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Ramanujan graphs on cosets of $\mathrm{PGL}_2(\mathbb{F}_q)$

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Abstract

In this paper, we study Cayley graphs on $\mathrm{PGL}_2(\mathbb{F}_q)$ mod the unipotent subgroup, the split and nonsplit tori, respectively. Using the Kirillov models of the representations of $\mathrm{PGL}_2(\mathbb{F}_q)$ of degree greater than one, we obtain explicit eigenvalues of these graphs and the corresponding eigenfunctions. Character sum estimates are then used to conclude that two types of the graphs are Ramanujan, while the third is almost Ramanujan. The graphs arising from the nonsplit torus were previously studied by Terras et al. We give a different approach here.

Keywords: Ramanujan graphs; Kirillov models; Character sum estimates

1. Introduction

A finite k-regular graph is called Ramanujan if its eigenvalues other than $\pm k$, called nontrivial eigenvalues, have absolute values at most $2\sqrt{k-1}$. Such graphs are good expanders and have broad applications in computer science. The first systematic explicit construction of an infinite family of k-regular Ramanujan graphs is given independently by Margulis [15] and Lubotzky et al. [14] for $k = p + 1$ with p a prime; their graphs are based on quaternion groups over $\mathbb{Q}$, and the nontrivial eigenvalues of these graphs can be interpreted as the eigenvalues of the Hecke operators on classical

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cusp forms of weight 2 so that the eigenvalue bound follows from the deep property that the Ramanujan conjecture holds for these cusp forms, established by Eichler [7] and Shimura [18]. The Ramanujan conjecture for cusp forms of GL_2 over a function field is proved by Drinfeld [6], hence the same method gives rise to infinite families of $(q+1)$-regular Ramanujan graphs, where q is a prime power. This is done by Morgenstern in [16].

On the other hand, there are explicit constructions of $(q+1)$-regular Ramanujan graphs for q a prime power whose nontrivial eigenvalues are expressed as character sums, which are shown to be bounded by $2\sqrt{q}$, as a consequence of the Riemann hypothesis for curves over finite fields. Such examples include Terras graphs [2] based on cosets of $PGL_2(\mathbb{F}_q)$ and norm graphs in [9] based on a finite field of q^2 elements. Later it is shown in [11] that these two kinds of graphs are in fact quotient graphs of Morgenstern graphs. This connection leads to very interesting relations between character sums and cusp forms for GL_2 over function fields, studied in detail in [3,4]. In particular, one obtains cusp forms whose Fourier coefficients are given by eigenvalues of the Terras/norm graphs in a systematic manner.

In this paper, we revisit Terras graphs and investigate two other types of graphs based on cosets of $G = PGL_2(\mathbb{F}_q)$ using a uniform method explained below. Up to conjugation, G contains three types of abelian subgroups: the unipotent subgroup

$$U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_q \right\},$$

which is a group of order q, the split torus

$$A = \left\{ \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} : y \in \mathbb{F}_q^\times \right\},$$

which is cyclic of order $q-1$, and the nonsplit torus K, which is an embedded image of $\mathbb{F}_{q^2}^\times/\mathbb{F}_q^\times$ in G as a cyclic subgroup of order $q+1$. It is well-known that $G = UAK$. Denote by H one of these three subgroups. For a double coset HsH which is its own inverse (i.e., symmetric) and which is the disjoint union of $|H|$ right H-cosets, consider the Cayley graph $X_{HsH} = \mathrm{Cay}(G/H, HsH/H)$, called an H-graph. It is undirected and $|H|$-regular. (When $H = K$, this is a Terras graph.) We shall prove

Main Theorem. (a) (cf. [20, p. 357]) *The nontrivial eigenvalues of X_{KsK} have absolute values at most $2\sqrt{q}$. Hence the K-graphs are $(q+1)$-regular Ramanujan graphs.*

(b) *The nontrivial eigenvalues of X_{UsU} are ± 1 and $\pm\sqrt{q}$. Thus the U-graphs are q-regular Ramanujan graphs.*

(c) *The nontrivial eigenvalues of X_{AsA} have absolute values at most $2\sqrt{q}$. Thus the A-graphs, being $(q-1)$-regular, are almost Ramanujan.*

Like representations of p-adic groups, the irreducible representations of G of degree greater than one also have a Kirillov model, in which the actions of U and A are standard, and the representations are distinguished by the action of the Weyl element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This is studied in [13]. Our approach is to use a Kirillov model to explicitly find functions in the space of each irreducible representation of G which are right H-invariant. This space has dimension at most 2. We then determine the eigenvalues and eigenfunctions of the adjacency operator of the graph X_{HsH}. For Terras graphs, the eigenvalues are obtained in [1,2] by computing the traces of the irreducible representations, expressed as character sums over $\mathbb{F}_q$, and then estimated. In our approach we obtain the same expression for eigenvalues arising from nondiscrete series representations, but an eigenvalue arising from a discrete series representation is expressed differently, namely, as the average of $q+1$ character sums over $\mathbb{F}_{q^2}$, one for each K-coset. Each character sum is associated with some idèle class character of the rational function field $\mathbb{F}_q(T)$ using the results obtained in Chapter 6 of [10] and [12], and then shown to have absolute value at most $2\sqrt{q}$ as a consequence of the Riemann hypothesis for curves.

It is worth pointing out that the U-graph X_{UwU} has two connected components, one of which can be identified with the Cayley graph $\mathrm{Cay}(\mathrm{PSL}_2(\mathbb{F}_q)/U, UwU/U)$. A suitable quotient of this graph may be interpreted as a graph on the cusps of a certain principal congruence subgroup of the Drinfeld modular group $\mathrm{GL}_2(\mathbb{F}_p[T])$. This kind of graph is first obtained by Gunnells [8] for principal congruence subgroups $\Gamma(p)$ of $\mathrm{SL}_2(\mathbb{Z})$. A generalization of this graph from p to prime power q with $q \equiv 1 \pmod 4$ is given in [5]. Both approaches rely on analyzing the graph structure, while ours is purely representation-theoretic.

The paper is organized as follows. The representation theory, including the Kirillov models, is reviewed in Section 2. Sections 3–5 are devoted to the K-, U-, and A-graphs, respectively. In each case, using Kirillov models, we determine the eigenvalues and compute the corresponding eigenfunctions; then character sum estimates are employed to bound the eigenvalues.

For convenience, the characteristic of $\mathbb{F}_q$ is assumed to be odd throughout the paper. Similar results are expected to hold for even characteristic.

2. Representations of $\mathbf{PGL_2}(\mathbb{F}_q)$

For brevity, write $\mathbb{F}$ for the finite field with q elements and $\mathbb{E}$ for its quadratic extension. The unipotent subgroup U acts on the space $\mathcal{L}(G)$ of complex-valued functions via left translations so that the space decomposes as

$$\mathcal{L}(G) = \bigoplus_{\psi \in \hat{\mathbb{F}}} \mathcal{L}_\psi(G),$$

where $\mathcal{L}_\psi(G) = \{f : G \to \mathbb{C} : f(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \psi(x) f(g) \text{ for all } g \in G\}$.

The irreducible representations of G are identified with the irreducible representations of $\mathrm{GL}_2(\mathbb{F})$ with trivial central character. Such representations are studied in the literature in detail (cf. [17]); those with degree greater than one are classified into three categories:

principal series, Steinberg, and discrete series representations. Fix a nontrivial additive character ψ of $\mathbb{F}$. Each irreducible representation π of G of degree greater than 1 has a Kirillov model $\mathcal{K}_\psi(\pi)$ and a Whittaker model $\mathcal{W}_\psi(\pi)$. According to π being a discrete series, Steinberg, or principal series representation, the space $\mathcal{K}_\psi(\pi)$ is spanned by $\hat{\mathbb{F}}^\times$ (the multiplicative characters of $\mathbb{F}$), $\hat{\mathbb{F}}^\times \cup \{D_0\}$, or $\hat{\mathbb{F}}^\times \cup \{D_0\} \cup \{D_\infty\}$, respectively, where D_0 (resp. D_∞) denotes the Dirac function at 0 (resp. ∞). The action of U on $\hat{\mathbb{F}}^\times$ is given in terms of ψ, and the action of UA on $\hat{\mathbb{F}}^\times$ is the same for all representations. The representation π is characterized by the action of $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In [13], to each π of degree greater than 1 a family of Gauss sums $\varepsilon(\pi, \chi, \psi)$, where $\chi \in \hat{\mathbb{F}}^\times$, is attached, with which the action of $\pi(w)$ is described. The details are as follows.

A discrete series representation $\pi = \pi_\Lambda$ arises from a character Λ of $\mathbb{E}^\times$ such that Λ is trivial on $\mathbb{F}^\times$ and $\Lambda \neq \mu \circ \mathrm{N}$ for all $\mu \in \hat{\mathbb{F}}^\times$. Here N denotes the norm map from $\mathbb{E}$ to $\mathbb{F}$. The last condition on Λ is equivalent to $\Lambda^{q+1} = 1$, $\Lambda^2 \neq 1$. Consequently, the inverse of Λ is $\bar{\Lambda} = \Lambda^q$. We associate π with

$$\varepsilon(\pi, \chi, \psi) = -\Gamma(\Lambda\chi \circ \mathrm{N}, \psi \circ \mathrm{Tr}) = -\sum_{z \in \mathbb{E}^\times} \Lambda(z)\chi(\mathrm{N}\, z)\psi(\mathrm{Tr}\, z), \quad \chi \in \hat{\mathbb{F}}^\times.$$

Here Tr is the trace map from $\mathbb{E}$ to $\mathbb{F}$. Observe that $\varepsilon(\pi_\Lambda, \chi, \psi) = \varepsilon(\pi_{\bar{\Lambda}}, \chi, \psi)$.

Characters μ of $\hat{\mathbb{F}}^\times$ give rise to principal series representations π_μ if $\mu^2 \neq 1$, and Steinberg representations π_μ if $\mu^2 = 1$. These are all nondiscrete series representations of G of degree greater than 1. For $\pi = \pi_\mu$ arising from a character μ of $\hat{\mathbb{F}}^\times$, we associate

$$\varepsilon(\pi, \chi, \psi) = \Gamma(\mu\chi, \psi)\Gamma(\mu^{-1}\chi, \psi), \quad \chi \in \hat{\mathbb{F}}^\times,$$

where $\Gamma(\xi, \psi) = \sum_{x \in \mathbb{F}^\times} \xi(x)\psi(x)$. Observe that $\varepsilon(\pi_\mu, \chi, \psi) = \varepsilon(\pi_{\mu^{-1}}, \chi, \psi)$.

Using $\varepsilon(\pi, \chi, \psi)$, we can describe the representation π on its Kirillov model $\mathcal{K}_\psi(\pi)$ by giving the action of the generators

$$h_r = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} (r \in \mathbb{F}^\times), \quad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} (s \in \mathbb{F}^\times), \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of G as follows:

$$\begin{aligned} \pi(h_r)\alpha &= \alpha(r)\alpha, & \alpha \in \hat{\mathbb{F}}^\times, \\ \pi(h_r)D_0 &= \mu(r)D_0, \pi(h_r)D_\infty = \mu^{-1}(r)D_\infty & \text{(when applicable)}, \\ \pi(u_s)\alpha &= (q-1)^{-1} \sum_{\beta \in \hat{\mathbb{F}}^\times} \beta\alpha^{-1}(s)\Gamma(\alpha\beta^{-1}, \psi)\beta, & \alpha \in \hat{\mathbb{F}}^\times, \\ \pi(u_s)D_0 &= D_0, \pi(u_s)D_\infty = D_\infty & \text{(when applicable)}, \end{aligned}$$

$$\begin{aligned}
\pi(w)\alpha &= q^{-1}\varepsilon(\pi,\alpha,\psi)\alpha^{-1} + e(\alpha), && \alpha \in \hat{\mathbb{F}}^{\times},\\
\pi(w)D_0 &= -q^{-1}\varepsilon(\pi,\mu^{-1},\psi)(\mu^{-1}+D_0) && \text{if } \pi \text{ is a Steinberg representation},\\
&= -q^{-1}\varepsilon(\pi,\mu,\psi)(\mu^{-1}+D_\infty) && \text{if } \pi \text{ is a principal series},\\
\pi(w)D_\infty &= -q^{-1}\varepsilon(\pi,\mu^{-1},\psi)(\mu+D_0) && \text{if } \pi \text{ is a principal series},
\end{aligned}$$

where

$$\begin{aligned}
e(\alpha) &= 0 && \text{if } \pi \text{ is a discrete series},\\
&= -q^{-1}\varepsilon(\pi,\alpha,\psi)(q^2-1)\delta_{\alpha\mu,1}D_0 && \text{if } \pi \text{ is a Steinberg representation},\\
&= -q^{-1}\varepsilon(\pi,\alpha,\psi)(q-1)\\
&\quad \times(\delta_{\alpha\mu,1}D_0 + \delta_{\alpha\mu^{-1},1}D_\infty) && \text{if } \pi \text{ is a principal series.}
\end{aligned}$$

Here $\delta_{\chi,1}$ is the Kronecker symbol, which is equal to 1 if $\chi = 1$, the trivial character, and 0 otherwise. It was shown in [13] that the relations on these generators are preserved, resulting from the identities satisfied by the Gauss sums with the main one called the Barnes' identity. The representations are characterized by the attached ε-factors. In particular, π_Λ and $\pi_{\Lambda^{-1}}$ are equivalent, and so are π_μ and $\pi_{\mu^{-1}}$.

