Generalized sketches as a framework for completeness theorems. Part I

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Abstract

The concept of sketch is generalized. Morphisms of finite (generalized) sketches are used as sketch-entailments. A semantics and a deductive calculus of sketch-entailments are developed. A General Completeness Theorem (GCT) shows that the deductive calculus is adequate for the semantics. In each of a number of categories of sketches, a particular set of sketch-entailments is singled out as a set of axioms used to specify a particular kind of structured category. The specification yields an adequate proof-system to derive sketch-entailments valid in structured categories of the given kind. Classical, Tarski-type semantics is related to the sketch-semantics of the paper. Specific completeness theorems are given in the sketch-based formalism, and they are related to representation theorems of categorical logic, and known completeness theorems of logic.

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0. Introduction

A completeness theorem asserts the equality of the formal deducibility relation and the semantic consequence relation in a particular logic. Algebraic Logic and, in particular, Categorical Logic replaces a completeness theorem by a representation theorem. Witness the examples of the Stone representation theorem for Boolean algebras, which is the algebraic equivalent of the (strong) completeness theorem for classical propositional logic; the representation theorems for cylindric [14] and polyadic [12] algebras that correspond similarly to the Gödel–Mal’cev completeness theorem for Classical First Order Logic (CFOL); or the categorical representation theorems, also to be mentioned below, similarly corresponding to the same completeness theorem and to other completeness theorems in logic. A common element in

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Research supported by NSERC Canada and FCAR Quebec. This is the first part of a three-part paper with the same title. The abstract, Sections 0–3 and the References section in the present first part serve all three parts. The numbering of the sections is consecutive throughout the three parts.
these correspondences is the fact that the algebraic and the logical formulations are translation-equivalent; given either the algebraic or the logical formulation, the other can be deduced by a formal argument that is a routine translation, all of whose elements are in view already when the algebraic version of the logic is introduced. In particular, it (the formal argument) can be carried out in a weak metatheory, not going, in strength, beyond a fragment of first order arithmetic.

Thus, categorical logic already has a framework for expressing completeness theorems. However, it can be legitimately argued that completeness-as-representation does not express the full intended meaning of completeness. The fact that truth is carried adequately by finite combinatorial objects, namely the formal deductions, has, apparently, disappeared in the statement of the representation theorem.

The Abstract Completeness Theorem (ACT) for CFOL, saying that the validities in CFOL form a recursively enumerable set, is an expression of the last-stated fact in an abstract form. Now, the ACT is not apparent on the form of either of the representation theorems related to CFOL as in fact it is on that of any of the usual variants of the completeness theorem. The fact remains that the ACT can be derived from representation; however, this derivation, going through the translation to completeness expressed in the usual syntax of CFOL, requires stepping out of the framework of categorical logic: apparently, categorical logic is not sufficient in itself for expressing the ACT.

To rectify this situation, what is needed is a syntax directly related to categorical logic, which provides an adequate notion of formal deduction. Such a syntax is described in this paper, not just for CFOL, but in a uniform way, for a whole family of doctrines, logics in a categorical formulation. As a result, we will obtain an analysis of the idea of completeness, applicable, again in a uniform manner, to several logics such as intuitionistic and modal logics, and generalized quantifiers. The question of whether the ACT is valid in a given situation becomes a precise question, provided the situation has been given a syntactic formulation as described here.

The syntax to be given is a natural codification of the idea of diagram manipulation, the informal idea of "syntax" for category theory. The syntax is based on a generalization of Ehresmann's concept of sketch [7, 11, 20, 10, 35].

Three things are done with the concept of sketch in this paper. One is a simple but sweeping generalization of the concept, the effect of which is that sketches become instrumental in essentially any (categorical) doctrine, not just ones based on limits and colimits as with the original, standard, notion of sketch. The second is the use of sketches in the specification of doctrines via sketch-axioms, which are morphisms of sketches. Finally, the third is the formulation of a proof-theory, specifically a notion of formal deduction. The new deductions deduce sketch-entailments, morphisms of sketches; the sketch-axioms are particular sketch-entailments. The new notion of deducibility serves to formulate a General Completeness Theorem (GCT), valid in any sketch-specified doctrine.

The "real", or Specific Completeness Theorems (SCTs) result from the combination of Representation Theorems (RTs) and the appropriate GCT. Each RT depends not just on the doctrine at hand, but also on the choice of the representing objects in the
doctrine. For instance, in the doctrine of Intuitionistic First Order Logic, we have the RT in which the representing objects are the presheaf categories (this is related to Kripke’s completeness theorem [19]), and another one in which the representing objects are the categories of sheaves over complete Heyting algebras (compare [32, 33]). One form of the RT related to the standard completeness theorem for CFOL refers to the doctrine of coherent categories, and Set, the category of sets, as the (single) representing object in it. Accordingly, a SCT refers to a doctrine as well as choice of semantical objects (as the “representing objects” can alternatively be called) in it; the GCT refers to a doctrine alone. Note that the role of representation theorems, the hallmark of Algebraic Logic, remains integral in the theory.

An Abstract Completeness Theorem (ACT) will express the property of a class of objects (sometimes a single object) within a doctrine, namely the fact that the exactness properties of the class for the operations codified in the doctrine form a recursively enumerable set. The ACT of this paper formulated for the categorical doctrine of first order logic, and the distinguished object the category of sets, is translation-equivalent to the usual ACT for CFOL.

There is an adequate expression of Compactness in this context as well.

Next, I give a more detailed account of the basic ideas of the paper, and their connections with the literature. The remarks that follow will perhaps be more easily understood after having seen the basic concepts in the first three sections.

According to Ehresmann, a sketch is a graph with certain cones and cocones distinguished in it. A simple-minded generalization is to allow arbitrary types of distinguished diagrams in place of cones and cocones. The intention is, as with Ehresmann, to associate certain definite qualities with the distinguished diagrams; in Ehresmann’s case these are “limit cone” and “colimit cocone”. In the case of Ehresmann’s sketches, this intention is realized by the fixing of the intended semantics, which stipulates that a model of a sketch, a graph map from the underlying category of the sketch into a category, should turn the distinguished cones and cocones into limit cones and colimit cocones, respectively.

The Ehresmannian semantics has an obvious and potentially vast extension to the generalized sketches, by associating essentially arbitrary qualities of diagrams in categories with the distinguished diagrams, the only restriction being that the quality associated with a diagram has to match the shape of the diagram. For example, in Cartesian closed categories, the operation of exponentiation involves a kind of diagram that contains all the data for any particular exponential, in much the same way as a limit cone contains the necessary data for a particular instance of limit; since a product is involved in the exponential, the data for that product are part of the diagram. Thus, we have a corresponding kind of (generalized) sketch for Cartesian closed categories. In this kind of semantic determination, it is not necessary to limit ourselves to qualities given by universal properties; the target categories may need to have additional structure to support the semantics.

This paper adopts the above-described simple-minded extension of the notion of sketch. However, it restricts the possible semantics to a certain “internal” form, in
contrast to the "purely external", and in principle arbitrary, determination in Ehresmann's style of the qualities talked about above.

Lair's *trames* (webs) [21] are generalized sketches, for the purposes of dealing with categorical operations defined by universal properties. These "adjoint operations", Lawvere's most important contribution to the foundational applications of category theory, are defined in a hierarchical fashion, one operation serving as a basis for another's adjoint definition. An example is the above-mentioned exponentiation, which uses binary product in its definition. Lair's *trames* are explicitly built in a hierarchical fashion conforming to the character of the operations they intend to specify. They represent a substantial generalization of the notion of sketch, sufficient for the specification of all adjoint operations. In relation to the (generalized) sketches of this paper, they are "special purpose" sketches. They have a specific form, among others with a negative part and a positive part distinguished, corresponding to "left-adjoint" and "right-adjoint" type operations. The present paper's generalized sketches are more general and more simple-minded; once again, in this paper the weight of the responsibility for specificity is placed on sketch-semantics in a new sense, rather than on the notion of sketch.

In the examples, this paper also adopts the hierarchical aspect of sketch-building, inasmuch sketches for Cartesian closed categories in this paper are based on sketches for Cartesian categories. However, in the general concept there is no need to carry along the hierarchical character.

The semantics of *trames* is of the same "external" kind as that of Ehresmann's sketches.

Another notion of generalized sketch is Wells' *forms* [34]. A form is derived from a pre-supposed specification of any particular doctrine (kind of structured category); this specification is assumed to be in the form of a finite limit sketch. Thus, if one has a finite limit sketch whose models are Cartesian closed categories (which does exist for the notion of Cartesian closed category with specified operations and morphisms preserving the operations on the nose), then Wells' theory gives a notion of "Cartesian closed form"; any one Cartesian closed form specifies (is the presentation of) a particular Cartesian closed category.

The chief difference between the sketches of this paper and the forms of Wells is that the former arise directly from the common definition of the doctrine considered, whereas the second needs the mediation of the finite-limit specification of the doctrine. Wells' forms use the language of finite-limit sketches as a specification language; this paper's sketches use a direct, and simple, specification language. Another difference is that the Wells forms apply only to doctrines that are finite-limit specifiable. The doctrine of Cartesian closed categories, with morphisms preserving the operations only up to isomorphism, is not finite-limit specifiable. The present paper's approach is not affected by this fact, and it applies equally well to the two versions of the doctrine of Cartesian closed categories.

As already indicated, the main idea of the paper is a sketch-semantics in a new sense. This is given by the well-known notion of relative injectivity (an object injective relative to an arrow, or a set of arrows; see [1]) in an arbitrary category. The category
in this case is the category of all sketches of a certain particular kind. Thus, we are talking about a sketch \( S \) satisfying an arrow \( r \) of sketches, the latter pictured as an entailment, in the sense of "the domain situation entailing the codomain situation". The definition is that \( S \models r \) iff all arrows from the domain of \( r \) to \( S \) factor through \( r \); that is, iff \( S \) is injective relative to \( r \).

