}) (b_2^{(2)}) \beta_2, (b_2^{(2)}) \bar{\beta}_2, (b_2^{(2)}) \beta_1, (b_2^{(2)}) \bar{\beta}_1)\).
- **Step3**: Repeating the same process by taking this time \( A \) to be the \( 3 \times 3 \) doubly stochastic matrix whose eigenvalues are \((1, \beta_3, \bar{\beta}_3)\) and \( B = C_6 \) of Step2. We then obtain the \( 9 \times 9 \) doubly stochastic matrix \( C_9 \) whose eigenvalues are

\[
(1, (a_3^{(3)}), (c_6^{(3)} \beta_3), (b_6^{(3)} \beta_3), (b_6^{(3)} \bar{\beta}_3), (c_6^{(3)} b_3^{(2)} \beta_2), (c_6^{(3)} b_3^{(2)} \bar{\beta}_2), (c_6^{(3)} b_3^{(2)}) \beta_1, (c_6^{(3)} b_3^{(2)} \bar{\beta}_1).\]

- **Step4**: Next repeating the same process with \( A \) as the \( 3 \times 3 \) doubly stochastic matrix whose eigenvalues are \((1, \beta_4, \bar{\beta}_4)\) and \( B = C_9 \) of Step3. We then obtain the \( 12 \times 12 \) doubly stochastic matrix \( C_{12} \) whose eigenvalues are given by the list

\[
(1, (a_4^{(4)}), (c_6^{(4)} a_6^{(3)}), (c_6^{(4)} c_6^{(3)} a_3^{(2)}), (b_4^{(4)} \beta_4), (b_4^{(4)} \bar{\beta}_4), (c_6^{(4)} b_6^{(3)} \beta_3), (c_6^{(4)} b_6^{(3)} \bar{\beta}_3),\right. \\
\left. (c_6^{(4)} c_6^{(3)} b_3^{(2)} \beta_2), (c_6^{(4)} c_6^{(3)} b_3^{(2)} \bar{\beta}_2), (c_6^{(4)} c_6^{(3)} b_3^{(2)}) \beta_1, (c_6^{(4)} c_6^{(3)} b_3^{(2)} \bar{\beta}_1).\right)
\]
We may continue in a similar fashion to obtain the matrix $C_{3k-3}$, and then the $k$th step is given by:

- **Stepk**: Let $A$ be the $3 \times 3$ doubly stochastic matrix whose eigenvalues are $(1, \beta_k, \bar{\beta}_k)$ and $B = C_{3k-3}$ which is found in the $(k-1)$th Step. We then obtain a $3k \times 3k$ doubly stochastic matrix $C_{3k}$ whose eigenvalues are given by the theorem.

Now for the case $n = 2k + 1$, another application of Theorem 2.6 with $A = 1$ and $B = C_{3k}$ which is the $3k \times 3k$ matrix obtained in Stepk, yields a $(3k + 1) \times (3k + 1)$ matrix $C_{3k+1}$ which satisfies the required conditions. For the first part of the case $n = 3k + 2$, we apply Theorem 2.6 with $A = 1$ and $B = C_{3k+1}$, and for the last part, we apply the same theorem with $A = \begin{pmatrix} 1+t & 1-t \\ 1+t & 1-t \end{pmatrix}$ whose spectrum is $(1, t)$ with $-1 \leq t \leq 1$ and $B = C_{3k}$ to complete the proof. □

Now using Theorems 2.6 and 3.2 instead of Theorem 3.3, then a virtually identical proof to that of the above theorem yields the following conclusion.

**Theorem 3.6.** For all $j = 1, 2, \ldots, k$, let $-1 \leq \lambda_j \leq 1$ and $-1 \leq \mu_j \leq 1$ with $|\lambda_j + 3\mu_j + 2| \geq 0$ and $3\lambda_j + \mu_j + 2 \geq 0$. In addition, let $\gamma$ be the $(3k - 1)$-list defined by:

$$
\gamma = \left( (a_{3k-3}^{(k)}), (c_{3k-3}^{(k)}a_{3k-6}^{(k-1)}), (c_{3k-3}^{(k)}c_{3k-6}^{(k-1)}a_{3k-9}^{(k-2)}), \ldots, (c_{3k-3}^{(k)}\cdots c_{6}^{(3)}a_{3}^{(2)}), (b_{3k-3}^{(k)}), (c_{3k-3}^{(k)}b_{3k-6}^{(k-1)}), (c_{3k-3}^{(k)}c_{3k-6}^{(k-1)}b_{3k-9}^{(k-2)}), \ldots, (c_{3k-3}^{(k)}\cdots c_{6}^{(3)}b_{3}^{(2)}) \right).
$$