In conclusion, there are $\frac{q-1}{2}$ (nonequivalent) discrete series representations π_Λ, each of degree $q-1$; there are two Steinberg representations π_μ from μ with $\mu^2 = 1$, each of degree q; and there are $\frac{q-3}{2}$ principal series representations π_μ with $\mu^2 \neq 1$, each of degree $q+1$. Together with the two degree one representations given by $\mu\circ\det$, $\mu^2=1$, this gives the complete list of the irreducible representations of G. Each irreducible representation occurs in $\mathcal{L}(G)$ with multiplicity equal to its degree.

The Whittaker model $\mathcal{W}_\psi(\pi)$ consists of functions on G obtained from the Kirillov model $\mathcal{K}_\psi(\pi)$ in the following way: for $v \in \mathcal{K}_\psi(\pi)$, define $W_v \in \mathcal{W}_\psi(\pi)$ via

$$W_v(g) = (\pi(g)v)(1), \quad g \in G.$$

The representation π of G on $\mathcal{W}_\psi(\pi)$ is by right translations. Because of the action of U on $\mathcal{K}_\psi(\pi)$, all functions are contained in $\mathcal{L}_\psi(G)$.

We proceed to discuss how each space $\mathcal{L}_\psi(G)$ decomposes. Fix a nontrivial additive character ψ of $\mathbb{F}$. We can describe all characters of $\mathbb{F}$ as ψ^a, $a \in \mathbb{F}$, where $\psi^a(x) = \psi(ax)$ for x in $\mathbb{F}$. Observe that for each function $f \in \mathcal{L}_\psi(G)$, the new function $f_a(g) := f(\left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right)g)$ lies in $\mathcal{L}_{\psi^a}(G)$ and $f \mapsto f_a$ is an isomorphism between the two G-modules $\mathcal{L}_\psi(G)$ and $\mathcal{L}_{\psi^a}(G)$. As noted above, the Whittaker model $\mathcal{W}_\psi(\pi)$ of each irreducible π of degree greater than 1 is contained in $\mathcal{L}_\psi(G)$ for ψ nontrivial, by counting dimension we find

$$\mathcal{L}_{\psi^a}(G) = \bigoplus_{\pi, \deg\pi>1} \mathcal{W}_{\psi^a}(\pi) \quad \text{for } a \neq 0.$$

We then conclude from checking the multiplicities that $\mathcal{L}_{\psi^0}(G)$ contains no discrete series representations; each principal series representation occurs there twice, and each

Steinberg representation and each degree 1 representation occur once. We record this in

Proposition 1. (1) *For nontrivial ψ's, the spaces $\mathcal{L}_\psi(G)$ are isomorphic and each irreducible representation of G of degree greater than 1 occurs in $\mathcal{L}_\psi(G)$ exactly once.*

(2) *For $\psi = \psi^0$ trivial, in $\mathcal{L}_{\psi^0}(G)$ each principal series representation occurs with multiplicity two, each Steinberg representation occurs with multiplicity one, as does each 1-dimensional representation.*

Remark. For each character μ of $\mathbb{F}^\times$, denote by

$$\operatorname{Ind}\mu = \left\{ f : G \to \mathbb{C} : f\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} g\right) = \mu(a) f(g) \text{ for all } g \in G \right\}.$$

The group G acts on $\operatorname{Ind}\mu$ by right translations. It is well-known that when $\mu^2 \neq 1$, this representation is the principal series representation π_μ, and it is isomorphic to $\operatorname{Ind}\mu^{-1}$. When $\mu^2 = 1$, this representation has two irreducible constituents, $\mu \circ \det$ and the Steinberg representation π_μ. Obviously,

$$\mathcal{L}_{\psi^0}(G) = \bigoplus_{\mu \in \hat{\mathbb{F}}^\times} \operatorname{Ind}\mu.$$

Let H be a subgroup of G. Then the space of functions on G right invariant by H has a similar decomposition

$$\mathcal{L}(G/H) = \bigoplus_{\psi} \mathcal{L}_\psi(G/H) = \bigoplus_{\psi} \bigoplus_{\pi} m_\psi(\pi) \mathcal{L}_\psi(\pi, G/H),$$

where π runs through irreducible representations of G, $m_\psi(\pi)$ is the multiplicity of π in $\mathcal{L}_\psi(G)$ as described in the proposition above, and $\mathcal{L}_\psi(\pi, G/H)$ consists of the right H-invariant functions in the space of π in $\mathcal{L}_\psi(G)$.

Let s be an element in G. Write $HsH = x_1 H \cup \cdots \cup x_k H$ as a disjoint union of k right H cosets. Define an operator T_{HsH} on $\mathcal{L}(G/H)$ by sending $f \in \mathcal{L}(G/H)$ to

$$(T_{HsH} f)(xH) = \sum_{i=1}^{k} f(x x_i H).$$

Clearly T_{HsH} preserves each space $\mathcal{L}_\psi(\pi, G/H)$. When $HsH = Hs^{-1}H$, we define an undirected Cayley graph $X_{HsH} = \operatorname{Cay}(G/H, HsH/H)$, called an H-graph, whose adjacency matrix may be identified with the operator T_{HsH}. We shall take $H = K, U$ and A, respectively, and study the eigenfunctions and the eigenvalues of T_{HsH}.

3. The K-graphs

In this section we take $H = K$. Since $G = UAK$, for each additive character ψ of $\mathbb{F}$, the space $\mathcal{L}_\psi(G/K)$ is $(q-1)$-dimensional. As remarked before, when $\psi = \psi^0$ is the trivial character,

$$\mathcal{L}_{\psi^0}(G) = \bigoplus_{\mu \in \hat{\mathbb{F}}^\times} \operatorname{Ind} \mu.$$

One sees immediately from the definition of $\operatorname{Ind}\mu$ that the right K-invariant space of $\operatorname{Ind}\mu$ is 1-dimensional, generated by

$$f_\mu\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} k\right) = \mu(y) \quad \text{for all } y \in \mathbb{F}^\times, \ \ x \in \mathbb{F} \text{ and } k \in K,$$

and the f_μ's form a basis of $\mathcal{L}_{\psi^0}(G)$. This shows that for a principal series representation π_μ, the space of right K-invariant vectors in any of its model is 1-dimensional for any ψ. When μ is the quadratic character, the 1-dimensional representation $\mu \circ \det$ does not contain nontrivial right K-invariant vectors since not all elements in K have square determinant. Therefore f_μ belongs to the Steinberg representation π_μ, and we arrive at the same conclusion that $\mathcal{L}_\psi(\pi_\mu, G/K)$ is 1-dimensional. When μ is the trivial character, the function f_μ lies in the space of the 1-dimensional representation $\mu \circ \det$, and hence $\mathcal{L}_\psi(\pi_\mu, G/K)$ is 0-dimensional for the Steinberg representation π_μ for μ trivial. We record this in

Proposition 2. *Let μ be a character of $\mathbb{F}^\times$. For any additive character ψ of $\mathbb{F}$, the space $\mathcal{L}_\psi(\pi_\mu, G/K)$ is 1-dimensional if $\mu \neq 1$, and 0-dimensional if $\mu = 1$.*

Our next goal is to show that, for ψ nontrivial, $\mathcal{L}_\psi(\pi_\Lambda, G/K)$ is 1-dimensional for each discrete series representation π_Λ. Since $\mathcal{L}_\psi(G/K)$ is $(q-1)$-dimensional and it contains a $\frac{q-1}{2}$-dimensional subspace $\bigoplus_{\mu \neq 1} \mathcal{L}_\psi(\pi_\mu, G/K)$, it suffices to show $\dim \mathcal{L}_\psi(\pi_\Lambda, G/K) \geqslant 1$ as there are $\frac{q-1}{2}$ discrete series representations.

Fix once and for all a nonsquare δ in $\mathbb{F}$ so that $\mathbb{E} = \mathbb{F}(\sqrt{\delta})$. We imbed $\mathbb{E}^\times$ in $\mathrm{GL}_2(\mathbb{F})$ as $\left\{\begin{pmatrix} b & a\delta \\ a & b \end{pmatrix} : a, b \in \mathbb{F} \text{ not both zero}\right\}$. Consequently, the elements in K are represented by

$$\begin{pmatrix} b & \delta \\ 1 & b \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & b(b^2-\delta)^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b^2-\delta \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{with } b \in \mathbb{F}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Fix a nontrivial additive character ψ of $\mathbb{F}$ and a character Λ of $\mathbb{E}^\times$ with $\Lambda^{q+1} = 1$ and $\Lambda^2 \neq 1$. Let $\pi = \pi_\Lambda$ be the associated discrete series representation with the attached ε-factor $\varepsilon(\pi,\chi,\psi)$. According to the description in §2 of the action of π on its Kirillov model $\mathcal{K}_\psi(\pi)$, we obtain, for $\theta \in \hat{\mathbb{F}}^\times$ and $b \in \mathbb{F}^\times$,

$$\pi\left(\begin{pmatrix} b & \delta \\ 1 & b \end{pmatrix}\right)\theta = (q-1)^{-2}q^{-1}\bar\theta(b)\sum_{\beta\in\hat{\mathbb{F}}^\times}\beta(b(b^2-\delta)^{-1})\Gamma(\theta\bar\beta,\psi)\varepsilon(\pi,\beta,\psi) \times\sum_{\gamma\in\hat{\mathbb{F}}^\times}(\gamma\beta)(b)\Gamma(\bar\gamma\bar\beta,\psi)\gamma,$$

$$\pi\left(\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}\right)\theta = \bar\theta(-\delta)\varepsilon(\pi,\theta,\psi)q^{-1}\bar\theta \quad\text{and}\quad \pi\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\theta = \theta.$$

Then $v_\theta := \sum_{b\in\mathbb{F}}\pi\left(\left(\begin{smallmatrix} b & \delta \\ 1 & b \end{smallmatrix}\right)\right)\theta + \theta$ is invariant by $\pi(K)$. Put

$$v_{\pi,\psi} = \sum_{\theta\in\hat{\mathbb{F}}^\times} v_\theta = \sum_\theta\sum_{b\in\hat{\mathbb{F}}^\times}(q-1)^{-2}q^{-1}\bar\theta(b)\sum_{\beta\in\hat{\mathbb{F}}^\times}\beta(b(b^2-\delta)^{-1})\Gamma(\theta\bar\beta,\psi)\varepsilon(\pi,\beta,\psi) \times\sum_{\gamma\in\hat{\mathbb{F}}^\times}(\gamma\beta)(b)\Gamma(\bar\gamma\bar\beta,\psi)\gamma + \sum_\theta\bar\theta(-\delta)\varepsilon(\pi,\theta,\psi)q^{-1}\bar\theta + \sum_\theta\theta.$$

Using

$$\sum_{\theta\in\mathbb{F}^\times}\bar\theta(b)\Gamma(\theta\beta,\psi) = \sum_\theta\bar\theta(b)\sum_{x\in\mathbb{F}^\times}(\theta\bar\beta)(x)\psi(x) = (q-1)\bar\beta(b)\psi(b),$$

we rewrite $v_{\pi,\psi}$ as

$$v_{\pi,\psi} = \sum_{b\in\mathbb{F}^\times}(q-1)^{-1}q^{-1}\sum_{\beta\in\hat{\mathbb{F}}^\times}\bar\beta(b^2-\delta)\psi(b)\varepsilon(\pi,\beta,\psi)\sum_{\gamma\in\hat{\mathbb{F}}^\times}(\gamma\beta)(b)\Gamma(\bar\gamma\bar\beta,\psi)\gamma + \sum_\theta\bar\theta(-\delta)\varepsilon(\pi,\theta,\psi)q^{-1}\bar\theta + \sum_\theta\theta.$$

Recall that $\varepsilon(\pi,\chi,\psi) = -\Gamma(\Lambda\chi\circ\mathrm{N},\psi\circ\mathrm{Tr}) = -\sum_{z\in\mathbb{E}^\times}\Lambda(z)\chi(\mathrm{N}\,z)\psi(\mathrm{Tr}\,z)$. Summing over β further simplifies the expression of $v_{\pi,\psi}$ as

$$v_{\pi,\psi} = -q^{-1}\sum_{\gamma\in\hat{\mathbb{F}}^\times}\sum_{b\in\mathbb{F}}\sum_{z\in\mathbb{E}^\times}\Gamma(z)\bar\gamma(\mathrm{N}\,z)\psi(\mathrm{Tr}\,z)\gamma(b^2-\delta)\psi((\mathrm{N}\,z)b(b^2-\delta)^{-1})\psi(b)\gamma + \sum_\gamma\gamma.$$