The novelty is certainly not in the general form of the semantics, which is injectivity. A strong form of injectivity, orthogonality ("factors through a unique arrow" above) appears already in [9] as a tool of semantics; e.g., the sheaves with respect to a given topology are selected by orthogonality conditions from among the presheaves. For a history of orthogonality and injectivity, see the historical remarks to Chs. I and 4 in [1]. As another example, Andrëka and Németi [2] use cone-injectivity (a generalized form of injectivity) as a formal tool for semantics, in their case explicitly for the purposes of first order logic. However, Andrëka's and Németi's application of injectivity and the ones in this paper take place in entirely different categories. In the Andrëka–Németi case, the application takes place in a category containing the category of models of a first order theory as a subcategory. In this paper, it takes place in the category of sketches. Note that sketches correspond to theories in the conceptual framework of logic. Thus, we have a semantics, where theories themselves are being models of statements that are morphisms of sketches, that is, morphisms of theories.

Sketch-axioms (sketch-entailments used as axioms) are used to specify a doctrine in the form of a full subcategory of sketches satisfying the sketch-axioms. Thus the structured categories that are members of the doctrine appear as sketches and they are selected by the axioms. Semantics in Ehresmann's sense, the concept of a model of a sketch, is recaptured simply in the form of a morphism of the sketch into a member of the doctrine.

I am not aware of the existence in the literature of an anticipation of the idea of using the category of sketches in the manner just described.

As was said above, sketch-semantics represents an internalization, and a consequent restriction, of a potentially arbitrary type of external sketch-semantics. It is rather clear that arbitrary external sketch-semantics as was indicated above will not be capable of being formulated in the sketch-semantics of this paper. The bulk of the paper consists in the demonstration, in individual examples and classes of examples, that, on the other hand, sketch-semantics is adequate for apparently all of the usual categorical doctrines, including ones that involve operations not defined by universal properties, such as monoidal categories. The fact that the sketch-semantic specifications are available for the doctrines in question is not entirely immediate. For example, the necessary presence of the rule (sketch-entailment) of the form \( \text{IsoTrans} \) for the specification of adjoint operations (see Section 4) was a surprise to me, and the treatment of monoidal categories (with not necessarily strict monoidal functors as morphisms!) and similar doctrines requires a new, mathematically interesting way of looking at these doctrines.

As was described in the first part of this introduction, the main contribution of the paper is a syntactic calculus, providing an adequate (sound and complete) formalization of sketch-semantics.
Although I had the basic ideas of the syntax of sketches for some years, the impetus
to work them out was provided by Charles Wells's talk at the October 1992 Montreal
Category Meeting about a sketch-theoretic approach to logic aiming at proof theory.
An extended abstract [4] of the projected work "Graph-based logic and logic" by him
and Bagchi was made available on the Category Theory Network. It seems to me that
the approach of the present paper is substantially different from that of Bagchi and
Wells, which is based on Wells' forms mentioned above.

The proof of the General Completeness Theorem, expressing the adequacy of the
deductive calculus, combines the classical construction of enough injectives in a
general setting, with a finiteness argument that exploits the specifics of construction.
The construction itself is, needless to say, far from being new. For example, it appears
in Barr [4a], where the author refers to Grothendieck's construction of enough
injectives in AB5 categories as the source for his own proof.

The technical aspects of this paper are rather minor. On the other hand, I am struck
by the conceptual importance of the scheme, which shows, to me at least, that
categorical logic is, to a great degree, autonomous, even in matters syntactical.
I believe that in the literature there is no general theory of completeness in logic (not
just categorical logic) comparable to the present one in scope and conceptual unity.

The paper [25] is a summary of related material. The present paper is a working out
of certain of the details of Sections 1, 2, 3 and 5 [25]. Ref. [25] contains ideas for work
in proof theory proper in the context of the sketch-based syntax. Another paper [26]
is in preparation, which, within the framework of the present one, gives an algebraic
theory of doctrines, concentrating on their bicategorical aspects.

When item \( m \) in Section \( n \) is referred to in another section, I use the notation \( n.m \).

1. Categories of sketches

Let \( G \) be a category, and \( K = \langle K \rangle_{K \in |K|} \) a (small) indexed set of objects \( K \) of \( G \). We
define a new category \( G|K \), the category of sketches over \( G \) with specification names the
elements of \( |K| \), and specification types the objects \( K \) of \( G \) for \( K \in |K| \). The objects of
\( G|K \) are entities of the form \( S = \langle |S|, \langle K[S] \rangle_{K \in |K|} \rangle \) with \( |S| \in \text{Ob}(G) \), and
\( K[S] \subseteq \text{hom}(\bar{K},|S|) \) \((K \in |K|)\). In other words, a sketch \( S \) is given by an underlying
\( G \)-object \( |S| \), and, for each specification name \( K \in |K| \), by a set \( K[S] \) of \( K \)-specifications,
each \( K \)-specification being a morphism \( \bar{K} \to |S| \) in \( G \). An arrow \( \varphi : S \to T \) in \( G|K \)
is an arrow \( |\varphi| : |S| \to |T| \) such that, for all \( K \in |K| \), \( s \in K[S] \) we have \( |\varphi| \cdot s \in K[T] \);
in other words, a map of sketches is a morphism on the underlying level that
transforms specifications into specifications. Composition of arrows in \( G|K \) is as in \( G \).

In our examples, we will always have that the map \( K \mapsto \bar{K} \) is an identity; each
specification type will be its own name. In this case, \( K \) is given by the set \( |K| \) of objects
of \( G \), and we write \( K \) for this set \( |K| \); now, the construction gives a category \( G|K \) on the
basis of a category \( G \) and a set \( K \) of objects of \( G \). The reason for the general definition
as it is given is to allow for the theoretical possibility of using the same specification
type with more than one name. For example, if \( G \) is the category of sets, and we want to use the empty set as a specification type with two different names, we have to use the first-stated definition with \( K \to \overline{K} \) not 1–1. However, with the same \( G \), if we avoid the empty set as a specification type, we can always find enough distinct isomorphic copies of prospective specification types to be able to use them as their own names. We will take advantage of the latter possibility in our examples, for underlying categories \( G \) different from, but related to, the category of sets.

The reader familiar with Ehresmann's sketches [7] will immediately see how they arise as an example of the general construction; in particular, \( G \) in that case is the category of graphs. To aid the reader's intuition, I now give an even simpler example, "sketches to specify categories". Sections 4, 5, 6 and 7 are devoted to the systematic presentation of many other examples.

*Graphs* are directed graphs. A (small) graph is the same as an object of the category \( \text{Set}^g \) (\( \text{Set} \) is the category of small sets and functions), with \( g \) the graph \( A \xrightarrow{d} O \); for \( G \in \text{Set}^g \), \( \text{Ob}(G) = G(O) = \) the set of objects of \( G \), \( \text{Arr}(G) = G(A) = \) the set of arrows of \( G \); for \( a \in \text{Arr}(G) \), \( d(a) \) and \( c(a) \) are the domain and codomain of \( a \), respectively.

We single out the following two specific graphs \( I \) and \( \text{CT} \) as specification types:

[Diagram: \( I \): \( \langle 0,0 \rangle \rightarrow^1 \langle 0,0 \rangle \) ("identity")

[Diagram: \( \text{CT} \): \( \langle 0,1 \rangle \rightarrow^1 \langle 1,2 \rangle \), \( \langle 0,1 \rangle \rightarrow^2 \langle 0,2 \rangle \) ("commutative triangle")

\( I \) has one object 0, and one arrow \( \langle 0, 0 \rangle : 0 \to 0 \). \( \text{CT} \) has three objects and three arrows as shown. Let us stress that \( I \) and \( \text{CT} \) are given individually, and not just up to isomorphism. They will serve as *names* for copies of themselves in other graphs, and have to be fixed absolutely.

The general definition of \( G|K \) is applied with \( G = \text{Graph} = \text{Set}^g \), the category of (small) graphs, and \( K = \{ I, \text{CT} \} \). The resulting category, \( \text{cSk} \), is called the category of *category-sketches*, or c-sketches for short for their role (to be described in the next section) in specifying categories. Spelled out, the definition is as follows. A *category-sketch* (c-skeleton) \( S \) is a graph \( |S| \), together with a set \( I[S] \) of graph-maps \( I \to |S| \) (the *identity specifications* in \( S \)) and a set \( \text{CT}[S] \) of graph-maps \( \text{CT} \to |S| \) (the *commutative
triangle specifications in $S)$. A map $\varphi : S \to T$ of $c$-sketches is a graph-map $|\varphi| : |S| \to |T|$ with the property that for any $i \in I[S]$, $\varphi \circ i \in I[T]$ and for any $\tau \in CT[S]$, $\varphi \circ \tau \in CT[T]$; a map of sketches takes an identity-specification or a commutative triangle specification in the domain into an entity of the same kind in the codomain.