Then the two $n$-lists $(1, \gamma)$ and $(1, W_{1,3k}^{(k+1)}, Z_{1,3k}^{(k+1)}\gamma)$ define two subregions of $\Theta^k_n$ (resp. $\Theta^k_n$) for the cases $n = 3k$ and $n = 3k + 1$, respectively. Moreover for the case $n = 3k + 2$, the following two $n$-lists

- $(1, W_{1,3k}^{(k+2)}, W_{1,3k}^{(k+1)}, Z_{1,3k}^{(k+2)}, Z_{1,3k}^{(k+1)}\gamma)$
- $(1, W_{3k}^{(k+1)}t, Z_{3k}^{(k+1)}\gamma)$ for all $-1 \leq t \leq 1

define also two subregions of $\Theta^k_n$ (resp. $\Theta^k_n$).

Now for better understanding and clarification of the above three theorems, we restate them as sufficient conditions for the resolution of the inverse eigenvalue problem for doubly stochastic matrices for a fixed $n$ such as the case $n = 6$. Indeed, the results in the preceding three theorems assert, respectively, the following consequences for $n = 6$.

- If $-1 \leq \lambda_j \leq 1$ for $j = 1, 2, 3$, then for any nonnegative numbers $\alpha_1, \rho_1, \alpha_2$ and $\rho_2$ such that $\alpha_1 + \rho_1 \neq 0$, and $\alpha_2 + \rho_2 \neq 0$, the list

$$
\left( 1, \frac{\alpha_2\sqrt{2} - \rho_2}{\alpha_2\sqrt{2} + 2\rho_2}, \frac{\alpha_2\sqrt{2} + \rho_2}{\alpha_2\sqrt{2} + 2\rho_2}(\alpha_1 - \rho_1), \frac{\alpha_2}{\alpha_2 + \rho_2\sqrt{2}}\lambda_3, \frac{\alpha_2\sqrt{2} + \rho_2}{\alpha_2\sqrt{2} + 2\rho_2}(\alpha_1 + \rho_1), \frac{\alpha_2}{\alpha_2 + \rho_2\sqrt{2}}\lambda_2, \frac{(\alpha_2\sqrt{2} + \rho_2)^2}{(\alpha_2\sqrt{2} + 2\rho_2)(\alpha_1 + \rho_1)}(\alpha_1)(\alpha_1 + \rho_1)\lambda_1 \right)
$$

is the spectrum of a $6 \times 6$ symmetric doubly stochastic matrix.

- If $z$ and $w$ are any points in $\Pi_3$, then for any nonnegative numbers $\alpha$ and $\rho$ with $\alpha + \rho \neq 0$, the list
(1, α − ρ, α, α + ρ, α + ρ, α + ρ, α + ρ, α + ρ, α + ρ)

is the spectrum of a 6 × 6 doubly stochastic matrix.

• If −1 ≤ λ_j ≤ 1, −1 ≤ μ_j ≤ 1 with λ_j + 3μ_j + 2 ≥ 0 and 3λ_j + μ_j + 2 ≥ 0 for j = 1, 2, then for any nonnegative numbers α and ρ such that α + ρ ≠ 0,

\[
\begin{pmatrix}
1, \frac{\alpha - \rho}{\alpha + \rho}, \frac{\alpha}{\alpha + \rho}, \frac{\alpha}{\alpha + \rho}, \frac{\alpha}{\alpha + \rho}, \frac{\alpha}{\alpha + \rho}
\end{pmatrix}
\]

is the spectrum of a 6 × 6 symmetric doubly stochastic matrix.