Let $W'_{\Lambda,\psi} = W_{v_{\pi,\psi}}$, that is, for $g \in G$, $W_{\Lambda,\psi}(g) = (\pi(g)v_{\pi,\psi})(1)$. Then $W'_{\Lambda,\psi}$ lies in the Whittaker model $\mathcal{W}_\psi(\pi)$ and it is right K-invariant. Hence $W'_{\Lambda,\psi}$ is determined by its values on the split torus A, which we now compute. For $y \in \mathbb{F}^\times$,

$$\begin{aligned} W'_{\Lambda,\psi}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) &= \left(\pi\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) v_{\pi,\psi}\right)(1) \\ &= -q^{-1} \sum_{\gamma \in \hat{\mathbb{F}}^\times} \sum_{b \in \mathbb{F}} \sum_{z \in \mathbb{E}^\times} \Gamma(z)\bar{\gamma}(\mathrm{N}\, z)\psi(\mathrm{Tr}\, z)\gamma(b^2 - \delta) \\ &\quad \times \psi((\mathrm{N}\, z)b(b^2-\delta)^{-1})\psi(b)\gamma(y) + \sum_{\gamma} \gamma(y) \\ &= -(q-1)q^{-1} \sum_{b \in \mathbb{F}} \sum_{\substack{z \in \mathbb{E}^\times, \\ \mathrm{N}\, z = y(b^2-\delta)}} \Lambda(z)\psi(\mathrm{Tr}\, z) \\ &\quad \times \psi((\mathrm{N}\, z)b(b^2-\delta)^{-1})\psi(b) + (q-1)\delta_{y,1}. \end{aligned}$$

Here $\delta_{y,1}$ is equal to 1 if $y = 1$ and 0 otherwise. Replacing the variable z by $z(b+\delta)$, we rewrite the above as

$$\begin{aligned} W'_{\Lambda,\psi}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) &= -(q-1)q^{-1} \sum_{b \in \mathbb{F}} \sum_{\substack{z \in \mathbb{E}^\times, \\ \mathrm{N}\, z = y}} \Lambda(z(b+\sqrt{\delta})) \\ &\quad \times \psi(\mathrm{Tr}(z(b+\sqrt{\delta})))\psi(b(y+1)) + (q-1)\delta_{y,1}. \end{aligned}$$

Note that $b(y+1) = \mathrm{Tr}(\frac{y+1}{2}(b+\sqrt{\delta}))$. We combine the two terms involving ψ to get

$$\begin{aligned} W'_{\Lambda,\psi}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) &= -(q-1)q^{-1} \sum_{b \in \mathbb{F}} \sum_{\substack{z \in \mathbb{E}^\times, \\ \mathrm{N}\, z = y}} \Lambda(z(b+\sqrt{\delta})) \\ &\quad \times \psi\left(\mathrm{Tr}\left(\left(z + \frac{y+1}{2}\right)(b+\sqrt{\delta})\right)\right) + (q-1)\delta_{y,1}. \end{aligned}$$

Set $W_{\Lambda,\psi} = (q-1)^{-1}W'_{\Lambda,\psi}$ and $W_\Lambda = \sum_{\psi \neq \psi^0} W_{\Lambda,\psi}$, which is a right K-invariant function on G belonging to $\bigoplus_{\psi \neq \psi^0} \mathcal{L}_\psi(\pi, G/K)$. We compute

$$\begin{aligned} W_{\Lambda,\psi}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) &= -q^{-1} \sum_{a \in \mathbb{F}^\times} \sum_{b \in \mathbb{F}} \sum_{\substack{z \in \mathbb{E}^\times, \\ \mathrm{N}\, z = y}} \Lambda(z(b+\sqrt{\delta})) \\ &\quad \times \psi^a\left(\mathrm{Tr}\left(\left(z + \frac{y+1}{2}\right)(b+\sqrt{\delta})\right)\right) + (q-1)\delta_{y,1} \end{aligned}$$

$$= -q^{-1} \sum_{a} \sum_{b} \sum_{z, \mathrm{N}\, z = y} \Lambda(za(b+\sqrt{\delta}))$$

$$\times \psi\left(\mathrm{Tr}\left(\left(z + \frac{y+1}{2}\right) a(b+\sqrt{\delta})\right)\right) + (q-1)\delta_{y,1}$$

since Λ is trivial on $\mathbb{F}^{\times}$. Observe that as a runs through all elements in $\mathbb{F}^{\times}$ and b runs through all elements in $\mathbb{F}$, $a(b+\sqrt{\delta})$ runs through all elements in $\mathbb{E}\backslash\mathbb{F}$. Therefore

$$\sum_{a \in \mathbb{F}^{\times}} \sum_{b \in \mathbb{F}} \Lambda(za(b+\sqrt{\delta}))\psi\left(\mathrm{Tr}\left(\left(z + \frac{y+1}{2}\right) a(b+\sqrt{\delta})\right)\right)$$

$$= \sum_{w \in \mathbb{E}^{\times}} \Lambda(zw)\psi\left(\mathrm{Tr}\left(\left(z + \frac{y+1}{2}\right) w\right)\right) - \sum_{a \in \mathbb{F}^{\times}} \Lambda(za)\psi\left(a\,\mathrm{Tr}\left(z + \frac{y+1}{2}\right)\right).$$

Since Λ is trivial on $\mathbb{F}^{\times}$, the last sum is equal to $(q-1)\Lambda(z)$ if $\mathrm{Tr}(z + \frac{y+1}{2}) = \mathrm{Tr}(z) + y + 1 = 0$, and to $-\Lambda(z)$ otherwise. This gives

$$W_{\Lambda}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = -q^{-1} \sum_{\substack{z \in \mathbb{E}, \\ \mathrm{N}\, z = y}} \sum_{w \in \mathbb{E}^{\times}} \Lambda(zw)\psi\left(\mathrm{Tr}\left(z + \frac{y+1}{z}\right) w\right)$$

$$+ \sum_{\substack{z \in \mathbb{E}, \mathrm{N}\, z = y, \\ \mathrm{Tr}(z)+y+1=0}} \Lambda(z) + q^{-1} \sum_{\substack{z \in \mathbb{E}, \\ \mathrm{N}\, z = y}} \Lambda(z) + (q-1)\delta_{y,1}$$

$$= -q^{-1} \sum_{\substack{z \in \mathbb{E}, \mathrm{N}\, z = y, \\ z + \frac{y+1}{2} \neq 0}} \Lambda(z)\bar{\Lambda}\left(z + \frac{y+1}{2}\right) \sum_{w \in \mathbb{E}} \Lambda\left(\left(z + \frac{y+1}{2}\right) w\right)$$

$$\times \psi\left(\mathrm{Tr}\left(\left(z + \frac{y+1}{2}\right) w\right)\right)$$

$$-q^{-1} \sum_{\substack{z \in \mathbb{E}, \mathrm{N}\, z = y, \\ z + \frac{y+1}{2} \neq 0}} \sum_{w \in \mathbb{E}} \Lambda(zw) + \sum_{\substack{z \in \mathbb{E}, \mathrm{N}\, z = y, \\ \mathrm{Tr}(z)+y+1=0}} \Lambda(z)$$

$$-q^{-1} \sum_{\substack{z \in \mathbb{E}, \\ \mathrm{N}\, z = y}} \Lambda(z) + (q-1)\delta_{y,1}.$$

Further, since Λ is a nontrivial character on the kernel of norm in $\mathbb{E}^\times$, we have $\sum_{\mathrm{N}\,z=y} \Lambda(z) = 0$ and $\sum_{w\in\mathbb{E}^\times} \Lambda(zw) = 0$. Therefore

$$W_\Lambda\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = -q^{-1}\Gamma(\Lambda, \psi\circ\mathrm{Tr}) \sum_{\substack{z\in\mathbb{E},\mathrm{N}\,z=y,\\ z+\frac{y+1}{2}\neq 0}} \bar{\Lambda}\left(1+\frac{y+1}{2z}\right) - \sum_{\substack{z\in\mathbb{E},\mathrm{N}\,z=y,\\ \mathrm{Tr}(z)+y+1=0}} \Lambda(z) + (q-1)\delta_{y,1}.$$

We discuss the second sum $\sum_{\substack{z\in\mathbb{E},\mathrm{N}\,z=y,\\ \mathrm{Tr}(z)+y+1=0}} \Lambda(z)$. Since the polynomial $x^2+(y+1)x+y$ factors as $(x+y)(x+1)$, there are no elements z in $\mathbb{E}\backslash\mathbb{F}$ with $\mathrm{N}\,z = y$ and $\mathrm{Tr}\,z = -y-1$. If $z\in\mathbb{F}$, then the conditions yield $y = z^2$ and $2z = -y-1 = -z^2-1$, implying the only nonvoid sum occurs when $z = -1$ and $y = 1$, in which case the sum is equal to 1. As for the first sum, in order that $z + \frac{y+1}{2} = 0$ for some z with $\mathrm{N}\,z = y$, we must have $z = -\frac{y+1}{2} \in \mathbb{F}^\times$. Then $y = \mathrm{N}(z) = \frac{y+1}{2}$ implies $y = 1$. In this case $z = -1$. Hence

$$\begin{aligned} W_\Lambda\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= -q^{-1}\Gamma(\Lambda, \psi\circ\mathrm{Tr}) \sum_{\substack{z\in\mathbb{E},z\neq -1,\\ \mathrm{N}\,z=1}} \bar{\Lambda}\left(1+\frac{1}{z}\right) + 1 + (q-1) \\ &= -q^{-1}\Gamma(\Lambda, \psi\circ\mathrm{Tr}) \sum_{\substack{z\in\mathbb{E},z\neq -1,\\ \mathrm{N}\,z=1}} \Lambda(1+z) + q, \end{aligned} \tag{3.1}$$

and for $y \neq 1$,

$$W_\Lambda\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) = -q^{-1}\Gamma(\Lambda, \psi\circ\mathrm{Tr}) \sum_{\substack{z\in\mathbb{E},\\ \mathrm{N}\,z=y}} \bar{\Lambda}\left(1+\frac{y+1}{2z}\right) = W_\Lambda\left(\begin{pmatrix} y^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Proposition 3. $\sum_{\substack{Nz=1\\ z\neq -1}} \Lambda(1+z) = -\Lambda(\sqrt{\delta}) = \pm 1.$

Proof. Write N for the subgroup of elements in $\mathbb{E}^\times$ with norm 1 to $\mathbb{F}$. It is cyclic of order $q+1$. The map $\phi : E^\times \to N$ given by $z \mapsto \frac{z}{z^q}$ is surjective with kernel $\mathbb{F}^\times$. Any character of $E^\times$ trivial on $\mathbb{F}^\times$ factors through N. Thus there is a character χ of N such that $\Lambda(z) = \chi(\phi(z)) = \chi(\frac{z}{z^q})$. Consider the restriction of ϕ to the subset $S = \{1+z : z\in N, z\neq -1\}$. We claim that ϕ is injective on S. Indeed, if $z, w \in N\setminus\{-1\}$ are such that $\phi(1+z) = \phi(1+w)$, then there exists $k\in\mathbb{F}^\times$ such that $1+z = k(1+w)$. Then

$z = kw + k - 1$ implies

$$\begin{aligned} 1 &= zz^q = k^2ww^q + (k-1)^2 + k(k-1)(w+w^q) \\ &= k^2 + (k-1)^2 + k(k-1)(w+w^q). \end{aligned}$$

If $z \neq w$, then $k \neq 1$, and the above implies $w + w^q = -2$, that is, $w = -1$, a contradiction. Thus $\phi(S)$ contains all elements in N except -1, for $1 + z = -(1 + z^q)$ would imply $z = -1$. Therefore

$$\sum_{\substack{\mathrm{N}\, z=1 \\ z\neq -1}} \Lambda(1+z) = \sum_{\substack{z\in N \\ z\neq -1}} \chi(z) = -\chi(-1) = -\Lambda(\sqrt{\delta}) = \pm 1$$

since $\Lambda^2(\sqrt{\delta}) = \Lambda(\delta) = 1$. $\square$

On the other hand,

$$\begin{aligned} \Gamma(\Lambda, \psi \circ \mathrm{Tr}) &= \sum_{z\in\mathbb{E}^\times} \Lambda(z)\psi(\mathrm{Tr}\, z) \\ &= \sum_{b\in\mathbb{F}^\times}\sum_{a\in\mathbb{F}^\times} \Lambda(a(b+\sqrt{\delta}))\psi(a\,\mathrm{Tr}(b+\sqrt{\delta})) \\ &\quad + \sum_{a\in\mathbb{F}^\times} \Lambda(a\sqrt{\delta})\psi(a\,\mathrm{Tr}\sqrt{\delta}) + \sum_{a\in\mathbb{F}^\times} \Lambda(a)\psi(a\,\mathrm{Tr}\,1) \\ &= -\sum_{b\in\mathbb{F}^\times} \Lambda(b+\sqrt{\delta}) - \Lambda(1) + (q-1)\Lambda(\sqrt{\delta}) = q\Lambda(\sqrt{\delta}). \end{aligned}$$

Therefore $\Gamma(\Lambda, \psi \circ \mathrm{Tr}) \sum_{\substack{N z=1 \\ z\neq -1}} \Lambda(1+z) = q\Lambda(\sqrt{\delta})(-\Lambda(\sqrt{\delta})) = -q$. Plugging into (3.1), we obtain

Proposition 4. $W_\Lambda\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = q + 1$.