We also consider a variant of the construction $G \downarrow K$. Given $G$ and $K = \langle K \rangle_{K \in |K|}$ as before, we define $G \parallel K$ to be the following category. An object $S$ of $G \parallel K$ is given by an object $|S|$ of $G$, and for each $K \in |K|$, an abstract set $K[S]$ and a $K[S]$-indexed family $\langle s \rangle_{s \in K[S]}$ of morphisms $s : K \to |S|$. A morphism $\varphi : S \to T$ of $G \parallel K$ is given by a morphism $|\varphi| : |S| \to |T|$ and functions $K[\varphi] : K[S] \to K[T]$ $(K \in |K|)$ such that for all $K \in |K|$ and for all $s \in K[S]$, we have $(K[\varphi](s))^{-} = |\varphi| \circ s$; diagramatically,

\[
\begin{array}{ccc}
K[S] & \xrightarrow{(\gamma)} & G(\overline{K}, |S|) \\
\downarrow \varphi \mid & \circ & \downarrow |\varphi| \circ (\gamma) \\
K[T] & \xrightarrow{(\gamma)} & G(\overline{K}, |T|)
\end{array}
\]

(a circle in a diagram signifies commutativity). Composition of morphisms in $G \parallel K$ is defined componentwise. Let us add that the identity morphisms have all their components identities.

Once again, in our examples, the naming-function $K \mapsto \overline{K}$ will be the identity; we have the category $G \parallel K$ for a set $K$ of objects of $G$.

$G \mid K$ is, essentially, the part of $G \parallel K$ in which each indexing $s \to \overline{s}$ above is 1–1. Indeed, the obvious inclusion $G \mid K \to G \parallel K$ is full and faithful, and its essential image consists of the objects of $G \parallel K$ with the said property of injectivity.

The second, more complicated, construction $G \parallel K$ is more basic than the first one, $G \mid K$. For one thing, the construction $G \parallel K$ is enough; the uses of the $G \mid K$ can be replaced by uses of the $G \parallel K$, but not (apparently) vice versa. For another, the categories obtained by the construction $G \parallel K$ are better behaved than the ones obtained by the other construction. (These remarks will be expanded on below). Nevertheless, the simpler construction $G \mid K$ will be the one used more frequently.

The two constructions will be employed to produce a large class of sketch-categories, which in turn will be used for the specification of kinds of structured categories. The constructions will be used iteratively; we start with a simple category $G$ (most commonly, Graph), obtain $G \mid K$ (less frequently, $G \parallel K$) for a suitable $K \in \text{Ob}(G)$, then use $G \mid K$ (or $G \parallel K$) as the category $G'$ to construct a further sketch-category $G' \mid K'$ (or $G' \parallel K'$), and possibly iterate the procedure.

As we will see, the structure, especially the colimit structure, of the sketch-categories will be important. In fact, all our sketch-categories will be very simple, familiar kinds of categories, with well-understood colimit structure, among other things. Specifically, if we only use the construction $G \parallel K$, and we start (as we do in the examples) with simple enough "base" categories, as sketch-categories we get, up to isomorphism, functor categories of the form $\text{Set}^X$. If certain finiteness requirements in the data are
met (as they are in almost all examples), here the category $X$ is finite. When we also use the construction $G|K$, we obtain reflective subcategories with special properties of categories of the form $\text{Set}^X$. In what follows, we elaborate on these remarks.

A presheaf topos is a category of the form $\text{Set}^X$, with $X$ any small category; a finite topos (the terminology is that of [3]) is any $\text{Set}^X$ where $X$ is a finite category (having finitely many objects and arrows). Every presheaf topos is a locally finitely presentable (lfp) category (for concepts in the Gabriel–Ulmer theory of lfp categories, see [9, 1]).

Note, to begin with, that the category $\text{Graph} = \text{Set}^g$ is a finite topos; the graph $g$ here can be replaced by the category it freely generates, which has only identity arrows in addition to those of $g$. Similarly, the category of 2-graphs is a finite topos: $2\text{-Graph} = \text{Set}^{c_2}$, where $c_2$ is the category

$$2 \xrightarrow{d'\, c'} 1 \xrightarrow{d\, c} 0; \quad dd' = dc', \quad cd' = cc'$$

(only the non-identity arrows are shown). 2-Graph gives rise to a sketch-category that can be used to specify the concept of 2-category (in the sense of sketch-semantics, see Section 2). We can make similar remarks on $n$-graphs and $n$-categories in general.

In a finite topos $\text{Set}^X$, the finitely presentable (fp) objects are exactly the finite ones, that is, the functors $G:X \to \text{Set}$ for which $G(X)$ is finite for all $X \in X$. The reason is as follows. Regardless of whether $X$ is finite, if $G:X \to \text{Set}$, and $\langle (X_i, x_i \in GX_i) \rangle_{i \in I}$ is a family of “elements” of $G$, then the subfunctor $F$ of $G$ generated by these elements has $F(Y) = \{(Gf)(x_i) : i \in I, (f:X_i \to Y) \in \text{Arr}(X)\}$ for all $Y \in \text{Ob}(X)$; if $X$ and $I$ are finite, $F$ is finite (independently of $G$). It is clear that every fp functor is finitely generated (that is, it is as $F$ above with $I$ finite); in fact, this would hold in a much greater generality. Thus, in a finite topos, every fp object is finite; the converse is fairly clear.

More generally, for any lfp category $A$, if $X$ is a finite category, we have

$$\text{fp}^A = (\text{fp}^X)^X.$$

I claim that

1. If $G$ is a presheaf topos, then $G\|K$ is isomorphic to a presheaf topos; if $G$ is a finite topos, $|K|$ is a finite set, and each $K_i (K \in |K|)$ is a finite object in $G$, then $G\|K$ is isomorphic to a finite topos.

Proof. Let $G = \text{Set}^X$, $K = \langle K_i \rangle_{K \in |K|}$ an indexed set of objects of $G$. We construct the following category $X^K$. We let

$$\text{Ob}(X^K) \overset{\text{def}}{=} \text{Ob}(X) \sqcup |K|$$

(for simplicity, we assume that $\text{Ob}(X)$ and $|K|$ are disjoint sets, $\text{Ob}(X^K) \overset{\text{def}}{=} \text{Ob}(X) \sqcup |K|$). The arrows of $X^K$ are as follows. The arrows $X \to Y$ in $X^K$ for $X, Y \in \text{Ob}(X)$ are those in $X$. For $K \in |K|$ and $X \in \text{Ob}(X)$, the arrows $K \to X$ in $X^K$ are the triples $\langle K, X, a \rangle$, with $a \in KX$. The only arrows with domain and codomain both in $|K| \subset \text{Ob}(X^K)$ are identities. There are no arrows of the form $X \to K$ for $K \in |K|, X \in \text{Ob}(X)$. 
Composition in $X^*K$ is given as follows. The arrows between objects in $\text{Ob}(X) \subseteq \text{Ob}(X^*K)$ are composed as in $X$. For composition of the form $K \to X \to Y$ ($X, Y \in \text{Ob}(X)$, $K \in |K|$), we have

$$(f: X \to Y) \circ \langle K, \alpha \rangle \overset{\text{def}}{=} \langle K, Y, K(f)(\alpha) \rangle.$$ (2)

All other compositions involve identities, and thus are obvious. The associativity of the composition in $X^*K$ is a consequence of the same fact for $X$, and the functoriality of the $K: X \to \text{Set}$, $K \in |K|$.

If $X, |K|$ and the functors $\bar{K}$ ($K \in |K|$) are all finite, then $X^*K$ is a finite category. With $\text{Set}^X$, the category $G || K$ is isomorphic (not just equivalent) to $\text{Set}^{X*K}$. The isomorphism

$$(\hat{\circ}): G || K \to \text{Set}^{X*K}$$ (3)

sends $S \in \text{Ob}(G || K)$ to $\hat{S}: X^*K \to \text{Set}$ where $\hat{S}(X) = |S|(X)$ ($X \in \text{Ob}(X)$), $\hat{S}(K) = K[S]$ ($K \in |K|$); for $f: X \to Y$, $\hat{S}(f) = |S|(f)$; and $\hat{S}(\langle K, X, \alpha \rangle): \hat{S}(K) \to \hat{S}(X)$, that is $\hat{S}(\langle K, X, \alpha \rangle): K[S] \to |S|(X)$, is the mapping $s \mapsto \delta_X(\alpha)$. The fact that $\hat{S}$ is a functor, in particular that it preserves the instance (2) of composition, is a consequence of the naturality of $S: R \to |S|$.

Informally, but clearly, the information contained in $\hat{S}$ is precisely that contained in $S$; the functor $|S|$ is the restriction of $\hat{S}$ to the full subcategory on the objects in $X$, and the map $\delta_X: K(X) \to |S|(X)$ is the same as $a \mapsto \delta_X(\langle K, X, a \rangle)(s)$.

The functor (3) sends $\phi : S \to T$ to $\hat{\phi} : \hat{S} \to \hat{T}$ for which $\hat{\phi}_X = \phi_X$ and $\hat{\phi}_K = K[\phi]$; the naturality of $\hat{\phi}$ with respect to the arrow $f: X \to Y$ is the naturality of $\phi$ with respect to the same $f$, and the naturality of $\hat{\phi}$ with respect to the arrow $\langle K, X, a \rangle : \bar{K} \to X$ is the condition $(|\phi|, \langle K[\phi] \rangle)_{\bar{K} \in |K|})$ has to satisfy to give rise to the arrow $\phi : S \to T$. The rest of the verification that (3) is an isomorphism of categories is left to the reader. $\square$

As I said above, the construction $G || K$ would suffice, at the expense of a slight added complexity of the specifications. According to the last-proved proposition, all the sketch-categories obtained by repeated use of the $G || K$ construction, based on starting categories that are finite toposes, and using finite $K$'s, are finite toposes. In any topos, coproducts are disjoint sums; colimits in general can be calculated in special ways; in presheaf toposes, colimits (and limits) are computed pointwise in Set.

The following easy propositions refer to more general situations. $G$ and $K$ are as in the first sentence of the section, otherwise arbitrary.

(4) The collection of functors $\{ | : G || K \to G \} \cup \{ [ ] : G || K \to \text{Set}; K \in |K| \}$ create limits and colimits in $G || K$.