To illustrate these results, we consider a numerical example that uses this last case with the following numerical data. Let α = 9, ρ = 1, λ_1 = μ_1 = \frac{1}{2} and λ_2 = 0 with μ_2 = −\frac{2}{3}, then the list

\[
\left(1, \frac{4}{5}, 0, -\frac{9}{20}, -\frac{9}{20}, -\frac{3}{5}\right)
\]

is realized by the matrix

\[
C = \begin{pmatrix}
0 & 9/2 & 9/2 & 1/3 & 1/3 & 1/3 \\
9/2 & 0 & 9/2 & 1/3 & 1/3 & 1/3 \\
9/2 & 9/2 & 0 & 1/3 & 1/3 & 1/3 \\
1/3 & 1/3 & 1/3 & 3 & 3 & 3 \\
1/3 & 1/3 & 1/3 & 3 & 0 & 6 \\
1/3 & 1/3 & 1/3 & 3 & 6 & 0
\end{pmatrix}.
\]

Here it is worth mentioning that we can repeat the same process with any known results concerning the three major problems to obtain new results. In addition, many more results of the kind we obtained in the last three theorems can be obtained by writing n as the sum of all possible combinations of 1, 2 and 3. To each possible such combination corresponds a new result which is similar in nature to the above theorems. Although for general n it appears to be no systematic way under which this can be done and the collection of all such results that can be obtained in this way seems messy and unattainable at the moment, however, it should be stressed that this can be easily done for a fixed n = n_0.

Finally we shall illustrate what we mentioned earlier that boundary points in lower dimension can be used to generate boundary points in higher dimension. But first recall the following two results appearing in [20] and are concerned with the 3 × 3 trace-zero (DIEP) and 4 × 4 trace-zero (SDIEP).

**Theorem 3.7.** Any 3 × 3 doubly stochastic matrix of trace-zero is the form

\[
\begin{pmatrix}
0 & a & 1 - a \\
1 - a & 0 & a \\
a & 1 - a & 0
\end{pmatrix}
\]

with 0 ≤ a ≤ 1, and whose spectrum is \((1, -\frac{1}{2} + l(a - \frac{1}{2})\sqrt{3}, -\frac{1}{2} - l(a - \frac{1}{2})\sqrt{3})\).

**Theorem 3.8.** Let 1 ≥ λ_1, μ_1, ν_1 ≥ −1. Then there exists a symmetric 4 × 4 doubly stochastic matrix of trace-zero with spectrum \((1, λ_1, μ_1, ν_1)\) if and only if \(1 + λ_1 + μ_1 + ν_1 = 0\).

**Proof.** It suffices to check that the 4 × 4 doubly stochastic matrix

\[
\begin{pmatrix}
0 & 1 + λ_1 & 1 + μ_1 & 1 + ν_1 \\
1 + λ_1 & 0 & 1 + ν_1 & 1 + μ_1 \\
1 + μ_1 & 1 + ν_1 & 0 & 1 + λ_1 \\
1 + ν_1 & 1 + μ_1 & 1 + λ_1 & 0
\end{pmatrix}
\]

has the required spectrum. □
Making use of Theorem 3.7 and applying the same technique as earlier, we have the following theorem that identifies boundary points for the region $\Theta_n$.

**Theorem 3.9.** For all $j = 1, 2, \ldots, k$, let $0 \leq a_j \leq 1$ and $\phi_j = -\frac{1}{2} + i(a_j - \frac{1}{2})\sqrt{3}$. Also define $\phi$ to be the $(3k - 1)$-list defined by:

$$
\phi = \left( (a^{(k)}_{3k-3}), \left( c^{(k)}_{3k-3}a^{(k-1)}_{3k-6} \right), \left( c^{(k)}_{3k-3}a^{(k-1)}_{3k-6} a^{(k-2)}_{3k-9} \right), \ldots, \left( c^{(k)}_{3k-3} \cdots c^{(3)}_{6} a^{(2)}_{3} \right), \left( b^{(k)}_{2k-3} \right) \right).
$$

Then the two $n$-lists $(1, \phi)$ and $(1, -\frac{1}{3k}, 3k-1 \phi)$ define boundary sets of $\Theta_n$ for the cases $n = 3k$ and $n = 3k + 1$, respectively. Moreover for $n = 3k + 2$, the three $n$-lists:

- $(1, -\frac{1}{3k+1}, -\frac{1}{3k+1}, 3k-1 \phi)$,
- $(1, -\frac{1}{2}, 0, \frac{k-1}{k} \phi)$,
- $(1, w^{(k+1)}_{3k}, -y^{(k+1)}_{3k}, z^{(k+1)}_{3k} \phi)$

are also boundary sets for the region $\Theta_n$.