Consequently $W_\Lambda \neq 0$, and hence $W_{\Lambda,\psi} \neq 0$ for some ψ. We have shown that the dimension of $\mathcal{L}_\psi(\pi_\Lambda, G/K)$ is at least one for some and hence for all $\psi \neq \psi^0$. We record this in

Proposition 5. *For each discrete series representation character π_Λ of G and each nontrivial additive character ψ of $\mathbb{F}$, the space $\mathcal{L}_\psi(\pi_\Lambda, G/K)$ is 1-dimensional.*

Let KsK be a K-double coset of G. The operator T_{KsK} preserves each space $\mathcal{L}_\psi(\pi, G/K)$ and the eigenvalue depends only on the representation, not its model. As computed in [1], the K-double cosets KsK with cardinality greater than $q+1$ are symmetric, and they are parameterized by $c \in \mathbb{F}$ with $c \neq \pm 1$ so that $KsK = \bigcup_{(y,x)} \left(\begin{smallmatrix} y & \delta x \\ 0 & 1 \end{smallmatrix}\right) K$ where (y, x) runs all solutions of $(y+c)^2 - \delta x^2 = c^2 - 1$ over $\mathbb{F}$. Here s may be any coset representative $\left(\begin{smallmatrix} y & \delta x \\ 0 & 1 \end{smallmatrix}\right)$, for instance. Denote this double coset by K_c for short. We proceed to compute the eigenvalues of T_{K_c} using the right K-invariant eigenfunctions obtained above.

For $\mu \in \mathbb{F}^\times$ with $\mu \neq 1$, we have $T_{K_c} f_\mu = \lambda_{\pi_\mu, c} f_\mu$. Hence

$$\begin{aligned} \lambda_{\pi_\mu, c} = \lambda_{\pi_\mu, c} f_\mu \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) &= \sum_{\substack{y \in \mathbb{F}, \\ (y+c)^2 - \delta x^2 = c^2 - 1}} f_\mu \left(\begin{pmatrix} y & \delta x \\ 0 & 1 \end{pmatrix} \right) \\ &= \sum_{\substack{y \in \mathbb{F}, \\ (y+c)^2 - \delta x^2 = c^2 - 1}} \mu(y), \end{aligned}$$

which is known to have absolute value bounded by $2\sqrt{q}$. See Theorem 10 in Chapter 9 of [10] for a proof using idèle class characters. As remarked before, the eigenvalue $\lambda_{\pi_\mu, c}$ depends only on the representation and the double coset K_c. We have shown

Proposition 6. *Let μ be a nontrivial character of $\mathbb{F}^\times$. The eigenvalue $\lambda_{\pi_\mu, c}$ on $\mathcal{L}_\psi(\pi_\mu, G/K)$ is $\sum_{\substack{(y,x) \\ (y+c)^2 - \delta x^2 = c^2 - 1}} \mu(y)$, which has absolute value at most $2\sqrt{q}$ for all additive character ψ* of $\mathbb{F}$. For $\mu = 1$, *the trivial character, the constant functions on G are the eigenfunctions of T_{K_c} with eigenvalue $q + 1$, coming from the trivial representation of G.*

Next we fix a discrete series representation π_Λ and discuss the eigenvalue of T_{K_c} on the 1-dimensional space $\mathcal{L}_\psi(\pi_\Lambda, G/K)$ for $\psi \neq \psi^0$. Since $W_{\Lambda,\psi} \in \mathcal{L}_\psi(\pi_\Lambda, G/K)$, we have $T_{K_c} W_{\Lambda,\psi} = \lambda_{\pi_\Lambda, c} W_{\Lambda,\psi}$ and hence $T_{K_c} W_\Lambda = \lambda_{\pi_\Lambda, c} W_\Lambda$, where $W_\Lambda = \sum_{\psi \neq \psi^0} W_{\Lambda,\psi}$ has the Fourier expansion

$$W_\Lambda \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = \sum_{\psi \neq \psi^0} W_{\Lambda,\psi} \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \psi(x).$$

Recall that $W_\Lambda \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \right) = q + 1 \neq 0$. We will use this value to compute the eigenvalue of T_{K_c} for $c \neq \pm 1$. By definition,

$$\begin{aligned} (T_{K_c} W_\Lambda) \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) &= \sum_{\substack{(y,x) \\ (y+c)^2 - \delta x^2 = c^2 - 1}} W_\Lambda \left(\begin{pmatrix} y & \delta x \\ 0 & 1 \end{pmatrix} \right) \\ &= \sum_{\substack{(y,x) \\ (y+c)^2 - \delta x^2 = c^2 - 1}} W_\Lambda \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \psi(\delta x) \end{aligned}$$

$$= \sum_{\substack{(y,x)\\(y+c)^2-\delta x^2=c^2-1}} \sum_{\psi\neq\psi^0} -q^{-1} \sum_{b\in\mathbb{F}} \sum_{\substack{w\in\mathbb{E}^\times\\ \mathrm{N}\,w=y(b^2-\delta)}} \Lambda(w)\psi(\mathrm{Tr}\,w) \times\psi(b(y+1))\psi(\delta x) + \delta_{y,1}\psi(\delta x).$$

For fixed (y, x), we compute

$$\begin{aligned}
&\sum_{\psi\neq\psi^0} -q^{-1} \sum_{b\in\mathbb{F}} \sum_{\substack{w\in\mathbb{E}^\times\\ \mathrm{N}\,w=y(b^2-\delta)}} \Lambda(w)\psi(\mathrm{Tr}\,w)\psi(b(y+1)\psi(\delta x)\\
&= q^{-1} \sum_{b\in\mathbb{F}} \sum_{\substack{w\in\mathbb{E}^\times\\ \mathrm{N}\,w=y(b^2-\delta)}} \Lambda(w) - \sum_{b\in\mathbb{F}} \sum_{\substack{w\in\mathbb{E}^\times,\mathrm{N}\,w=y(b^2-\delta)\\ \mathrm{Tr}\,w=-(y+1)b-\delta x}} \Lambda(w)\\
&= -\sum_{b\in\mathbb{F}} \sum_{\substack{w\in\mathbb{E}^\times,\mathrm{N}\,w=y(b^2-\delta)\\ \mathrm{Tr}\,w=-(y+1)b-\delta x}} \Lambda(w)
\end{aligned}$$

since $\sum_{\mathrm{N}\,w=y(b^2-\delta)} \Lambda(w) = 0$ for all $b \in \mathbb{F}$. Further,

$$\sum_{\psi\neq\psi^0} \delta_{y,1}\psi(\delta x) = -\delta_{y,1} + q\delta_{y,1}\delta_{x,0} = -\delta_{y,1}$$

as $(y, x) = (1, 0)$ does not satisfy the equation $(y+c)^2-\delta x^2 = c^2-1$ because $c \neq -1$. Therefore we may write

$$(T_{K_c} W_\Lambda)\left(\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right) = \sum_{\substack{(y,x)\\(y+c)^2-\delta x^2=c^2-1}} S_{(y,x)},$$

where

$$S_{(y,x)} = -\sum_{b\in\mathbb{F}} \sum_{\substack{w\in\mathbb{E}^\times,\mathrm{N}\,w=y(b^2-\delta)\\ \mathrm{Tr}\,w=-(y+1)b-\delta x}} \Lambda(w) - \delta_{y,1}. \tag{3.2}$$

To proceed, we prove

Theorem 7. *Given* $c \in \mathbb{F}, c \neq \pm 1$, *let* $y, x \in \mathbb{F}$ *satisfy* $(y + c)^2 - \delta x^2 = c^2 - 1$. *Then* $|S_{(y,x)}| \leqslant 2\sqrt{q}$.

The following two results, adapted from Theorem 4 in Chapter 6 of [10] and Theorem 5 of [11], respectively, will be used repeatedly in the proof.

Theorem 8. *Let ν be the quadratic character of $\mathbb{F}^\times$. Let $f(T)$ be a quadratic polynomial over $\mathbb{F}$ with two distinct roots. Denote by $v_1, \ldots, v_r, r \leqslant 2$, the places of $\mathbb{F}(T)$ containing the roots of f. Then there exists an idèle class character $\eta = \eta_{\nu,f}$ of $\mathbb{F}(T)$ such that*

(1) *The conductor of η is $v_1 + \cdots + v_r$;*
(2) *At each place v of degree 1 with uniformizer $\pi_v = T - v$ where η is unramified, we have $\eta_v(\pi_v) = \nu(f(v))$, and $\eta_\infty(\pi_\infty) = 1$.*

Theorem 9. *Let $h(T)$ be a nonconstant polynomial over $\mathbb{E}$ with distinct roots. Let $w_1, \ldots, w_r$ be the places of $\mathbb{F}(T)$ containing the roots of h. Then there exists an idèle class character $\omega = \omega_{\Lambda,h}$ of $\mathbb{F}(T)$ such that*

(1) *The conductor of ω is $w_1 + \cdots + w_r$;*
(2) *At each place v of degree 1 with uniformizer $\pi_v = T - v$ where ω is unramified, we have $\omega_v(\pi_v) = \Lambda(h(v))$.*

The character ω in Theorem 9 is unramified at ∞ since Λ is trivial on $\mathbb{F}^\times$. But the value of $\omega_\infty(\pi_\infty)$ depends on Λ and h. In case $h(0) \neq 0$, it is equal to

$$\begin{aligned}\omega_\infty(\pi_\infty) &= \prod_{v \neq \infty} \omega_v(T) = \Lambda(h(0))\Lambda(\text{product of roots of } h)^{-1} \\ &= \Lambda\ (\text{the leading coefficient of } h). \end{aligned} \tag{3.3}$$

following the proof of Theorem 5 in [11].

The following character sum estimate results from the Riemann hypothesis for curves, as explained in Section 1, Chapter 6 of [10]. It will be used repeatedly to derive character sum estimates.

Proposition 10. *Let χ be an idèle class character of $\mathbb{F}(T)$ such that its conductor has degree m. Then*

$$\left| \sum_{\substack{\deg v = 1 \\ \chi_v \text{ unramified}}} \chi_v(\pi_v) \right| \leqslant (m-2)\sqrt{q}.$$

We now begin the proof of Theorem 7. We distinguish three cases.

Case 1: $y = -1$. Then $\delta x^2 = 2 - 2c$ implies $x \neq 0$ since $c \neq 1$. In this case

$$S_{(-1,x)} = -\sum_{b \in \mathbb{F}} \sum_{\substack{w \in \mathbb{E}^\times, \mathrm{N}\, w = -(b^2 - \delta) \\ \mathrm{Tr}\, w = -\delta x}} \Lambda(w).$$

Write $w = -\frac{\delta x}{2} + v\sqrt{\delta}$. Then $\mathrm{N}\, w = \frac{\delta^2 x^2}{4} - v^2\delta = \delta - b^2$ amounts to $\delta + v^2\delta - \frac{\delta^2 x^2}{4} = b^2$ being a square, or equivalently, $1 + v^2 - \frac{\delta x^2}{4} = 1 + v^2 - \frac{1-c}{2} = v^2 + \frac{1+c}{2}$ not a square in $\mathbb{F}$. Denote by ν the quadratic character of $\mathbb{F}^\times$, extended to a function on $\mathbb{F}$ by letting $\nu(0) = 0$. We rewrite $S_{(-1,x)}$ as

$$\begin{aligned} S_{(-1,x)} &= -\sum_{\substack{v \in \mathbb{F} \\ v^2 + \frac{1+c}{2} \text{ is a nonsquare}}} \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right) \\ &= -\frac{1}{2} \sum_{v \in \mathbb{F}} \left(1 - \nu\left(v^2 + \frac{1+c}{2}\right)\right) \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right) \\ &\quad + \frac{1}{2} \sum_{\substack{v \in \mathbb{F} \\ v^2 + \frac{1+c}{2} = 0}} \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right) \\ &= \frac{1}{2} \sum_{v \in \mathbb{F}} \nu\left(v^2 + \frac{1+c}{2}\right) \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right) - \frac{1}{2} \sum_{v \in \mathbb{F}} \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right) \\ &\quad + \frac{1}{2} \sum_{\substack{v \in \mathbb{F} \\ v^2 + \frac{1+c}{2} = 0}} \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right). \end{aligned}$$

As v runs through all elements in $\mathbb{F}$, no two elements of the form $-\frac{\delta x}{2} + v\sqrt{\delta}$ differ by a multiple in $\mathbb{F}^\times$, hence $-\frac{1}{2} \sum_{v \in \mathbb{F}} \Lambda(-\frac{\delta x}{2} + v\sqrt{\delta}) = \frac{1}{2} \Lambda(\sqrt{\delta})$ and

$$\begin{aligned} S_{(-1,x)} &= \frac{1}{2} \sum_{v \in \mathbb{F}} \nu\left(v^2 + \frac{1+c}{2}\right) \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right) + \frac{1}{2} \Lambda(\sqrt{\delta}) \\ &\quad + \frac{1}{2} \sum_{\substack{v \in \mathbb{F} \\ v^2 + \frac{1+c}{2} = 0}} \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right). \end{aligned}$$

Let $f(T) = T^2 + \frac{1+c}{2}$ and $h(T) = -\frac{\delta x}{2} + T\sqrt{\delta}$. Let $\eta = \eta_{\nu, f}$ and $\omega = \omega_{\Lambda, h}$ be the idèle class characters of $\mathbb{F}(T)$ as described in Theorems 8 and 9, respectively. The conductor of ω is w_1 with the uniformizer $\pi_{w_1} = T^2 - \frac{1-c}{2}$, which is disjoint from the conductor of η. Hence the conductor of $\eta\omega$ has degree 4. Moreover, $\omega_\infty(\pi_\infty) = \Lambda(\sqrt{\delta})$ by Eq. (3.3). Therefore

$$\left| \sum_{\substack{\deg v=1 \\ \eta_v\omega_v \text{ unramified}}} \eta_v(\pi_v)\omega_v(\pi_v) \right| = \left| \sum_{v\in\mathbb{F}} v(f(v))\Lambda(h(v)) + \Lambda(\sqrt{\delta}) \right|$$

$$= \left| \sum_{v\in\mathbb{F}} v\left(v^2 + \frac{1+c}{2}\right)\Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right) + \Lambda(\sqrt{\delta}) \right|$$

$$= \left| 2S_{(-1,v)} - \sum_{\substack{v\in\mathbb{F} \\ v^2+\frac{1+c}{2}=0}} \Lambda\left(-\frac{\delta x}{2} + v\sqrt{\delta}\right) \right| \leqslant (4-2)\sqrt{q}$$

implies

$$|S_{(-1,x)}| \leqslant \sqrt{q} + 1 < 2\sqrt{q},$$

as desired.