(5) The inclusion $i \in G[K] \to G || K$ has a left adjoint $\delta$ for which $|\delta(S)| = |S|$ and $K[\delta(S)] = [\delta : S \in K[S]]$; $G || K$ is a reflective subcategory of $G || K$.

(6) $i$ is powerful: it induces bijections $\text{Sub}_{G[K]}(S) \to \text{Sub}_{G || K}(iS)$ on the subobject lattices. $\delta$ preserves monomorphisms. The unit components $\eta_S: S \to \delta(S)$ are regular epimorphisms (in $G || K$; $G[K]$ is regular-epi reflective in $G || K$).
Assume that \( G \) is an lfp category, and for every \( K \in |K|, \bar{K} \) is an fp object of \( G \) (in this case, we call the system \( K = \langle \bar{K} \rangle_{K \in |K|}, \) finitary). Then \( G || K \) is an lfp category, and an object \( S \) of \( G || K \) is fp iff \(|S|\) is fp in \( G \), each specification-set \( K[S] \) (\( K \in |K| \)) is finite, and for all but finitely many \( K \in |K|, K[S] \) is empty. (When in addition \(|K|\) is finite, in which case we call the system \( K \) finite, the last clause is superfluous.)

**Proof of (7).** By (4), if the category \( G \) has (small) colimits, then so does \( G || K \), and the forgetful functor \( |: G || K \rightarrow G \) and the functors \( K[ ]: G || K \rightarrow \text{Set} \) preserve them.

We will show that \( G || K \) is \( \aleph_0 \)-accessible (see \([1,28]\)); since by the above, \( G || K \) is cocomplete, it will follow that \( G || K \) is lfp.

Let us call \( S \in G || K \) finite if \(|S|\) is fp, each \( K[S] \) (\( K \in |\mathcal{K}| \)) is a finite set, and all but finitely many \( K[S] \) (\( K \in |\mathcal{K}| \)) are empty. Using what filtered colimits are in \( G || K \), it is easy to check that every finite object is fp. Next, I claim that every object of \( G || K \) is a filtered colimit of finite objects. In fact, writing \( A \) for \( G || K \), and \( A_f \) for the full subcategory of \( A \) consisting of the finite objects, we can show that the canonical diagram

\[
(\varphi: A \rightarrow S) \mapsto A_f \downarrow S \rightarrow A
\]

is filtered, and has \( S \) as a colimit, with coprojections \( \ell_\varphi = \varphi \).

To see that \( A_f \downarrow S \) is a filtered category, let \( \mathcal{A} = \langle x_i: A_i \rightarrow S, \alpha_i: A_i \rightarrow A_j \rangle_{i:j} \) be a finite diagram in \( A_f \downarrow S \). Since \(|S|\) is a canonical filtered colimit of fp objects in \( G \), there is \( f: U \rightarrow |S| \) in \( G \), with \( U \in G \) fp, such that for all \( i \), we have a commutative diagram

\[
\begin{array}{ccc}
|A_i| & \xrightarrow{b_i} & U \\
|S| & \xrightarrow{f} & \\
\end{array}
\]

and such that \( \langle b_i \rangle_i \) provides a cocone on the diagram \( |\mathcal{A}| \), \( \mathcal{A} \) composed with the forgetful functor \( G || K \rightarrow G \). For each \( i, K \in |K| \) and \( s \in K[A_i] \), and for \( s_i \overset{\text{def}}{=} K[x_i](s) \), we have \( s_i: \bar{K} \rightarrow |S| \). Since there are altogether finitely many \( s_i \)'s (\( \bigcup_{i,K} K[A_i] \) is a finite set), and each \( \bar{K} \) is fp, there is a factorization \( f = hg \) with \( g: U \rightarrow V \), fp, such that each \( s_i \) factors as \( s_i = g\tilde{s}_i \) for a suitable \( \tilde{s}_i: \bar{K} \rightarrow V \). Further, we may choose \( g: U \rightarrow V \) (by composing the given \( g \) with a suitable \( V \rightarrow V' \)) so that, in addition, for every \( i: i \rightarrow j \), and every \( K \in |K|, s \in G[A_i] \), we have \( (K[x_i](s))_j = \tilde{s}_i \).

Define \( B \in G || K \) by \(|B| = V \), \( K[B] = \{\tilde{s}_i: i \in I, s \in K[A_i]\} \), and \( (s_i) \overset{\text{def}}{=} \bar{K} \rightarrow |B| \) as \( \tilde{s}_i \). Define \( \psi: B \rightarrow S \) by \( \psi = h \), each \( K[B] \rightarrow K[S] \) as an inclusion. Define \( \beta_i: A_i \rightarrow B \) by \(|\beta_i| = gh_i \), each \( K[A_i] \rightarrow K[B] \) as \( s \mapsto \tilde{s}_i \). Then we have

\[
\begin{array}{ccc}
A_i & \xrightarrow{\beta_i} & B \\
|S| & \xrightarrow{\psi} & \\
\end{array}
\]

and \( \langle \beta_i \rangle_i \) gives a cocone on \( \mathcal{A} \) in \( A_f \downarrow S \) as desired.

Having given a flavor of the arguments involved, I leave the verification that \( S \) is canonically the colimit of (8) to the reader. \( \square \)
What we know now contains the information that $G/K$ is lfp; also that every fp object is a retract of a finite one; since the retract of an fp object of $G$ is fp, it is immediate that a retract of a finite object of $G/K$ is finite.

(9) Under the same hypotheses as in (7), the inclusion $i: G/K \to G \| K$ preserves filtered colimits; thus, by (5), $G/K$ is lfp as well. Moreover, the fp objects of $G/K$ coincide with the fp objects of $G \| K$ that are in $G/K$.

Proof of (9). By (5), or seen directly, $G/K$ has colimits; for $S = \text{colim}_{i \in I} S_i$ we have $|S| = \text{colim}_{i \in I} |S_i|$ as before, and $K[S] = \{ \phi: g \in I (g: \tilde{K} \to |S_i|) \in K[S_i] \}$, where the $\phi_i: |S_i| \to |S|$ are colimit coprojections. It is not hard to check that the inclusion $i: G/K \to G \| K$ preserves filtered colimits; note however that this needs the hypothesis that $K$ is finitary. As a reflective subcategory of an lfp category, with the inclusion preserving filtered colimits, $G/K$ is lfp. Moreover, the fp objects of $G/K$ are retracts of the ones of the form $\delta(A) (\delta$ from (5)), $A$ fp in $G/K$. However, any subobject of a finite object of $G/K$ is finite; the desired conclusion concerning the fp objects of $G/K$ follows. \qed

Let us call a category $G$ Noetherian if it is lfp, the product of two fp objects is fp, and every subobject of an fp object is fp. It is clear that finite toposes are Noetherian. The characterization of fp objects in (7) shows that if $G$ is Noetherian, and $K$ is finitary, then $G/K$ and $G\| K$ are Noetherian as well.

Another property being inherited from $G$ to $G/K$ and $G\| K$ is being a regular category; if $G/K$ is regular, so are $G\| K$ and the functor $i$; this easily follows from (4), (5) and (6).

One notices that in a regular Noetherian category, any quotient of an fp object is fp, and that every object is the (canonical) filtered colimit of its fp subobjects.

Finally, we observe that if we have

$$G \xleftarrow{i} G' \xrightarrow{p} G'/i,$$

with $i$ full, powerful and faithful and $p \dashv i$, then, for any system $K = \langle \tilde{K} \rangle_{K \in [K]}$ of objects of $G$, and for $iK = \langle (i\tilde{K}) \rangle_{K \in [K]}$, we have functors

$$G/K \xleftarrow{i'} G'/i,$$

such that $p' \dashv i'$, and $i'$ is full, powerful and faithful. In fact, when $i$ is an inclusion, then so is $i'$; and $|p'(S)| = p(|S|)$, $K[p'(S)] = K[S]$, and $\tilde{s}^{(p')}(S): G \to |p'(S)| = \eta_S \cdot \tilde{s}^{(S)}$, for $\tilde{s}^{(S)}: K \to |S|$ and $\eta_S$ the unit of the adjunction $p \dashv i$. Also, if $p$ preserves monomorphisms, so does $p'$, and if $p \dashv i$ is a regular-epi reflection, so is $p' \dashv i'$.

By a finite (resp., finitary) sketch-category, we mean a category obtained by starting with a finite topos (resp., presheaf topos), and applying the constructions $G/K, G\| K$...
iteratively in a finite number of times, always with a finite (resp., finitary) system $K$. We conclude

(10) Every finitary sketch-category $S$ is regular and Noetherian, and it is a full regular-epi-reflective subcategory of a presheaf topos $E$ such that the inclusion is powerful and preserves filtered colimits, the reflector preserves monomorphisms, and the finite objects of $S$ are the finite objects of $E$ that lie in $S$. If $S$ is a finite sketch-category, $E$ can be chosen, in addition, to be a finite topos.

Finally in this section, I will describe a further analysis of the construction of sketch-categories that was kindly communicated to me by F.W. Lawvere.

First of all, Lawvere points out that the foregoing constructions form a special case of a more general situation. Given any adjoint pair

$$
\begin{array}{c}
\text{C} \\
\text{D}, \\
\text{F} \\
\text{G}
\end{array}
\text{G} \downarrow \text{D}, \text{F} \downarrow \text{G}
$$

of functors, we have that the comma-categories $\text{F} \downarrow \text{Id}_\text{D}$, $\text{Id}_\text{C} \downarrow \text{G}$ are isomorphic; in fact an adjunction between $F$ and $G$ is the same thing as an isomorphism of these categories making the diagram

$$
\begin{array}{c}
\text{F} \downarrow \text{Id}_\text{D} \\
\text{C} \times \text{D} \\
\text{Id}_\text{C} \downarrow \text{G}
\end{array}
$$

containing the forgetful functors commute. The category $\text{F} \downarrow \text{Id}_\text{D}$, equivalently $\text{Id}_\text{C} \downarrow \text{G}$, is called the adjunction category for the given adjunction. Lawvere observes that

(11) The categories $\text{G} \parallel \text{K}$ are precisely the adjunction categories for adjunctions

$$
\text{Set}^{[K]} \xleftarrow{\text{F}} \text{G}^\text{op} \text{G}^\text{op}
$$

with a discrete exponent $[K]$ on the left side.