**Proof.** The idea of the proof relies again on successive applications of Theorem 2.6. Indeed, the case $n = 3k$ can be done by using an identical proof of Theorem 3.5 with the exception that the starting $3 \times 3$ matrix in this case is the one giving by Theorem 3.7, and then we obtain the $3k \times 3k$ matrix $C_{3k}$ whose eigenvalues are $(1, \phi)$. As for the case $n = 3k + 1$, it suffices to apply Theorem 2.6 with $\alpha = 0$, $A = 1$ and $B = C_{3k}$ to obtain the $(3k + 1) \times (3k + 1)$ matrix $C_{3k+1}$ with the desired spectrum. Finally, in the case $n = 3k + 2$, we apply Theorem 2.6, where for the first part, we take $\alpha = 0$, $A = 1$ and $B = C_{3k+1}$ and for the second part, we let $\alpha = 0$ and $A$ be any $2 \times 2$ matrix with $B = C_{3k}$ and similarly for the last part, we let

$$
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

whose spectrum is $(1, -1)$ and $B = C_{3k}$ to obtain the required spectra. \hfill \Box

Similarly using Theorem 3.8 and defining the following new notations $W^{(i)}_{4,n} = f^{(i)}_{4,n}$, $V^{(i)}_{4,n} = g^{(i)}_{4,n}$ and $Z^{(i)}_{4,n} = h^{(i)}_{4,n}$ to ease down the technical issue, we obtain the following theorem that also help identifying boundary points for $\Theta_n^5$.

**Theorem 3.10.** Let $-1 \leq \lambda_j, \mu_j, v_j \leq 1$ be such that $1 + \lambda_j + \mu_j + v_j = 0$ for all $j = 1, 2, \ldots, k$. Define $\delta$ to be the $(4k - 1)$-list given by:

$$
\delta = \left( (h^{(k)}_{4k-4}), (h^{(k)}_{4k-4}h^{(k-1)}_{4k-8}), (h^{(k)}_{4k-4}h^{(k-1)}_{4k-8}h^{(k-2)}_{4k-12}), \ldots, (h^{(k)}_{4k-4} \cdots h^{(4)}_{4} h^{(2)}_{4}) \right),\left( g^{(k)}_{4k-4} \right),
$$

then

$$
(4k-4)^{\lambda_{k-1}}, (h^{(k)}_{4k-4}g^{(k-1)}_{4k-8})^{\mu_{k-1}}, (h^{(k)}_{4k-4}g^{(k-1)}_{4k-8}g^{(k-2)}_{4k-12})^{v_{k-1}},
$$

and

$$
(h^{(k)}_{4k-4}h^{(k-1)}_{4k-8}h^{(k-2)}_{4k-12})^{\lambda_{k-2}}, (h^{(k)}_{4k-4}h^{(k-1)}_{4k-8}h^{(k-2)}_{4k-12}h^{(k-3)}_{4k-16})^{\mu_{k-2}}, (h^{(k)}_{4k-4}h^{(k-1)}_{4k-8}h^{(k-2)}_{4k-12}g^{(k-1)}_{4k-24})^{v_{k-2}},
$$

and

$$
(h^{(k)}_{4k-4} \cdots h^{(3)}_{4} g^{(2)}_{4})^{\lambda_{2}}, (h^{(k)}_{4k-4} \cdots h^{(3)}_{4} g^{(2)}_{4} h^{(2)}_{4})^{\mu_{2}}, (h^{(k)}_{4k-4} \cdots h^{(3)}_{4} g^{(2)}_{4} g^{(2)}_{4})^{v_{2}},
$$

and

$$
(h^{(k)}_{4k-4} \cdots h^{(3)}_{4} g^{(2)}_{4} g^{(2)}_{4})^{\lambda_{1}}, (h^{(k)}_{4k-4} \cdots h^{(3)}_{4} g^{(2)}_{4} g^{(2)}_{4})^{\mu_{1}}, (h^{(k)}_{4k-4} \cdots h^{(3)}_{4} g^{(2)}_{4} g^{(2)}_{4})^{v_{1}}.
$$
Then the two \( n \)-lists \((1, \delta)\) and \(\left(1, -\frac{1}{4k}, \frac{4k-1}{4k}, \delta\right)\) define boundary sets of \(\Theta_n^4\) for the cases \(n = 4k\) and \(n = 4k + 1\), respectively. In addition, for \(n = 4k + 2\), the three \( n \)-lists \(\alpha = \left(1, -\frac{1}{4k+1}, \frac{4k}{4k+1}, \frac{4k}{4k+1}, \delta\right), \eta = \left(1, -\frac{1}{2k}, 0, \frac{2k-1}{2k}, \delta\right)\) and \(\xi = \left(1, w_{4k}^{(k+1)}, -y_{4k}^{(k+1)}, z_{4k}^{(k+1)}, 4k\delta\right)\) define also boundary sets for the region \(\Theta_n^4\). Finally, for \(n = 4k + 3\), the following \( n \)-lists:

\[
\begin{align*}
\left(1, -\frac{1}{4k+2}, \frac{2k+1}{2k+2}, 0, \frac{2k+1}{2k+2}, \frac{4(k+2)}{4k+2}\right), \\
\left(1, -\frac{2}{2k+1}, 0, \frac{2k+1}{2k+2}, \frac{4(k+2)}{4k+2}\right), \\
\left(1, -\frac{1}{4k+2}, 0, \frac{4k+2}{4k}, \frac{4k+2}{4k}, \delta\right), \\
\left(1, -\frac{1}{4k+2}, 0, \frac{4k+2}{4k}, \frac{4k+2}{4k}, \delta\right),
\end{align*}
\]

are also boundary subregions of \(\Theta_n^4\).

**Proof.** First the case \(n = 4k\) can be done by using an identical proof of Theorem 3.5 but taking into account that the initial matrix that we start with in this case is the \(4 \times 4\) matrix giving by Theorem 3.8, and then we obtain the \(4k \times 4k\) matrix \(C_{4k}\) whose eigenvalues are \((1, \delta)\). For the case \(n = 4k + 1\) we again apply Theorem 2.6 with \(\alpha = 0, A = 1\) and \(B = C_{4k}\) to obtain the \((4k + 1) \times (4k + 1)\) matrix \(C_{4k+1}\) with the desired spectrum. As for the case \(n = 4k + 2\), the process can be repeated where for the first part, we take \(\alpha = 0, A = 1\) and \(B = C_{4k+1}\) and for the second part, we let \(\alpha = 0\) and \(A\) be any \(2 \times 2\) matrix with \(B = C_{4k}\) and similarly for the last part, we let

\[
A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

whose spectrum is \((1, -1)\) and \(B = C_{4k}\) we then obtain the three \((4k + 2) \times (4k + 2)\) doubly stochastic matrices \(C_{4k+2}^{(1)}, C_{4k+2}^{(2)}\) and \(C_{4k+2}^{(3)}\) whose spectra are, respectively, \(\sigma, \eta\) and \(\xi\). Finally, for the case \(n = 4k + 3\), we may continue the proof by using the same iterative process where for the first list, we take \(\alpha = 0, A = 1\) and \(B = C_{4k+2}^{(2)}\) and for the second list, we let \(\alpha = 0\) and \(A\) be any \(2 \times 2\) matrix with \(B = C_{4k+1}\) and for the third list, \(A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) and \(B = C_{4k+1}\). The fourth and fifth and sixth lists can be obtained by first taking \(\alpha = 0, A = 1\) and \(B = C_{4k+2}^{(1)}\) and secondly by letting \(\alpha = 0\) and \(A\) be any \(3 \times 3\) matrix with \(B = C_{4k}\) and thirdly by taking \(\alpha = 0, A = 1\) and \(B = C_{4k+2}^{(3)}\) in Theorem 2.6, respectively.

We conclude by stressing again as we mentioned earlier that it is possible to obtain more results of this kind by writing \(n\) as the sum of all possible combinations of \(1, 2, 3\) and \(4\). However to account for all such results is not an easy task though it can be done for any fixed \(n = n_0\).

4. Conclusion

We presented a useful theorem concerning the spectral properties of doubly stochastic matrices and showed how it can be used to yield many sufficient conditions for three inverse eigenvalue problems
concerning doubly stochastic matrices as the applications in Section 3 suggests. Although at this stage we are not able to offer complete solutions using this technique, perhaps its major importance lies in the fact that it can bring us one step closer to complete solutions as it can be used to generate boundary points for which complete solutions to any of these problems relies heavily on characterizing all such points. However besides the two facts that it offers many new partial results, and any known results concerning these problems can lead to new results, the main importance of this theorem lies maybe in the fact that it can be used as a checking tool in case of a conjecture concerning complete solutions is given for any of the three major problems.

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References