It remains to deal with the case $y \neq -1$. We eliminate the variable b in the expression of $S_{(y,x)}$. Write $w = u + v\sqrt{\delta}$ with $u, v \in \mathbb{F}$. The conditions $\mathrm{N}\, w = y(b^2 - \delta)$ and $\mathrm{Tr}\, w = -(y+1)b - \delta x$ can be combined as

$$\frac{\mathrm{N}\, w}{y} + \delta = b^2 = \left(\frac{\mathrm{Tr}\, w + \delta x}{y+1}\right)^2.$$

In other words,

$$u^2 - v^2\delta + y\delta = \frac{y}{(y+1)^2}(2u + \delta x)^2 = \frac{4y}{(y+1)^2}u^2 + \frac{4yu\delta x}{(y+1)^2} + \frac{y\delta^2 x^2}{(y+1)^2},$$

which in turn yields

$$\left(\frac{y-1}{y+1}\right)^2 u^2 - \frac{4y\delta x}{(y+1)^2}u - v^2\delta = -y\delta + \frac{y\delta^2 x^2}{(y+1)^2}. \tag{3.4}$$

Case 2: $y = 1$. Then $\delta x^2 = 2 + 2c \neq 0$ since $c \neq -1$. The above relation can be simplified as

$$-\delta x u - v^2\delta = -\delta + \frac{\delta^2 x^2}{4} = -\delta + \delta\,\frac{1+c}{2} = \delta\,\frac{c-1}{2},$$

which allows us to express u in terms of $v : u = \frac{1}{x}\left(\frac{1-c}{2} - v^2\right)$. Then

$$S_{(1,x)} = -\sum_{v\in\mathbb{F}} \Lambda\left(\frac{1}{x}\left(\frac{1-c}{2} - v^2\right) + v\sqrt{\delta}\right) - 1.$$

Let

$$\begin{aligned} h(T) &= \frac{1}{x}\left(\frac{1-c}{2} - T^2\right) + T\sqrt{\delta} = -\frac{1}{x}\left(T^2 - x\sqrt{\delta}T - \frac{1-c}{2}\right) \\ &= -\frac{1}{x}\left(\left(T - \frac{x\sqrt{\delta}}{2}\right)^2 - \frac{x^2\delta}{4} + \frac{c-1}{2}\right) = -\frac{1}{x}\left(\left(T - \frac{x\sqrt{\delta}}{2}\right)^2 - 1\right). \end{aligned}$$

Let $\omega = \omega_{\Lambda,h}$ be the idèle class character of $\mathbb{F}(T)$ attached to Λ and h as described in Theorem 9. The conductor of ω is $w_1 + w_2$, where w_1 and w_2 are two degree two places of $\mathbb{F}(T)$ containing $\frac{x\sqrt{\delta}}{2} - 1$ and $\frac{x\sqrt{\delta}}{2} + 1$ as roots, respectively. So the conductor of ω has degree 4. By Eq. (3.3), $\omega_\infty(\pi_\infty) = 1$. Put together, we have

$$-S_{(1,x)} = \sum_{v\in\mathbb{F}} \Lambda(h(v)) + 1 = \sum_{\deg v=1} \omega_v(\pi_v),$$

which satisfies $|S_{(1,x)}| \leqslant 2\sqrt{q}$.

Case 3: $y \neq \pm 1$. We have $y^2 + 2yc + 1 = \delta x^2$. The relation (3.4) can be rewritten as

$$\left(\frac{y-1}{y+1}u - \frac{2y\delta x}{y^2-1}\right)^2 - v^2\delta = \delta y^2\,\frac{2+2c}{(y-1)^2}$$

so that

$$S_{(y,x)} = -\sum_{\mathrm{N}\left(\frac{y-1}{y+1}u - \frac{2y\delta x}{y^2-1} + v\sqrt{\delta}\right) = \delta y^2\,\frac{2+2c}{(y-1)^2}} \Lambda(u + v\sqrt{\delta}).$$

Replacing u by $\frac{y+1}{y-1}u + \frac{2y\delta x}{(y-1)^2}$, we rewrite the above as

$$S_{(y,x)} = -\sum_{\mathrm{N}(u+v\sqrt{\delta}) = \delta y^2\,\frac{2+2c}{(y-1)^2}} \Lambda\left(\frac{y+1}{y-1}u + \frac{2y\delta x}{(y-1)^2} + v\sqrt{\delta}\right).$$

Set $z = u + v\sqrt{\delta}$. Then

$$2u = z + z^q = z + \frac{\mathrm{N}\, z}{z} = z + \frac{1}{z}\,\delta y^2\,\frac{2+2c}{(y-1)^2}.$$

Using this, we may express the argument of Λ as a rational function in z:

$$\begin{aligned}\frac{y+1}{y-1}u + \frac{2y\delta x}{(y-1)^2} + v\sqrt{\delta} &= z + u\frac{2}{y-1} + \frac{2y\delta x}{(y-1)^2}\\ &= \frac{y}{y-1}z + \frac{1}{z}\,\frac{2+2c}{(y-1)^3}\,\delta y^2 + \frac{2y\delta x}{(y-1)^2} =: Q(z).\end{aligned}$$

By choosing an element $w \in \mathbb{E}\backslash\mathbb{F}$ with $\mathrm{N}\, w = \delta y^2 \frac{2+2c}{(y-1)^2}$, we rewrite $S_{(y,x)}$ as

$$S_{(y,x)} = -\sum_{\mathrm{N}\, z = \delta y^2 \frac{2+2c}{(y-1)^2}} \Lambda(Q(z)) = -\sum_{\mathrm{N}\, z_1 = 1} \Lambda(Q(wz_1)).$$

Here

$$\begin{aligned}Q(wz_1) &= \frac{y}{y-1}wz_1 + \frac{1}{z_1 w}\,\frac{\mathrm{N}\, w}{y-1}\,\delta y^2 + \frac{2y\delta x}{(y-1)^2} = \frac{1}{y-1}\left(ywz_1 + \frac{w^q}{z_1} + \frac{2y\delta x}{y-1}\right)\\ &=: R(z_1).\end{aligned}$$

Consider

$$\begin{aligned}R\left(\frac{T-\sqrt{\delta}}{T+\sqrt{\delta}}\right) &= \frac{1}{y-1}\left(yw\,\frac{T-\sqrt{\delta}}{T+\sqrt{\delta}} + \frac{w^q(T+\sqrt{\delta})}{T-\sqrt{\delta}} + \frac{2y\delta x}{y-1}\right)\\ &= \frac{1}{(y-1)(T^2-\delta)}\left(yw(T-\delta)^2 + w^q(T+\sqrt{\delta})^2 + \frac{2y\delta x}{y-1}(T^2-\delta)\right)\\ &=: \frac{h(T)}{(y-1)(T^2-\delta)},\end{aligned}$$

where $h(T) = (yw + w^q + \frac{2y\delta x}{y-1})T^2 + (2w^q\sqrt{\delta} - 2yw\sqrt{\delta})T + yw\delta + w^q\delta - \frac{2y\delta^2 x}{y-1} \in \mathbb{E}[T]$. As T runs through elements in $\mathbb{F}$, $\frac{T-\sqrt{\delta}}{T+\sqrt{\delta}}$ runs through all elements z_1 in E with norm 1 except $z_1 = 1$. Observe that for $T \in \mathbb{F}$, $\Lambda(R(\frac{T-\sqrt{\delta}}{T+\sqrt{\delta}})) = \Lambda(\frac{h(T)}{(y-1)(T^2-\delta)}) = \Lambda(h(T))$ since Λ is trivial on $\mathbb{F}^\times$. Thus we have

$$S_{(y,x)} = -\sum_{\mathrm{N}\, z_1 = 1} \Lambda(R(z_1)) = -\sum_{v\in\mathbb{F}} \Lambda(h(v)) - \Lambda(R(1)).$$

Notice that in the course of deriving various expressions of $S_{(y,x)}$, the character Λ is always evaluated at nonzero elements in $\mathbb{E}$. In particular, this means that $h(v) \neq 0$ for all $v \in \mathbb{F}$. In other words, the roots of h are outside $\mathbb{F}$. Since $y \neq 1$ and $w \notin \mathbb{F}$ by choice, both the leading coefficient and the constant term of $h(T)$ lie in $\mathbb{E}\backslash\mathbb{F}$. Moreover, their ratio also lies in $\mathbb{E}\backslash\mathbb{F}$. This implies that if $h(T)$ has a root t of multiplicity 2, then t lies in $\mathbb{E}\backslash\mathbb{F}$. In this case, for $v \in \mathbb{F}$,

$$\Lambda(h(v)) = \Lambda\left(yw + w^q + \frac{2y\delta x}{y-1}\right)\Lambda^2(v-t).$$

As v runs through all elements in $\mathbb{F}$, no two elements of the form $v - t$ differ by a multiple in $\mathbb{F}^\times$. Since the order of Λ is greater than 2, we have

$$-\sum_{v\in\mathbb{F}} \Lambda(h(v)) = \Lambda\left(yw + w^q + \frac{2y\delta x}{y-1}\right)\Lambda^2(1)$$

and consequently $|S_{(y,x)}| \leqslant 2 < 2\sqrt{q}$.

Finally we discuss the case where h has two distinct roots. Either they are nonconjugate over $\mathbb{F}$, and hence contained in two distinct places v_1, v_2 of $\mathbb{F}(T)$ of degree 2, or they are conjugate over $\mathbb{F}$ and contained in a degree 4 place v_1 of $\mathbb{F}(T)$. At any rate, the idèle class character $\omega = \omega_{\Lambda,h}$ attached to Λ and h as described in Theorem 9 has conductor of degree 4. Further, ω is unramified at all places of degree 1, and

$$\omega_\infty(\pi_\infty) = \Lambda\left(yw + w^q + \frac{2y\delta x}{y-1}\right) = \Lambda(R(1)).$$

Therefore

$$S_{(y,x)} = -\sum_{v\in\mathbb{F}} \Lambda(h(v)) - \Lambda(R(1)) = -\sum_{\deg v=1} \omega_v(\pi_v),$$

and $|S_{(y,x)}| \leqslant 2\sqrt{q}$. This completes the proof of the theorem.

Since

$$\begin{aligned}(T_{K_c}W_\Lambda)\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) &= \sum_{\substack{(y,x)\\ (y+c)^2-\delta x^2=c^2-1}} S_{(y,x)} = \lambda_{\pi_\Lambda,c}W_\Lambda\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\\ &= \lambda_{\pi_\Lambda,c}(q+1)\end{aligned}$$

is a sum of $q+1$ terms $S_{(y,x)}$ and each $S_{(y,x)}$ is of absolute value at most $2\sqrt{q}$, we conclude that the eigenvalue $\lambda_{\pi_\Lambda,c}$ of T_{K_c} on the space $\mathcal{L}_\psi(\pi_\Lambda, G/K)$ associated to the discrete series π_Λ satisfies $|\lambda_{\pi_\Lambda,c}| \leqslant 2\sqrt{q}$ for each nontrivial additive character ψ. This proves

Proposition 11. *Let Λ be a character of $\mathbb{E}^\times$ trivial on $\mathbb{F}^\times$ and of order greater than 2. The eigenvalue $\lambda_{\pi_\Lambda,c}$ of the operator T_{K_c} on the 1-dimensional space $\mathcal{L}_\psi(\pi_\lambda, G/K)$ is equal to $\frac{1}{q+1}\sum_{\substack{(y,x)\\(y+c)^2-\delta x^2=c^2-1}} S_{(y,x)}$ and satisfies $|\lambda_{\pi_\Lambda,c}| \leqslant 2\sqrt{q}$ for all nontrivial additive character ψ.*

Combined with Proposition 6, we obtain another proof of the following result established by Terras et al. in [1,2].