(Here, we are assuming the $G$ has colimits; the functors $\text{Set}^{[K]} \rightarrow G$ having a right adjoint are precisely the left Kan extensions of functors $[K]^{\text{op}} \rightarrow G$.) In fact, with a given $K = \langle K \rangle_{K \in [K]}$ as before, the functor $F$ is the left Kan extension (or, unique colimit preserving extension) of the functor $K \mapsto \bar{K}$ from the discrete category $[K]^{\text{op}} = [K]$ to $G$ along the Yoneda functor $[K] \rightarrow \text{Set}^{[K]}$. Indeed, an object $S$ of the adjunction category is a triple $\langle \phi, [S], \psi : F\phi \rightarrow [S] \rangle$; here, we may write $K \mapsto K[S]$ ($K \in [K]$) for $\phi$; then $\phi$ can be written in the form $\bigsqcup_{K \in [K]} K[S] \cdot [K]$; accordingly, $F\phi$ is $\bigsqcup_{K \in [K]} K[S] \cdot \bar{K}$; and an arrow $F\phi \rightarrow [S]$ is a family $\langle s \mapsto s \cdot K[S] \rightarrow \text{hom}(\bar{K}, [S]) \rangle_{K \in [K]}$, which is precisely the data for a sketch $S \in G \parallel K$. One sees that the notions of morphism also correspond to each other.
Next, Lawvere points out that

(12) The adjunction category for an adjunction between presheaf toposes is isomorphic to a presheaf topos.

Here is the construction for (12). Consider an adjunction

\[
\text{Set}^A \xrightarrow{F} \text{Set}^X, \quad F \dashv G
\]

between presheaf toposes. The adjunction category is \( \text{Set}^B \) for \( B = A^*FX \) constructed as follows. \( \text{Ob}(B) = \text{Ob}(A) \cup \text{Ob}(X) \). Both \( A \) and \( X \) are full subcategories of \( B \). The arrows \( A \to X \), with \( A \in A \), \( X \in X \), are the elements of \((F\bar{A})(X)\), with \( \bar{A} = A(A, -) \). There are no arrows of the form \( X \to A \), with \( A \in A \), \( X \in X \). The composites of \( B \xrightarrow{f} A \xrightarrow{a} X \), \( A \xrightarrow{i} X \xrightarrow{u} Y \) with \( B, A \in A \), \( X, Y \in X \) and \( t \in (F\bar{A})(X) \) are

\[
tf = (@x(t)), \quad ur = ((\text{Yoneda}(4))(t)).
\]

To explain the isomorphism between the adjunction category and \( \text{Set}^B \), start with an object \((C, D, FCD)\) of the adjunction category; \( C \in \text{Set}^A \), \( D \in \text{Set}^X \). The Yoneda correspondence assigns \( \hat{a}: \bar{A} \to C \) to any \( a \in C(A) \); we have \( F\hat{a}: F\bar{A} \to FC \), and composing with \( h, h \circ F\hat{a}: F\bar{A} \to D \) and the function

\[
(h \circ F\hat{a})_X:(F\bar{A})(X) \to D(X).
\]

The restrictions to \( A \) and \( X \) of the functor \( \Phi \) corresponding to \((C, D, FC \xrightarrow{h} D)\) are \( C \) and \( D \), respectively, and for \( t:A \to X \), that is, \( t \in (F\bar{A})(X) \), the value \( \Phi t: \Phi(A) \to \Phi(X) \), i.e., \( \Phi t:C(A) \to D(X) \), is the map

\[
(a \in C(A)) \mapsto (h \circ F\hat{a})_X(t).
\]

Of course, these determinations generalize the ones going into the proof of (1).

Further, Lawvere points out a special property of the categories \( A^*FX \) thus constructed. A category is called one-way (see [22]) if its endomorphism monoids are trivial (the only element of \( \text{hom}(X, X) \) is \( 1_X \)). The skeletal one-way categories are those without circuits: without positive-length composable paths of non-identity arrows starting and ending in the same object. If \( X \) is a finite skeletal one-way category, then for any \( X \in X \), we can set \( \ell(X) \), the level of \( X \), the maximal integer \( n \) such that there is a length-\( n \) composable path starting in \( X \) of non-identity arrows. It is clear that all arrows go from an object to a lower level object. Conversely, if the set of objects of a category is partitioned into levels so that all arrows go from an object to a lower level object, then the category is clearly skeletal and one-way. Let the height of the category be the maximal level of any object in it.

In Lawvere's construction, if the categories \( A \) and \( X \) are one-way, then so is \( A^*FX \); the property "skeletal" is similarly inherited. In fact, if \( F \) is finite meaning that each \((F\bar{A})(X)\) \((A \in A, X \in X)\) is a finite set, and if \( A \) and \( X \) are both finite, skeletal and one-way, then so is \( A^*FX \), and the height of \( A^*FX \) is less than or equal to the sum of heights of \( A \) and \( X \).

As was pointed out in (11), the construction of the sketch-category \( \text{Set}^X \parallel K \) corresponds to the construction \( A^*FX \) with \( A = |K| \) a discrete category. Thus, every finite
sketch-category built up by using only the construction $G \parallel K$ is of the form $\text{Set}^X$ for a finite, skeletal one-way category $X$. Conversely, every such $\text{Set}^X$ is a finite sketch-category via the $\parallel$ construction starting with the terminal category $1$; this is shown by an obvious induction on the height of $X$. In conclusion: the finite sketch-categories via the $\parallel$ construction are precisely the finite toposes with a finite, skeletal, one-way exponent.

Note finally that Graph, 2-Graph and, indeed, $n$-Graph in general are one-way toposes, finite toposes with one-way exponent categories.

2. Sketch semantics

Let $S$ be an arbitrary category; we talk about “sketches” when referring to objects of $S$ since in the applications $S$ will be a sketch-category obtained by the constructions described in Section 1. Let $r: R \to R'$ be an arbitrary arrow in $S$. We say that a sketch $S \in S$ satisfies $r$, in symbols $S \Rightarrow r$, if

(*) any $\varphi: R \to S$ factors through $r$; that is, for any $\varphi: R \to S$, there is $\varphi': R' \to S$ such that

$$
\begin{array}{c}
R \\
\varphi \\
\downarrow r \\
S \\
\varphi' \\
\downarrow r' \\
R'
\end{array}
$$

commutes.

Because of their use in this definition, the maps of $S$ are sometimes called sketch-axioms, or sketch-entailments.

To show the use of the definition, let us return to the example of $S = \text{cSk}$ in Section 1; objects of $\text{cSk}$ are called c-sketches. We define seven specific maps in $\text{cSk}$, the s(keletal)-axioms for category. Here and later, the specifications will occasionally be sketchy (pun unintended), but I hope that there is no serious possibility of misunderstanding. The maps are always" almost inclusions"; entities in the domain are mapped to the same-named entities in the codomain, unless there is a collapse of two entities into one.

$$
\text{ExId} : \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} \to \begin{array}{c}
\begin{array}{c}
\langle 0,0 \rangle \\
I[ ] = \{\text{id}_1\}
\end{array}
\end{array}
$$

("Existence of identity". Here, we have two c-sketches, and a map between them. The domain sketch has no arrows, and has both specification-sets $I[ ]$, $\text{CT}[ ]$ empty. The codomain sketch has $I[ ] = \{\text{id}_1\}$. )
UnId:

\[
\begin{array}{c}
\langle 0,0 \rangle \\
0 \\
\langle 0,1 \rangle
\end{array}
\]

\[I[\ ] = \{i_0, i_1\}\]

\[i_0(\langle 0,0 \rangle) = \langle 0,0 \rangle\]

\[i_1(\langle 0,0 \rangle) = \langle 0,0,1 \rangle\]

("Uniqueness of identity"; actually, superfluous.)

ExComp:

\[
\begin{array}{c}
\langle 0,1 \rangle \\
0 \\
\langle 1,2 \rangle \\
\langle 0,2 \rangle \\
2
\end{array}
\]

\[CT[\ ] = \{id_{CT}\}\]

("Existence of composite"; in the domain, \(I[\ ]\) and \(CT[\ ]\) are empty; in the codomain \(I[\ ] = \emptyset\), and \(CT[\ ] = \{id_{CT}\}\).)

UnComp:

\[
\begin{array}{c}
0 \\
1 \\
\langle 1,2 \rangle \\
2
\end{array}
\]

\[CT[\ ] = \{[\langle 0,1 \rangle \mapsto f, \langle 1,2 \rangle \mapsto g, \langle 0,2 \rangle \mapsto h],\]

\[\{\langle 0,1 \rangle \mapsto f, \langle 1,2 \rangle \mapsto g, \langle 0,2 \rangle \mapsto i\}\}\]

("Uniqueness of composite". The domain has two commutative triangles, the codomain one; the map is the obvious one, in particular, it identifies the parallel arrows. The letters \(f, g, h, i\) stand for completely fixed entities such as the ones used before (e.g., \(\langle 0,1 \rangle\)), and they are employed only as a shorthand.)

LId:

\[
\begin{array}{c}
\langle 0,0 \rangle \\
0 \\
\langle 0,1 \rangle \\
\langle 0,0 \rangle
\end{array}
\]

\[I[\ ] = \{i_1\}\]

\[CT[\ ] = \emptyset\]

\[CT[\ ] = \{\tau\}\]

("Left identity law". In both the domain and the codomain, there is an identity specification \(\in I[\ ];\) in the codomain \(C\), we have \(CT[C] = \{\tau\}\) with the map \(\tau: CT \to |C|\) which maps \(0,1 \mapsto 0,2 \mapsto 1, \langle 0,1 \rangle \mapsto \langle 0,0 \rangle, \langle 1,2 \rangle \mapsto \langle 0,1 \rangle, \langle 0,2 \rangle \mapsto \langle 0,1 \rangle\). Here, and below, a notation like \(i_1\) means an inclusion-map of the entity in the subscript.)
RId:

("Right identity law"; similar to LId.)

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
1 & 2 & 3 & 0 \\
\end{array}
\]

\(\text{CT} = \{012, 023, 123\}\)

("Associative law". The abbreviations used are self-explanatory.)

Any category \(C\) gives rise to a c-sketch, denoted by \(C_{(S)}\), whose underlying graph is the underlying graph of \(C\), and for which \(I[C_{(S)}], CT[C_{(S)}]\) are the set of all identities and the set of all commutative triangles in the category, in the usual sense, construed as graph-arrows \(I \rightarrow |C|, CT \rightarrow |C|\) in the obvious ways. Also, with a functor \(F: C \rightarrow D\), we have the corresponding sketch-map \(F_{(S)}: C_{(S)} \rightarrow D_{(S)}\); as a mapping, \(F_{(S)}\) is identical to \(F\). If a c-sketch \(S\) is obtained from a(n obviously unique) category, we say \(S\) is a category. Let \(\mathcal{R}_{cat}\) denote the set of the seven sketch-axioms \(\text{ExId, UnId, ExComp, UnComp, LId, RId, Assoc}\); these are called the sketch-axioms for category.

Now, here is a proposition:

A c-sketch is a category if and only if it satisfies each of the sketch-axioms for category. More precisely: the mapping \(C \mapsto C_{(S)}, F \mapsto F_{(S)}\) is a full and faithful functor of the category \(\text{Cat}\) of small categories into the category of c-sketches; it is also 1–1 on objects, and the range of it consists exactly of those sketches that satisfy the s-axioms for category.

This is practically clear. Indeed, a c-sketch \(S\) is (arises from) a category iff (1) to any object \(A \in |S|\), there is an arrow \(1_A: A \rightarrow A\) assigned to \(A\), (2) which is unique; (3) to any pair of arrow \(A \xrightarrow{f} B \xrightarrow{g} C\) as shown, there is assigned a composite \(gf: A \rightarrow C\), (4) which is unique; (5 and 6) composites with identities behave in the right way; and (7) composition is associative. Now, here each of the conditions \((i)\) for \(i = 1\) to 7 is equivalent to saying that \(S \models r_i\), with \(r_i\) for \(i = 1\) to 7 being the maps listed above in this order.

The contents of the last sentence are, of course, completely elementary; still, together with the examples to be shown later, it contains the main point of the paper: the fact that there is a natural way of expressing the individual parts of specifications of various notions (that of "category" in this case) in the form of sketch-axioms.

A sketch-axiom \(R \rightarrow R'\) may be read as an entailment that says: "the situation described in \(R\) entails that described in \(R'\)". For instance, \(\text{ExComp}\) "says" that the situation consisting of two composable arrows entails that which extends the first one by having (at least one) composite; \(\text{UnComp}\) "says" that the situation of having two arrows which are composites of the same pair of arrows entails the one in which those two arrows are equal. The entailment \(R \rightarrow R'\) being satisfied by \(S\), \(S \models (R \rightarrow R')\), means
that no matter how we have a situation of the type $R$ in $S$, we have another one in $S$ of type $R'$ extending the first one.

For a category $\mathcal{S}$ of sketches, and a sketch-entailment $r$, "$\mathcal{S} \models r$" stands for "for all $S \in \mathcal{S}$, $S \models r$". With a set $\mathcal{A}$ of sketch-entailments (arrows), $S \models \mathcal{A}$ means $S \models r$ for all $r \in \mathcal{A}$; $\mathcal{S} \models \mathcal{A}$ means that $S \models r$ for all $S \in \mathcal{S}$; and finally, $\vdash_{\mathcal{A}} r$ means that for all $S$ such that $S \models \mathcal{A}$, we have $S \models r$. These notations follow usual practice in logic.

Given a category $\mathcal{S}$ and a set $\mathcal{A}$ of arrows in it, by $\mathcal{S} : \mathcal{A}$ we denote the full subcategory of $\mathcal{S}$ consisting of $S \in \mathcal{S}$ for which $S \models \mathcal{A}$. $\mathcal{S} : \mathcal{A}$ is called the category (or doctrine) specified by $\mathcal{S}$ and $\mathcal{A}$. Above, up to isomorphism we exhibited $\mathsf{Cat}$, the category of small categories, as $\mathcal{S} : \mathcal{P}_2$ for $\mathcal{S} = \mathsf{Sk}$ and $\mathcal{A} = \mathcal{A}_{\mathsf{cat}}$. Our contention is that many categories of structured categories can be naturally and profitably exhibited in the form $\mathcal{S} : \mathcal{A}$. For brevity, a doctrine specification is a pair $(\mathcal{S}, \mathcal{A})$, with $\mathcal{S}$ a sketch-category, and $\mathcal{A}$ a set of arrows in $\mathcal{S}$; it specifies $\mathcal{S} : \mathcal{A}$.

A doctrine specification $(\mathcal{S}, \mathcal{A})$ is finitary if $\mathcal{S}$ is a finitary sketch-category (see Section 1), and $\mathcal{A}$ is a set of arrows between fp objects of $\mathcal{S}$. The same is a finite doctrine specification if $\mathcal{S}$ is a finite sketch-category, and $\mathcal{A}$, in addition to the previous requirement, is also a finite set. In this paper, we will restrict ourselves to finitary doctrine specification, although "infinitary" ones are equally relevant, in relation to doctrines of infinitary logic. In fact, with a single exception, all our examples will be finite doctrine specifications.

Note that the general construction $\mathcal{S} : \mathcal{A}$ is far from being new; $\mathcal{S} : \mathcal{A}$ is the category of objects that are injective relative to the class $\mathcal{A}$ of arrows; see, e.g., Ch. 4 of [1]. Also, the construction has a very general scope that is well-known. For example, the categories that can be obtained in the form $\mathcal{S} : \mathcal{A}$, with $\mathcal{S}$ a presheaf category, and $\mathcal{A}$ a set of arrows, are, up to equivalence of categories, those accessible categories that have (small) products. These categories are called weakly locally presentable in [1]; see 4.8 Theorem, p. 178 in loc. cit. for a multiple characterization of weakly locally presentable categories, including the ones in the previous sentence. Thus, it is not the mere possibility of presenting a category in the form $\mathcal{S} : \mathcal{A}$ that matters; rather, what does is the actual presentation itself.

Let us note that, for the purposes of specification, the construction $\mathcal{G} \restriction \mathcal{K}$ is not necessary, the construction $\mathcal{G} \parallel \mathcal{K}$ suffices. The most immediate thing to say is that the former is equivalent to a full subcategory of the latter singled out by sketch-axioms. With any $K \in |\mathcal{K}|$, consider the sketch-map $\mathsf{Uni}[\mathcal{K}] : \langle \langle K \rangle \rangle \to \langle K \rangle$ in $\mathcal{G} \parallel \mathcal{K}$ as follows. $\langle K \rangle$ has $|\langle K \rangle| = K$, and $K[\langle K \rangle] = \{0\}$, with $0 = \text{id}_K$; all other $K'[\langle K \rangle] = \emptyset$; $\langle \langle K \rangle \rangle$ also has $|\langle \langle K \rangle \rangle| = K$, but $K[\langle \langle K \rangle \rangle]$ equal to $\{0, 1\}$, with $0 = 1 = \text{id}_K$; the map $\mathsf{Uni}[\mathcal{K}]$ is the identity on the level of $\mathcal{K}$. Clearly, the sketches satisfying each $\mathsf{Uni}[\mathcal{K}] (K \in \mathcal{K})$ are exactly the ones in which maps $s \mapsto \tilde{s}$ are each one-to-one; the latter form the essential image of the inclusion of $\mathcal{G} \restriction \mathcal{K}$ in $\mathcal{G} \parallel \mathcal{K}$. (The existence in the satisfaction of the axiom $\mathsf{Uni}[\mathcal{K}]$ is automatically a unique existence; this means that the objects of $\mathcal{G} \restriction \mathcal{K}$ are singled out in $\mathcal{G} \parallel \mathcal{K}$ by the condition of being orthogonal (see [9]) to given arrows between fp objects; this also shows that $\mathcal{G} \restriction \mathcal{K}$ is lfp if $\mathcal{G} \parallel \mathcal{K}$ is.)

Accordingly, we can restate our specifications, in the one example above, and the many examples later, to take place always in the context of a sketch-category of the
form $G \parallel K$. The $\text{cSk}_1$ (as opposed to $\text{cSk}$) is $\text{Graph} \parallel \{1, \text{CT}\}$. Then, we have to redefine the sketch-axioms for category, in the obvious way, by replacing each by its image under the inclusion functor $G \parallel K \to G \parallel K$. Finally, we add the sketch-axioms $\text{Uni}[1], \text{Uni}[\text{CT}]$. The resulting specification gives $\text{Cat}$ up to equivalence.