Theorem 12. *Given $c \in \mathbb{F}$ with $c \neq \pm 1$, denote by K_c the K-double coset $K\left(\begin{smallmatrix} y & \delta x \\ 0 & 1 \end{smallmatrix}\right)K = \cup_{(y,x)}\left(\begin{smallmatrix} y & \delta x \\ 0 & 1 \end{smallmatrix}\right)K$ where (y,x) satisfies $(y+c)^2 - \delta x^2 = c^2 - 1$. Then the Cayley graphy $X_c = \mathrm{Cay}(G/K, K_c)$ is an undirected $(q+1)$-regular Ramanujan graph.*

4. The *U*-graphs

In this section, we let $H = U$. We first analyze the space $\mathcal{L}_{\psi^0}(G/U)$. Note that $G = AU \cup AU\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}U$ is a disjoint union of two double cosets. One sees immediately that for each $\mu \in \hat{\mathbb{F}}^\times$, the right U-invariant subspace of Ind_μ is 2-dimensional, generated by

$$g_\mu\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}U\right) = \mu(y) \text{ for all } y \in \mathbb{F}^\times \text{ and } g_\mu\left(AU\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}U\right) = 0.$$

and

$$h_\mu\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}U\right) = \mu(y) \text{ for all } y \in \mathbb{F}^\times,\ x \in \mathbb{F}, \text{ and } h_\mu(AU) = 0.$$

Thus $\mathcal{L}_{\psi^0}(G/U)$ is $2(q-1)$-dimensional. Next we fix a nontrivial additive character ψ of $\mathbb{F}$. For each irreducible representation π of G with $\deg \pi > 1$ and each $v \in \mathcal{K}_\psi(\pi)$, the Whittaker function

$$\overline{W}_v(g) = q^{-1}\sum_{s \in \mathbb{F}}(\pi(gu_s)v)(1)$$

is right U-invariant. Here $u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ as in §2. Using the actions of $\pi(u_s)$ on $\theta \in \hat{\mathbb{F}}^\times$, D_0, and D_∞ described in §2, one gets

$$\overline{W}_\theta(g) = 0, \overline{W}_{D_0}(g) = (\pi(g)D_0)(1) \quad \text{and} \quad \overline{W}_{D_\infty}(g) = (\pi(g)D_\infty)(1)$$

for all $g \in G$. Since dim $\mathcal{L}(G/U) = (q+1)(q-1)$, by dimension counting one finds that the right U-invariant space is 0-dimensional for discrete series representations; it is 1-dimensional generated by $\{\overline{W}_{D_0}\}$ for the Steinberg representations π_μ with $\mu^2 = 1$, and it is 2-dimensional generated by $\{\overline{W}_{D_0}, \overline{W}_{D_\infty}\}$ for principal series representations π_μ with $\mu^2 \neq 1$. We record this in

Proposition 13. (1) *The space* $\mathcal{L}_{\psi^0}(\pi_\mu, G/U)$ *is* 2*-dimensional for all* $\mu \in \hat{\mathbb{F}}^\times$.

(2) *For nontrivial* ψ*'s, the space* $\mathcal{L}_\psi(\pi, G/U)$ *is* 0*-dimensional if* π *is a discrete series representation*; *it is* 1*-dimensional if* π *is a Steinberg representation, and* 2*-dimensional if* π *is a principal series representation*.

The $2(q-1)$ U-double cosets of G fall in two categories. The first consists of $U\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}U, r \in \mathbb{F}^\times$, which are not symmetric if $r \neq \pm 1$ and which are contained in the Borel subgroup $(= AU)$ of G, hence not interesting; the second consists of $U\begin{pmatrix} 0 & t \\ -1 & 0 \end{pmatrix}U, t \in \mathbb{F}^\times$, which are symmetric and of interest to us. Write $U_t = U\begin{pmatrix} 0 & t \\ -1 & 0 \end{pmatrix}U$ for short. Then

$$U_t = \bigcup_{c \in \mathbb{F}} \begin{pmatrix} c & t \\ -1 & 0 \end{pmatrix} U$$

is a disjoint union of q U-cosets. The operator T_{U_t} preserves each space $\mathcal{L}_\psi(\pi, G/U)$. To study its eigenvalues and eigenfunctions, we start with ψ^0. Note that

$$\begin{pmatrix} c & t \\ -1 & 0 \end{pmatrix} U = \begin{pmatrix} t & -c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U$$

for all $c \in \mathbb{F}$ and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & t \\ -1 & 0 \end{pmatrix} U = \begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} U \quad \text{and}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c & t \\ -1 & 0 \end{pmatrix} U = \begin{pmatrix} tc^{-2} & tc^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U$$

for all $c \in \mathbb{F}^\times$. Let $\mu \in \hat{\mathbb{F}}^\times$. Then for $l_\mu \in \{g_\mu, h_\mu\}$, we have

$$\begin{aligned} (T_{U_t} l_\mu)(U) &= \sum_{c \in \mathbb{F}} l_\mu\left(\begin{pmatrix} c & t \\ -1 & 0 \end{pmatrix} U\right) = \sum_{c \in \mathbb{F}} l_\mu\left(\begin{pmatrix} t & -c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U\right) \\ &= q\mu(t) l_\mu\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U\right) \end{aligned}$$

and

$$(T_{U_t} l_\mu)\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U\right) = \sum_{c\in\mathbb{F}} l_\mu\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c & t \\ -1 & 0 \end{pmatrix} U\right) = l_\mu\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix} U\right)$$
$$+ \sum_{c\in\mathbb{F}^\times} l_\mu\left(\begin{pmatrix} tc^{-2} & c^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U\right)$$
$$= \mu(t^{-1}) l_\mu(U) + \mu(t) \sum_{c\in\mathbb{F}^\times} \mu^2(c) l_\mu\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U\right).$$

Thus

$$T_{U_t} g_\mu = \mu(t^{-1}) h_\mu \quad \text{and} \quad T_{U_t} h_\mu = q\mu(t) g_\mu + \mu(t) \sum_{c\in\mathbb{F}^\times} \mu^2(c) h_\mu.$$

In other words, with respect to the basis $\{g_\mu, h_\mu\}$ of $\mathcal{L}_{\psi^0}(\pi_\mu, G/U)$, the operator T_{U_t} can be represented by the matrix

$$\begin{pmatrix} 0 & q\mu(t) \\ \mu(t^{-1}) & \mu(t)\sum_{c\in\mathbb{F}^\times} \mu^2(c) \end{pmatrix}.$$

Consequently, if $\mu^2 \neq 1$, the eigenvalues of T_{U_t} are $\pm\sqrt{q}$ with eigenfunctions $\pm\sqrt{q}\mu(t) g_\mu + h_\mu$; while if $\mu^2 = 1$, the eigenvalues are $\mu(t)q$ and $-\mu(t)$ with eigenfunctions $g_\mu + h_\mu$ and $qg_\mu - h_\mu$, respectively.

Next we deal with the case $\psi \neq \psi^0$. Observe that

$$\begin{pmatrix} c & t \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

and $\Gamma(1,\psi) = -1$. Recall that for a Steinberg representation $\pi = \pi_\mu$ with $\mu^2 = 1$, the space $\mathcal{L}_\psi(\pi, G/U)$ is 1-dimensional generated by $\{\overline{W}_{D_0}\}$. The action of T_{U_t} on $\overline{W}_{D_0}$ is given by, according to the Kirillov model of π_μ,

$$(T_{U_t}\overline{W}_{D_0})(g) = \sum_{c\in\mathbb{F}} \overline{W}_{D_0}\left(g\begin{pmatrix} c & t \\ -1 & 0 \end{pmatrix}\right) = \sum_{c\in\mathbb{F}} \left(\pi\left(g\begin{pmatrix} c & t \\ -1 & 0 \end{pmatrix}\right) D_0\right)(1)$$
$$= \sum_{c\in\mathbb{F}} (\pi(g)\pi(u_c)\pi(w)\pi(h_{t^{-1}}) D_0)(1)$$
$$= \sum_{c\in\mathbb{F}} (\pi(g)\pi(u_c)\pi(w)\mu(t) D_0)(1) \quad (\text{since } \mu = \mu^{-1})$$

$$= \sum_{c\in\mathbb{F}} (\pi(g)\pi(u_c)(-q^{-1}\varepsilon(\pi,\mu^{-1},\psi)\mu(t))(\mu^{-1}+D_0))(1)$$

$$= -q^{-1}\varepsilon(\pi,\mu^{-1},\psi)\mu(t)\pi(g)$$

$$\times\left[\sum_{c\in\mathbb{F}^\times}(q-1)^{-1}\sum_{\beta\in\hat{\mathbb{F}}^\times}\beta\mu(c)\Gamma(\mu^{-1}\beta^{-1},\psi)\beta+\mu^{-1}+\sum_{c\in\mathbb{F}}D_0\right](1)$$

$$= -q^{-1}\varepsilon(\pi,\mu^{-1},\psi)\mu(t)\pi(g)$$

$$\times[(q-1)^{-1}(q-1)\Gamma(1,\psi)\mu^{-1}+\mu^{-1}+qD_0](1)$$

$$= -\varepsilon(\pi,\mu^{-1},\psi)\mu(t)[\pi(g)D_0](1) = -\varepsilon(\pi,\mu^{-1},\psi)\mu(t)\overline{W}_{D_0}(g)$$

for all $g\in G$, which implies that the eigenvalue is

$$-\varepsilon(\pi,\mu^{-1},\psi)\mu(t) = -\Gamma(\mu\mu^{-1},\psi)\Gamma(\mu^{-1}\mu^{-1},\psi)\mu(t) = -\Gamma(1,\psi)\Gamma(1,\psi)\mu(t) = -\mu(t).$$

Recall that for a principal series representation $\pi=\pi_\mu$ with $\mu^2\neq 1$, the space $\mathcal{L}_\psi(\pi_\mu, G/U)$ is 2-dimensional generated by $\{\overline{W}_{D_0},\overline{W}_{D_\infty}\}$. Using the Kirillov model of π, we get

$$(T_{U_t}\overline{W}_{D_0})(g) = \sum_{c\in\mathbb{F}}\overline{W}_{D_0}\left(g\begin{pmatrix}c & t\\ -1 & 0\end{pmatrix}\right) = \sum_{c\in\mathbb{F}}\left(\pi\left(g\begin{pmatrix}c & t\\ -1 & 0\end{pmatrix}\right)D_0\right)(1)$$

$$= \sum_{c\in\mathbb{F}}(\pi(g)\pi(u_c)\pi(w)\pi(h_{t^{-1}})D_0)(1) = \sum_{c\in\mathbb{F}}(\pi(g)\pi(u_c)\pi(w)\mu(t^{-1})D_0)(1)$$

$$= \sum_{c\in\mathbb{F}}(\pi(g)\pi(u_c)(-q^{-1}\varepsilon(\pi,\mu,\psi)\mu(t^{-1}))(\mu^{-1}+D_0))(1)$$

$$= -q^{-1}\varepsilon(\pi,\mu,\psi)\mu(t^{-1})\pi(g)$$

$$\times\left[\sum_{c\in\mathbb{F}^\times}(q-1)^{-1}\sum_{\beta\in\hat{\mathbb{F}}^\times}\beta\mu(c)\Gamma(\mu^{-1}\beta^{-1},\psi)\beta+\mu^{-1}+\sum_{c\in\mathbb{F}}D_\infty\right](1)$$

$$= -q^{-1}\varepsilon(\pi,\mu,\psi)\mu(t^{-1})\pi(g)$$

$$\times[(q-1)^{-1}(q-1)\Gamma(1,\psi)\mu^{-1}+\mu^{-1}+qD_\infty](1)$$

$$= -\varepsilon(\pi,\mu,\psi)\mu(t^{-1})[\pi(g)D_\infty](1) = -\varepsilon(\pi,\mu,\psi)\mu(t^{-1})\overline{W}_{D_\infty}(g)$$

and

$$
\begin{aligned}
(T_{U_t}\overline{W}_{D_\infty})(g) &= \sum_{c\in\mathbb{F}} \overline{W}_{D_\infty}\left(g\begin{pmatrix} c & t \\ -1 & 0\end{pmatrix}\right) = \sum_{c\in\mathbb{F}}\left(\pi\left(g\begin{pmatrix} c & t \\ -1 & 0\end{pmatrix}\right)D_\infty\right)(1)\\
&= \sum_{c\in\mathbb{F}} (\pi(g)\pi(u_c)\pi(w)\pi(h_{t^{-1}})D_\infty)(1)\\
&= \sum_{c\in\mathbb{F}} (\pi(g)\pi(u_c)\pi(w)\mu^{-1}(t^{-1})D_\infty)(1)\\
&= \sum_{c\in\mathbb{F}} (\pi(g)\pi(u_c)(-q^{-1}\varepsilon(\pi,\mu^{-1},\psi)\mu(t))(\mu + D_0))(1)\\
&= -q^{-1}\varepsilon(\pi,\mu^{-1},\psi)\mu(t)\pi(g)\\
&\quad\times\left[\sum_{c\in\mathbb{F}^\times}(q-1)^{-1}\sum_{\beta\in\hat{\mathbb{F}}^\times}\beta\mu^{-1}(c)\Gamma(\mu\beta^{-1},\psi)\beta + \mu = \sum_{c\in\mathbb{F}} D_0\right](1)\\
&= -q^{-1}\varepsilon(\pi,\mu^{-1},\psi)\mu(t)\pi(g)[(q-1)^{-1}(q-1)\Gamma(1,\psi)\mu + \mu + qD_0](1)\\
&= -\varepsilon(\pi,\mu^{-1},\psi)\mu(t)[\pi(g)D_0](1) = -\varepsilon(\pi,\mu^{-1},\psi)\mu(t)\overline{W}_{D_0}(g)
\end{aligned}
$$

for all $g\in G$. With respect to the basis $\{\overline{W}_{D_0}, \overline{W}_{D_\infty}\}$, the operator T_{U_t} is represented by the matrix

$$
\begin{pmatrix} 0 & -\varepsilon(\pi,\mu^{-1},\psi)\mu(t) \\ -\varepsilon(\pi,\mu,\psi)\mu(t^{-1}) & 0 \end{pmatrix}.
$$

As

$$
\varepsilon(\pi,\mu,\psi)\varepsilon(\pi,\mu^{-1},\psi) = \Gamma(\mu^2,\psi)\Gamma(1,\psi)\Gamma(1,\psi)\Gamma(\mu^{-2},\psi) = q\mu^2(-1) = q,
$$

the eigenvalues are $\pm\sqrt{q}$ with corresponding eigenfunctions $\pm\sqrt{q}\varepsilon(\pi,\mu,\psi)^{-1}\mu(t)$ $\overline{W}_{D_0} - \overline{W}_{D_\infty}$.