The category (doctrine) $S: \mathcal{R}$ specified by $(S, \mathcal{R})$ is an accessible category having small products as remarked above. In general, we cannot say much more; e.g., $S: \mathcal{R}$ is not locally presentable in general; actually, most of our examples are not locally presentable (equivalently, they do not have all (small) limits). (Most of the examples become \textit{locally presentable as a bicategory}, with an additional structure of 2-cells; this is the main subject of the sequel [26]; here, we will not have anything to do with the higher dimensional aspects of doctrines.)

Assume that $S$ is an lfp category, and $\mathcal{R}$ is a set of arrows between fp objects of $S$ (these are the assumptions on $(S, \mathcal{R})$ under which the general theory of this section, and Sections 3 and 8 go through; a finitary doctrine specification satisfies these assumptions). Consider the following cardinal invariant of $(S, \mathcal{R})$. Let $\lambda = \lambda_{S, \mathcal{R}}$ be the maximum of the following three cardinals: $\aleph_0$, the cardinality of $\mathcal{R}$, and the cardinality of the set of the isomorphism classes of fp objects in $S$. When $(S, \mathcal{R})$ is a finite doctrine specification, $\lambda$ is equal to $\aleph_0$; now, the fp objects are the finite objects; any horn-set of two finite objects is finite.

For any regular cardinal $\kappa > \lambda$ (hence, in the finite case, any uncountable regular cardinal), an object $S \in S$ is $\kappa$-presentable iff $\text{hom}(R, S)$ is of cardinality $< \kappa$ for all $R \in S_{\aleph_0}$. This follows from 2.3.13 (p. 30) in [28]; notice (to make the connection with loc. cit.) that, by 2.3.4 (p. 22) loc. cit. $\aleph_0 < \kappa$. For example, in the finite case a sketch is $\kappa$-presentable, for any infinite regular cardinal $\kappa$, iff its cardinality (in the usual sense of the cardinality of a functor $C \to \text{Set}$) is $< \kappa$; this is true for $\kappa = \aleph_0$ since in a finite topos, the finitely presentable objects are exactly the finite objects; and for $\kappa > \aleph_0$ by what we just said. We have

$$\text{Let } \kappa \text{ be a regular cardinal, } \kappa > \lambda_{S, \mathcal{R}}. \text{ Then } S: \mathcal{R} \text{ is } \kappa\text{-accessible, and an object of } S: \mathcal{R} \text{ is } \kappa\text{-presentable in } S: \mathcal{R} \text{ iff it is } \kappa\text{-presentable in } S. \text{ Moreover, } S: \mathcal{R} \text{ has filtered colimits and arbitrary products, both preserved by the inclusion into } S.$$ 

Thus, for a finite doctrine specification $(S, \mathcal{R})$, $S: \mathcal{R}$ is $\aleph_1$-accessible, and $\kappa$-accessible for all uncountable regular cardinals $\kappa$.

I will show the last-stated proposition only for a finitary $(S, \mathcal{R})$. In this case, I will show that one can write down a set of axioms in "regular logic", in a language for first order logic, whose category of models is isomorphic to $S: \mathcal{R}$; both the language and the set of axioms are of cardinality $\leq \lambda_{S, \mathcal{R}}$, and they are finite in the case of a finite doctrine specification. (An axiom in regular logic is a sentence of the form $\forall \bar{x}(\varphi(\bar{x}) \to \psi(\bar{x}))$, with $\varphi$ and $\psi$ positive primitive formulas, of the form $\exists \bar{y} \land_{1 \leq i \leq n} \theta_i(\bar{x}, \bar{y})$, with the $\theta_i$ atomic.)

The above conclusions on accessibility and filtered colimits will then follow by 3.3.5(ii) in [28, p. 56]. (In fact, loc. cit. gives the desired conclusion on $\kappa$-accessibility
only when \( \kappa \) is a successor cardinal. However, in the remaining case when \( \kappa \) is weakly inaccessible, \( \kappa > \lambda_{S,R} \), one directly checks, using that \( S:R \) is \( \kappa' \)-accessible for all successors \( \kappa' \) such that \( \lambda_{S,R} < \kappa' < \kappa \), that the objects that are \( \kappa' \)-presentable for some \( \kappa' < \kappa \) form a dense subcategory of \( S:R \) via \( \kappa \)-filtered colimits. The proof of 3.3.5(ii) in [28] gives that the objects of \( S:R \) that are \( \kappa \)-presentable in \( S \) are dense in \( S:R \), from which the conclusion on presentability follows. The assertion about products is best verified directly from the definitions.

Let \( S = \text{Set}^C \), \( C \) a small category. Let \( L \) be the multi-sorted language which is the underlying graph of \( C \): its sorts are the objects of \( C \), its unary sorted operation symbols the arrows of \( C \). The axioms include the statements of the form \( \forall x (g(f(x)) = (g \circ f)(x)) \), for every composable pair \( (f, g) \) of arrows of \( C \), and the ones of the form \( \forall x (1_C(x) = x) \) for each \( C \in \text{Ob}(C) \). Each of the rest of the axioms \( \alpha_r \) is associated with a sketch-axiom \( r: R \to R' \) in \( R \), as follows:

\[
\forall \langle x(C,a) \rangle_{C \in C, a \in R(C)} \left( \bigwedge_{f:C_1 \to C_2} f(x(C_1,a_1)) = x(C_2,a_2) \to \right)
\exists \langle y(C,b) \rangle_{C \in C, b \in R'(C)} \left( \bigwedge_{f:C_1 \to C_2} f(x(C_1,b_1)) = x(C_2,b_2) \right.
\left. \bigwedge_{a \in RC_1, \; b \in RC_2, \; (Rf)(a) = b} \right)
\]

The class of objects of \( S:R \) is precisely the class of \( L \)-models satisfying the axioms; the arrows of \( S:R \) are the \( L \)-homomorphisms.

### 3. Formal deductions of sketch-entailments

For motivation, we first consider the simple example of a possible way the category \( \langle S \rangle \) presented by a c-sketch \( S \) may be constructed. \( \langle S \rangle \) comes with a canonical "embedding" \( \gamma: S \to \langle S \rangle \); note that we are identifying the category \( \langle S \rangle \) with the corresponding sketch as in the previous section. \( \gamma \) is defined by the familiar universal property: \( \gamma: S \to \langle S \rangle \) factors uniquely through any \( S \to D \), with \( D \) a (sketch derived from \( a \)) category. The construction of \( \langle S \rangle \) can be performed in steps, providing successive extensions of sketches. In one step, e.g., the composite of two arrows, \( f: A \to B, g: B \to C \), with matching codomain and domain, has to be added to \( T \), the sketch of the previous step. This can be accomplished by taking the pushout

\[
\begin{array}{ccc}
R & \xrightarrow{r} & R' \\
\downarrow{\psi} & & \downarrow{\psi'} \\
T & \xrightarrow{t} & T'
\end{array}
\] (1)
where \( r \) is "composition", the map \( \text{ExComp} \) of \( \ref{1.1} \) and where \( \varphi : R \to T \) picks out the two arrows \( \langle 0, 1 \rangle \mapsto f, \langle 1, 2 \rangle \mapsto g \):

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\uparrow & & \downarrow & & \uparrow \\
A & \xrightarrow{h} & C
\end{array}
\]

This is a superfluous move if there already is in \( T \) a commutative triangle with \( f \) and \( g \) as two of the sides. But, applying another pushout

\[
\begin{array}{ccc}
R_1 & \xrightarrow{r} & R'_1 \\
\varphi & \downarrow & \varphi' \\
T' & \xrightarrow{q} & T''
\end{array}
\]

this time with \( r_1 \) being \( \text{UnComp} \) in \( \ref{1.1} \), and \( \varphi_1 \) picking out the two commutative triangles concerned, \( T'' \) has the two composites identified; \textit{nothing else happened} to \( T'' \):

\[
\begin{array}{ccc}
0 & \xrightarrow{f} & 1 & \xrightarrow{g} & 2 \\
\uparrow & & \downarrow & & \uparrow \\
A & \xrightarrow{h} & B & \xrightarrow{g} & C
\end{array}
\]

With \( r : R \to R' \) any map of c-sketches, and \( \varphi : R \to T \) an arbitrary map, we write \( \varphi^*r \) for the pushout (1).

We apply the operations \( \varphi^*r \) iteratively; \( r \) ranges over the seven s-axioms for category in \( \ref{1.1} \). We are tempted to call those seven sketch-maps the \textit{templates} of the construction. The process goes through a possibly transfinite number of steps. Having made a limit-number of steps, we take the colimit of the chain of sketches of the previous steps as the construction of the next step. We will arrive at the desired category after a suitable number of steps without any particular care for what templates we use in what order \textit{provided} we take care of unwanted "infinite loops".
Note that we may get into the rut of adding more and more composites of \( f \) and \( g \) in the above example, not getting the opportunity for anything else, if we do not take the care to avoid doing so. Here is a general recipe:

apply the operation \( \varphi*r \) only if \( \varphi \) does not factor through \( r \).

Certainly, this is reasonable since if \( \varphi \) does factor through \( r \), the job that we want to do with the operation \( \varphi*r \) is already done; e.g., if \( r \) is the "uniqueness" map UnComp, the operation \( \varphi*r \) will identify two arrows that are already equal. I claim that

Any transfinite sequence of completion steps as described above that avoids superfluous steps as specified will terminate in \( \langle S \rangle \) in an ordinal number of steps whose cardinality is less than or equal to \( \max(K_0, \#(S)) \).

I will not take the time to prove the claim; the essence of the argument will be contained in the proof of Corollary 6 below. It suffices to say that the finite segments of the construction of \( \langle S \rangle \) will act as the formal deductions for the doctrine specified by \( \text{cSk} \) and \( \mathcal{R}_{\text{cat}} \).