We have shown

Theorem 14. *For $t\in\mathbb{F}^\times$, the Cayley graph $X_{U_t} = \mathrm{Cay}(G/U, U_t/U)$ is a q-regular Ramanujan graph with the following eigenvalues: $\mu(t)q$ of multiplicity one, $-\mu(t)$ of multiplicity q for $\mu^2 = 1$, and $\pm\sqrt{q}$ of multiplicity $(q+1)(q-3)/2$.*

Therefore the graph X_{U_t} is bipartite for t nonsquare since $\mu(t) = -1$ for μ of order 2. If t is a square, the graph has two connected components, one with square determinants and one with nonsquares. In case $t = 1$, one component is isomorphic to the graph $\mathrm{Cay}(\mathrm{PSL}_2(\mathbb{F})/U, UwU/U)$, which, for the case $q = p$ a prime, is a cover of the Ramanujan graph on cusps of the principal congruence subgroup $\Gamma(p)$ of $\mathrm{SL}_2(\mathbb{Z})$ studied by Gunnells in [8]. When $q = p^e$ is a power of the prime p, a similar interpretation holds, as we explain below.

The cusps of the Drinfeld upper half plane attached to the rational function field $\mathbb{F}_p(T)$ and the (degree one) place at infinity is represented by $\mathbb{F}_p(T) \cup \{\infty\} = \mathbb{P}^1(\mathbb{F}_p(T))$, on which the group $\Gamma = \mathrm{GL}_2(\mathbb{F}_p[T])$ acts via fractional linear transformations. Endow a graph structure, called X, on $\mathbb{P}^1(\mathbb{F}_p(T))$ by defining two cusps u, v to be adjacent if there exists $\gamma \in \Gamma$ sending the cusp 0 to u and the cusp ∞ to v. Let P be an irreducible polynomial of degree e over $\mathbb{F}_p$, and let $\Gamma(P)$ be the principal congruence subgroup of Γ consisting of matrices congruent to the identity matrix modulo P. Denote by X_P the quotient graph $\Gamma(P)\backslash X$. This is a positive characteristic analog of the graph $G(p)$ in Gunnells's paper [8].

We proceed to show the connection between X_P and $\mathrm{Cay}(\mathrm{PSL}_2(\mathbb{F})/U, UwU/U)$. First observe the following lemma, which can be deduced by the same argument as in the proof of Lemma 1.42 in Shimura's book [19].

Lemma 15. *Let $t = \frac{a}{b}$ and $t' = \frac{c}{d}$ be two cusps in $\mathbb{P}^1(\mathbb{F}_p(T))$, where $a, b, c, d \in \mathbb{F}_p[T]$ and $\gcd(a,b) = \gcd(c,d) = 1$. Then t and t' are equivalent under $\Gamma(P)$ if and only if $\binom{a}{b} \equiv \lambda \binom{c}{d} \bmod P$ for some $\lambda \in \mathbb{F}_p^\times$.*

Note that $\mathbb{F}_p[T]$ modulo P is a finite field with $p^e = q$ elements, which we identify with $\mathbb{F}$. By the lemma above, the vertices in X_P may be expressed as the column vectors $\binom{a}{b}$ modulo scalar multiplications by $\mathbb{F}_p^\times$, where $a, b \in \mathbb{F}$, not both zero. The cusp 0 is represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and cusp ∞ by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The image of Γ modulo P is

$$G' = \{\gamma \in \mathrm{GL}_2(\mathbb{F}) : \det \gamma \in \mathbb{F}_p^\times\}$$

and the action of Γ on the cusps of $\Gamma(P)$ becomes (matrix) left multiplication by G'.

Observe that the vertices of X_P are also the first (or second) columns of the matrices in G' modulo $\mathbb{F}_p^\times$. Since

$$G' = \mathrm{SL}_2(\mathbb{F}) \cdot \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_p^\times \right\}$$

acts transitively on the vertices of X_P and the stabilizer of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in G' is the Borel subgroup of G', the vertices of X_P are represented by $\mathrm{PSL}_2(\mathbb{F})/B$, where B denotes the Borel subgroup of $\mathrm{PSL}_2(\mathbb{F})$. Suppose that vertices u, v of X_P are adjacent, that is, there is a matrix $\gamma \in G'$ such that $\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} = u$ and $\gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v$. Then v and u are the first and second columns of γ. In particular, this shows that the neighbors of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are the second columns of the elements in the unipotent subgroup U of G', which is also the unipotent subgroup of $\mathrm{PSL}_2(\mathbb{F})$. Note that they are the first columns of the coset representatives of the double coset $UwU = \bigcup_{c\in\mathbb{F}} \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} U$. As the edge structure on X_P is defined by transporting the edges out of the vertex $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to other vertices using left multiplications by G', we have shown

Proposition 16. *The graph X_P on cusps of $\Gamma(P)$ is a quotient of* $\mathrm{Cay}(\mathrm{PSL}_2(\mathbb{F})/U, UwU/U)$, *hence also a quotient of* $\mathrm{Cay}(G/U, UwU/U)$. *Consequently, it is a q-regular Ramanujan graph.*

5. The *A*-graphs

In this section, the group H is A. We start with the space $\mathcal{L}_{\psi^0}(G/A)$. The group $G = UA \cup UA \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A \cup UA \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} A$ is a disjoint union of three double cosets. For $\mu \neq 1$, one can easily see that the right A-invariant space of $\mathrm{Ind}\,\mu$ is 1-dimensional, generated by

$$f_\mu\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} A\right) = \mu(y) \quad \text{for all } y \in \mathbb{F}^\times,\ x \in \mathbb{F} \quad \text{and}$$

$$f_\mu(UA) = f_\mu\left(UA\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A\right) = 0.$$

If $\mu = 1$, then the right A-invariant space is 3-dimensional generated by $\{f_1, f_2, f_3\}$ where f_1, f_2 and f_3 are the characteristic functions of $UA, UA\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} A$ and $UA\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A$, respectively. Clearly, $\mathcal{L}_{\psi^0}(1 \circ \det, G/A)$ is 1-dimensional. So $\mathcal{L}_{\psi^0}(\pi_1, G/A)$ is 2-dimensional. Next we fix a nontrivial additive character ψ of $\mathbb{F}$. For each irreducible representation π of G of degree greater than 1, the function

$$W_v(g) := (q-1)^{-1} \sum_{r\in\mathbb{F}^\times} (\pi(gh_r)v)(1)$$

is right A-invariant for all $v \in \mathcal{K}_\psi(\pi)$. From the action of $\pi(h_r)$ on $\theta \in \hat{\mathbb{F}}^\times$, D_0 and D_∞ described in §2, we find

$$W_\theta(g) = \begin{cases} (\pi(g)\theta)(1) & \text{if } \theta = 1; \\ 0 & \text{if } \theta \neq 1, \end{cases} \quad W_{D_0}(g) = \begin{cases} (\pi(g)D_0)(1) & \text{if } \pi = \pi_1; \\ 0 & \text{if } \pi = \pi_\mu \text{ with } \mu \neq 1, \end{cases}$$

and $W_{D_\infty}(g) = 0$ for all $g \in G$. Since $\dim \mathcal{L}(G/A) = q(q-1)$, we conclude by dimension counting that for a principal series representation π_μ, the Steinberg representation π_μ with μ quadratic, and for a discrete series representation π_Λ, the space of right A-invariant vectors in its Whittaker model is 1-dimensional generated by $\{W_1\}$, and the space of right A-invariant vectors is 2-dimensional generated by $\{W_1, W_{D_0}\}$ for the Steinberg representation π_1.

We have proven

Proposition 17. (1) *The space* $\mathcal{L}_{\psi^0}(\pi_\mu, G/A)$ *is* 1*-dimensional if* $\mu \neq 1$ *and* 2*-dimensional if* $\mu = 1$; *and the space* $\mathcal{L}_{\psi^0}(1 \circ \det, G/A)$ *is* 1*-dimensional*.

(2) *For nontrivial* ψ*'s, the space* $\mathcal{L}_\psi(\pi, G/A)$ *is* 2*-dimensional if* π *is the Steinberg representation* π_1, *and* 1*-dimensional otherwise*.

The group C has $q+4$ A-double cosets. Except for $A\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}A = A$, $A\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}A$, each of the remaining $q+2$ double cosets $A\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}A$, $A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}A$ and $A\begin{pmatrix} 1 & \delta-c \\ 1 & 1-c \end{pmatrix}A$, $c \in \mathbb{F}$, is the disjoint union of q A-cosets. Write $A_c = A\begin{pmatrix} 1 & \delta-c \\ 1 & 1-c \end{pmatrix}A$ and $A_\infty = A\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}A$ for short. Among these, we rule out A_δ and A_∞ since they are contained in conjugates of the Borel subgroup of G, and A_1 and $A(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix})A$ since they are not their own inverse. The remaining symmetric double cosets $A_c, c \in \mathbb{F}\backslash\{1, \delta\}$ are of interest to us. For such c, we express A_c as the union of A-cosets:

$$A_c = \bigcup_{x \in \mathbb{F}^\times} \begin{pmatrix} x & x(\delta - c) \\ 1 & 1-c \end{pmatrix} A.$$

Our computations on the space $\mathcal{L}_{\psi^0}(G/A)$ will use the following facts repeatedly:

(i) $\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix} A = \begin{pmatrix} \frac{x(\delta-1)}{1-c} & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} A$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix} A = \begin{pmatrix} \frac{1-\delta}{x(\delta-c)} & -\frac{\delta}{x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} A$ for all $x \in \mathbb{F}^\times$,

(ii) $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix} A = \begin{pmatrix} \frac{x(\delta-1)}{(x-1)(x(\delta-c)-(1-c))} & -\frac{1}{1-x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} A$ for all $x \in \mathbb{F}^\times, x \neq 1, \frac{c-1}{c-\delta}$, $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \delta-c \\ 1 & 1-c \end{pmatrix} A = \begin{pmatrix} \frac{1}{\delta-1} & \frac{c-1}{\delta-1} \\ 0 & 1 \end{pmatrix} A$ and $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1-c}{\delta-c} & 1-c \\ 1 & 1-c \end{pmatrix} A = \begin{pmatrix} \frac{(1-c)(\delta-c)}{\delta-1} & \frac{\delta-c}{\delta-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A.$

For $\mu \neq 1$, $\mathcal{L}_{\psi^0}(\pi_\mu, G/A)$ is 1-dimensional generated by $\{f_\mu\}$ with $f_\mu((\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix})) = 1$, so it is an eigenspace of T_{A_c} with the eigenvalue

$$\lambda = (T_{A_c} f_\mu)\left(\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right) = \sum_{x \in \mathbb{F}^\times} f_\mu\left(\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix}\right)$$

$$= f_\mu\left(\begin{pmatrix} \frac{1}{\delta-1} & \frac{c-1}{\delta-1} \\ 0 & 1 \end{pmatrix}\right) + f_\mu\left(\begin{pmatrix} \frac{(1-c)(\delta-c)}{\delta-1} & \frac{\delta-c}{\delta-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

$$+ \sum_{\substack{x\in\mathbb{F}^\times \\ x\neq 1, \frac{1-c}{\delta-c}}} f_\mu\left(\begin{pmatrix} \frac{x(\delta-1)}{(x-1)(x(\delta-c)-(1-c))} & -\frac{1}{x-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right)$$

$$= \sum_{\substack{x\in\mathbb{F}^\times \\ x\neq 1, \frac{1-c}{\delta-c}}} \mu\left(\frac{x(\delta-1)}{(x-1)(x(\delta-c)-(1-c))}\right).$$

Therefore $|\lambda|\leqslant 2\sqrt{q}$ by Theorem 3 in Chapter 6 of [10]. If $\mu=1$, then $\mathcal{L}_{\psi^0}(\pi_1, G/A)\oplus \mathcal{L}_{\psi^0}(1\circ\det, G/A)$ is 3-dimensional, generated by $\{f_1, f_2, f_3\}$. For $i\in\{1,2,3\}$, we have

$$(T_{A_c} f_i)(UA) = \sum_{x\in\mathbb{F}^\times} f_i\left(\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix}\right)$$

$$= \sum_{x\in\mathbb{F}^\times} f_i\left(\begin{pmatrix} \frac{x(\delta-1)}{1-c} & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right),$$

$$(T_{A_c} f_i)\left(UA\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}A\right)$$

$$= \sum_{x\in\mathbb{F}^\times} f_i\left(\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-\delta \end{pmatrix}\right)$$

$$= f_i\left(\begin{pmatrix} \frac{1}{1-\delta} & \frac{c-1}{\delta-1} \\ 0 & 1 \end{pmatrix}\right) + f_i\left(\begin{pmatrix} \frac{(1-c)(\delta-c)}{\delta-1} & \frac{\delta-c}{\delta-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$$

$$+ \sum_{\substack{x\in\mathbb{F}^\times \\ a\neq 1, \frac{1-c}{\delta-c}}} f_i\left(\begin{pmatrix} \frac{x(\delta-1)}{(x-1)(x(\delta-c)-(1-c))} & -\frac{\delta}{x-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right),$$

$$(T_{A_c} f_i)\left(UA\begin{pmatrix} 0 & \delta \\ 1 & 0 \end{pmatrix}A\right) = \sum_{x\in\mathbb{F}^\times} f_i\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix}\right)$$

$$= \sum_{x\in\mathbb{F}^\times} f_i\left(\begin{pmatrix} \frac{\delta-1}{x(\delta-c)} & -\frac{1}{x} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\right).$$

This shows that

$$(T_{A_c} f_1) = f_2,\ (T_{A_c} f_2) = (q-1)f_1 + (q-3)f_2 + (q-1)f_3 \quad \text{and} \quad (T_{A_c} f_3) = f_2.$$

So with respect to the basis f_1, f_2, f_3, the operator T_{A_c} is represented by the matrix

$$\begin{pmatrix} 0 & q-1 & 0 \\ 1 & q-3 & 1 \\ 0 & q-1 & 0 \end{pmatrix}.$$

Clearly, 0 is an eigenvalue of T_{A_c}. Further, $\mathcal{L}_{\psi^0}(1 \circ \det, G/A)$ is a 1-dimensional eigenspace with the eigenvalue $q-1$. Hence 0, $q-1$ and -2 (from trace computation) are eigenvalues of T_{A_c} with the eigenfunctions $f_1 - f_3$, $f_1 + f_2 + f_3$ and $\frac{1-q}{2} f_1 + f_2 + \frac{1-q}{2} f_3$, respectively.

Now we turn to a nontrivial additive character ψ. Recall that $\mathcal{L}_\psi(\pi, G/A)$ is generated by $\{W_1\}$ for $\pi \neq \pi_1$ and by $\{W_1, W_{D_0}\}$ for $\pi = \pi_1$. We shall compute the actions of T_{A_c} on these two functions. Note that

$$\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1(1-c) & (1-\delta)^{-1} \\ 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} (1-\delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta-1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all } x \in \mathbb{F}^\times.$$

Using the Kirillov model of π given in §2, we compute first

$$\begin{aligned}
(T_{A_c}W_1)(g) &= \sum_{x\in\mathbb{F}^\times} W_1\left(g\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix}\right) = \sum_x \left[\pi\left(g\begin{pmatrix} x & x(\delta-c) \\ 1 & 1-c \end{pmatrix}\right)1\right](1) \\
&= \sum_x [\pi(g)\pi(h_x)\pi(u_1)\pi(w)\pi(u_{(1-c)(1-\delta)^{-1}})\pi(h_{(1-\delta)^{-1}})1](1) \\
&= (q-1)^{-2}q^{-1}\sum_{\beta\in\hat{\mathbb{F}}^\times} \beta((1-c)(1-\delta)^{-1})\Gamma(\beta^{-1},\psi)\varepsilon(\pi,\beta,\psi) \\
&\quad \times \sum_{\gamma\in\hat{\mathbb{F}}^\times} \Gamma(\beta^{-1}\gamma^{-1})\left[\pi(g)\sum_x \gamma(x)\gamma\right](1) \\
&\quad + (q-1)^{-1}\sum_{\beta\in\hat{\mathbb{F}}^\times} \beta((1-c)(1-\delta)^{-1})\Gamma(\beta^1,\psi)\left[\pi(g)\sum_x \pi(h_x)e(\beta)\right](1) \\
&= (q-1)^{-1}q^{-1}\sum_\beta \beta((1-c)(1-\delta)^{-1})\Gamma(\beta^{-1},\psi)^2\varepsilon(\pi,\beta,\psi)[\pi(g)1](1) \\
&\quad + (q-1)^{-1}\sum_\beta \beta((1-c)(1-\delta)^{-1})\Gamma(\beta^{-1},\psi)\left[\pi(g)\sum_x \pi(h_x)e(\beta)\right](1).
\end{aligned}$$

The sum $\sum_{x\in\mathbb{F}^\times}\pi(h_x)e(\beta) = -q^{-1}(q-1)\varepsilon(\pi,\beta,\psi)(q^2-1)\delta_{\beta,1}D_0$ if π is the Steinberg representation π_1, and 0 otherwise. In case π is the Steinberg representation π_1, we also need to find the action of T_{A_c} on W_{D_0}, which is, for $g\in G$,

$$
\begin{aligned}
(T_{A_c}W_{D_0})(g) &= \sum_{x\in\mathbb{F}^\times} W_{D_0}\left(g\begin{pmatrix} x & x(\delta-c)\\ 1 & 1-c\end{pmatrix}\right) = \sum_x \left[\pi\left(g\begin{pmatrix} x & x(\delta-c)\\ 0 & 1-c\end{pmatrix}\right)D_0\right](1)\\
&= \sum_x [\pi(g)\pi(h_r)\pi(u_1)\pi(w)\pi(u_{(1-c)(1-\delta)^{-1}})\pi(h_{(1-\delta)^{-1}})D_0](1)\\
&= \sum_x [\pi(g)\pi(h_{x^{-1}})\pi(u_1)\pi(w)\pi(u_{(1-c)(1-\delta)^{-1}})D_0](1)\\
&= \sum_x [\pi(g)\pi(h_{x^{-1}})\pi(u_1)[-q^{-1}\varepsilon(\pi_1,1,\psi)(1+D_0)](1)\\
&= -q^{-1}\sum_x \pi(g)\pi(h_{x^{-1}})\left[(q-1)^{-1}\sum_{\beta\in\hat{\mathbb{F}}^\times}\Gamma(\beta^{-1},\psi)\beta + D_0\right](1)\\
&= -q^{-1}\pi(g)\left[(q-1)^{-1}\sum_\beta \Gamma(\beta^{-1},\psi)\sum_x\beta(x)\beta+\sum_x D_0\right](1)\\
&= q^{-1}[\pi(g)1](1) - q^{-1}(q-1)[\pi(g)D_0](1)\\
&= q^{-1}W_1(g) - q^{-1}(q-1)W_{D_0}(g)
\end{aligned}
$$

since $\varepsilon(\pi_1,1,\psi)=1$.

To compute the eigenvalues and the eigenfunctions of T_{A_c}, we begin with the 2-dimensional space $\mathcal{L}_\psi(\pi_1, G/A)$ generated by $\{W_1, W_{D_0}\}$. The identities

$$
\varepsilon(\pi_1,\beta,\psi)=\Gamma(\beta,\psi)^2 \quad\text{and}\quad \Gamma(\beta,\psi)\Gamma(\beta^{-1},\psi)=\begin{cases}1 & \text{if } \beta=1;\\ q & \text{if } \beta\neq 1,\end{cases}
$$

yield the expression

$$
(T_{A_c}W_1)(g) = -(q-1)^{-1}q^{-1}(q^2-1)W_1(g) + q^{-1}(q^2-1)W_{D_0}(g).
$$

Hence with respect to the basis $\{W_1, W_{D_0}\}$, the operator T_{A_c} is represented by the matrix

$$
\begin{pmatrix} -\dfrac{q+1}{q} & \dfrac{1}{q}\\[2mm] \dfrac{q^2-1}{q} & -\dfrac{q-1}{q}\end{pmatrix}.
$$

Thus 0 and -2 are the eigenvalues of T_{A_c} with eigenfunctions $W_1 + (q+1)W_{D_0}$ and $(q-1)W_1 - W_{D_0}$, respectively.

Next we consider the case where $\pi \neq \pi_1$. Then $\mathcal{L}_\psi(\pi, G/A)$ is 1-dimensional, so it is an eigenspace generated by $\{W_1\}$ with the eigenvalue

$$\lambda = (q-1)^{-1} q^{-1} \sum_{\beta \in \hat{\mathbb{F}}^\times} \beta((1-c)(1-\delta)^{-1}) \Gamma(\beta^{-1}, \psi)^2 \varepsilon(\pi, \beta, \psi).$$

When π is a discrete series representation π_Λ, we have $\varepsilon(\pi, \beta, \psi) = -\Sigma_{z \in \mathbb{E}^\times} \Lambda(z)\beta(\mathrm{N}\, z)\psi(\mathrm{Tr}\, z)$ so that

$$\begin{aligned}
&\sum_{\beta \in \hat{\mathbb{F}}^\times} \beta((1-c)(1-\delta)^{-1}) \Gamma(\beta^{-1}, \psi)^2 \varepsilon(\pi, \beta, \psi) \\
&\quad = -\sum_{\beta \in \hat{\mathbb{F}}^\times} \beta((1-c)(1-\delta)^{-1}) \sum_{x \in \mathbb{F}^\times} \beta^{-1}(x)\psi(x) \sum_{y \in \mathbb{F}^\times} \beta^{-1}(y)\psi(y) \\
&\qquad \times \sum_{z \in \mathbb{E}^\times} \Lambda(z)\beta(\mathrm{N}\, z)\psi(\mathrm{Tr}\, z) \\
&\quad = -(q-1) \sum_{x \in \mathbb{F}^\times} \sum_{z \in \mathbb{E}^\times} \psi(z)\psi\left(\frac{(1-c)\mathrm{N}\, z}{(1-\delta)x}\right) \Lambda(z)\psi(\mathrm{Tr}\, z) \\
&\quad = -(q-1) \sum_{z \in \mathbb{E}^\times} \Lambda(z)\psi(\mathrm{Tr}\, z) \sum_{x \in \mathbb{F}^\times} \psi\left(x + \frac{(1-c)\mathrm{N}\, z}{(1-\delta)x}\right).
\end{aligned}$$

Thus $|\lambda| \leqslant (q-1)^{-1} q^{-1} (q-1) q (2\sqrt{q}) = 2\sqrt{q}$ Corollary 5 in Chapter 6 of [10].

When π is a principal series representation or a Steinberg representation π_μ with $\mu \neq 1$, we have $\varepsilon(\pi, \beta, \psi) = \Gamma(\mu\beta, \psi)\Gamma(\mu^{-1}\beta, \psi)$ so that

$$\begin{aligned}
&\sum_{\beta \in \hat{\mathbb{F}}^\times} \beta((1-c)(1-\delta)^{-1}) \Gamma(\beta^{-1}, \psi)^2 \varepsilon(\pi, \beta, \psi) \\
&\quad = \sum_{\beta \in \hat{\mathbb{F}}^\times} \beta((1-c)(1-\delta)^{-1}) \sum_{x \in \mathbb{F}^\times} \beta^{-1}(x)\psi(x) \sum_{y \in \mathbb{F}^\times} \beta^{-1}(y)\psi(y) \sum_{u \in \mathbb{F}^\times} \mu\beta(u)\psi(u) \\
&\qquad \times \sum_{z \in \mathbb{F}^\times} \mu^{-1}\beta(z)\psi(z)
\end{aligned}$$

$$= (q-1) \sum_{x,u,z \in \mathbb{F}^\times} \psi(x)\psi\left(\frac{uz(1-c)}{(1-\delta)x}\right)\mu(u)\psi(u)\mu^{-1}(z)\psi(z)$$

$$= (q-1) \sum_{u \in \mathbb{F}^\times} \mu(u)\psi(u) \sum_{z \in \mathbb{F}^\times} \mu^{-1}(z)\psi(z) \sum_{x \in \mathbb{F}^\times} \psi\left(x + \frac{uz(1-c)}{(1-\delta)x}\right).$$

Thus $|\lambda| \leqslant (q-1)^{-1}q^{-1}(q-1)\sqrt{q}\sqrt{q}(2\sqrt{q}) = 2\sqrt{q}$ by the same corollary.

We have found all eigenvalues and eigenfunctions of the operator T_{A_c} on the space $\mathcal{L}(G/A)$. The eigenvalues are the spectrum of the Cayley graph $\mathrm{Cay}(G/A, A_c/A)$; the estimates indicate that it is almost a Ramanujan graph. We record this in

Theorem 18. *For $c \neq 1, \delta$, all nontrivial eigenvalues of the Cayley graph $X_{A_c} = \mathrm{Cay}(G/A, A_c/A)$ have absolute value at most $2\sqrt{q}$.*

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