Let \( S \) be a category having pushouts, \( \mathcal{R} \) a set of arrows in it. We will talk about objects of \( S \) as "sketches", because of the applications. In what follows, \( S \) and \( \mathcal{R} \) are fixed.

An entailment (an arrow in \( S \)) \( u: U \rightarrow U' \) is immediately deducible (from \( \mathcal{R} \)) if it is a pushout of some \( r \in \mathcal{R} \): there is a pushout diagram of the form

\[
\begin{array}{ccc}
R & \xrightarrow{r} & R' \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
U & \xrightarrow{u} & U'
\end{array}
\]

in \( S \), with \( r \) some element of \( \mathcal{R} \). Further, \( t:S \rightarrow T \) is directly deducible (from \( \mathcal{R} \)) if it is the composite of finitely many immediately deducible ones. Finally, \( s:S \rightarrow S' \) is deducible (from \( \mathcal{R} \)), in notation \( \vdash s \), or more specifically \( \vdash \varphi s \), if an extension of \( s \) is directly deducible: there is a \( t':S' \rightarrow T \) such that \( t' \circ s:S \rightarrow T \) is directly deducible.

For the notion of formal deduction, there are a couple of slightly different candidates. In the most explicit version, a (formal) deduction of \( s:S \rightarrow S' \) (from \( \mathcal{R} \)) consists of a map \( t':S' \rightarrow T \) together with a finite sequence of pushout squares (I') with each \( r \) from \( \mathcal{R} \) such that the \( u \)-legs of the squares are composable, and they give \( t' \circ s \) as their composite. In a less explicit, but more economical, version, a deduction of \( s \) is given by a sequence \( \langle U_i \rangle_{i \leq n} \) of objects and a sequence \( \langle u_i: U_i \rightarrow U_{i+1} \rangle_{i+1 \leq n} \) of arrows such that each \( u_i \) is immediately deducible, and such that there is a map \( t':S' \rightarrow T \) satisfying \( t' \circ s = u_{n-1} \circ u_{n-2} \circ \cdots \circ u_0 \).

We discuss effectiveness notions (decidable, effectively enumerable (e.e.), \ldots) for the use with sketch-based syntax. We take as the basic space of finite entities the set \( \mathbb{HF} \) of hereditarily finite (hf) sets; \( \mathbb{HF} = V_\omega \), the \( \omega \)th level in the von Neumann hierarchy of pure sets. We have the standard notions of decidable (= recursive), e.e. (= recursively
enumerable), etc. for subsets of $\mathcal{H}E$; they coincide with the sets that are $\Delta_1$-definable, resp. $\Sigma_1$-definable in $(\mathcal{H}E, \in \uparrow \mathcal{H}E)$. The good thing about $\mathcal{H}E$ is that it is closed under the operations of taking ordered pairs, finite sets, and finite sequences of elements; any morphism $f: A \to B$ between hf sets $A, B$ is an hf set ($f = (A, B, \text{graph}(f))$, etc.

In our determination of specific entities (specification types and sketch axioms), we take care of making them hf sets. Notice that the specification types for $\text{cSk}$, and the axioms in $\mathcal{H}en$ are hf sets; similar control will be exercised in the later examples too.

The general theory of sketches, as any reasonable categorical theory, has all its concepts invariant under isomorphism. For example, the notions of "finite", "exactness property" (a notion introduced later), "deducible from given set of axioms", etc. are all invariant under isomorphism in suitable senses. This invariance helps us give a precise meaning to effectiveness notions. In general, we say that a collection $\mathcal{C}$ of finite entities (sketches, sketch-entailments, sequences of sketch-entailments (deductions), ...) is decidable, or e.e., if the subcollection $\mathcal{C}_{hf}$ of its hf elements is decidable, resp. e.e. Since $\mathcal{C}$ is (usually) closed under isomorphism, the subcollection $\mathcal{C}_{hf}$ is representative: every entity in $\mathcal{C}$ has an isomorphic copy in $\mathcal{C}_{hf}$.

For example, if the finite category $\mathcal{C}$ is in fact an hf set (and clearly, every finite category is isomorphic to one which is an hf set), then every finite object of Set$^C$ is isomorphic to an (in fact, several) hf object (a functor $\mathcal{C} \to \text{Set}$ that takes hf values; such a functor can clearly be identified with an appropriate hf set). Thus, every finite sketch, and every finite sketch-entailment, in Set$^C$ is isomorphic to one which is hf.

I have made explicit certain aspects of effectiveness in the algebraic context. When people talk, e.g., about an r.e. set of finite groups, they have in mind (or should have in mind) a concept like the one described above. One difference might be that people would tend to talk about Godel numbering by natural numbers "in the usual way"; using hf sets removes this kind of vagueness.

Suppose that $\mathcal{S}$ is a finite sketch-category, with all specification types being hf sets. Then the predicate $\text{Ded}(d, s, \mathcal{R})$ with the variables $d, s$ and $\mathcal{R}$ ranging over $\mathcal{H}E$,

"$\mathcal{R}$ is a (finite) set of sketch-entailments in $\mathcal{S}$, $s$ is a sketch-entailment in $\mathcal{S}$, and $d$ is a deduction of $s$ from $\mathcal{R}$",

in either sense of "deduction", is decidable. This is seen by inspection, given our experience with similar statements in many contexts (among others, in "effective algebra").

It immediately follows that the set of syntactical consequences $\mathcal{D}[\mathcal{R}] = \{s : \#s\}$ of a finite set $\mathcal{R}$ of sketch-entailments is r.e. In fact, since

$$\mathcal{D}[\mathcal{R}] = \bigcup_{\mathcal{R} \text{ finite}} \mathcal{D}[\mathcal{R}]$$

the same conclusion follows for any r.e. set $\mathcal{R}$ of axioms.

Similar assertions can be made in the more general case of finitary doctrine specifications, provided they are "countably and effectively" given.
The notion of formal deduction is a codification of that of diagram manipulation. In a sketch-category, a deduction (with a finite premise) involves a succession of finite sketches, that is, finite diagrams with certain distinguished parts designated to have special qualities (product, exponential, etc.). Each sketch in the succession is obtained by an application of a rule, that is, an arrow in $\mathcal{A}$, where an application of the rule $r: R \rightarrow R'$ to a sketch $U$ consists in selecting an $R$-figure $\varphi: R \rightarrow U$ in $U$, and amalgamating $R'$ with $U$ with respect to the maps $r$ and $\varphi$ via a pushout. The last amalgamation, a pushout effects only a local change in $U$, because of the nature of pushouts in a finite topos, deriving as they do from pushouts in Set. These local changes consist in adding a bounded number of new entities (objects, arrows, designations of parts as "product", etc.), and/or identifying a certain bounded number of entities (here "bounded" means "bounded by an integer depending only on $\mathcal{A}$, not on $S$").

For an entailment (arrow in $S$) $s$, let us say that $s$ is valid if $\vdash_s s$, and deducible if $\vdash s$; let $V = V^\vdash$ be the class of valid entailments, $D = D^\vdash$ that of the deducible ones.

**Lemma 1.** For either of the classes $\mathcal{X} = V$, $\mathcal{X} = D$, the following properties hold:

(i) Every isomorphism belongs to $\mathcal{X}$.

(ii) The composite of two composable elements of $\mathcal{X}$ belongs to $\mathcal{X}$.

(iii) If $g \circ f \in \mathcal{X}$, then $f \in \mathcal{X}$.

(iv) Any pushout of a member of $\mathcal{X}$ is again a member of $\mathcal{X}$: if in the pushout $(1') r \in \mathcal{X}$, then $u \in \mathcal{X}$.

The easy proof is left to the reader.

The class $D$ is clearly contained in the least class $\mathcal{X}$ closed under (i), (iii) and (iv); by the lemma, applied to $\mathcal{X} = D$, it is the same as that least class, as well as the least class satisfying all four of conditions in the lemma. The lemma, applied to $\mathcal{X} = V$ now, thus gives

**Corollary 2** (Soundness Theorem). Each deducible arrow is valid; $D \subseteq V$.

In what follows, we assume that $S$ is a locally finitely presentable (lfp) category. Next, we recall some elementary facts concerning comma-categories. Let $S \in \mathcal{S}$. The comma-category $S \downarrow \mathcal{S}$ (with objects arrows $S \rightarrow T$, and arrows $(S \rightarrow T) \rightarrow (S \rightarrow U)$ the arrows $T \rightarrow U$, making the triangle commute; see [23]) is also lfp, and the forgetful functor $S \downarrow \mathcal{S} \rightarrow \mathcal{S}$ creates (small) colimits.

An fp object of $S \downarrow \mathcal{S}$ is called a relatively finite (rf) arrow in $\mathcal{S}$. The rf arrows are pushouts of fp arrows of $\mathcal{S}$: if $S \rightarrow T$ is rf, there exists a pushout diagram

$$
\begin{array}{ccc}
R & \longrightarrow & R' \\
\downarrow & & \downarrow \\
S & \longrightarrow & T \\
\end{array}
$$

with $R, R'$ fp objects.
Connecting two comma-categories, we note that the composite of two (composable) rf arrows in \( S \) is rf again.

**Theorem 3** (General Completeness Theorem). Let \( S \) be an lfp category, and \( \mathcal{R} \) a set of arrows between fp objects in \( S \). For any relatively finite entailment \( s: S \to S', \vdash s \iff \vdash s' \); that is, for such \( s \), validity and deducibility coincide.

For the proof, we need some preliminaries. Throughout this section, \( S \) and \( \mathcal{R} \) are as in the statement of Theorem 3.

Recall the cardinal \( \lambda = \lambda_{S,\mathcal{R}} \) from Section 2. Let \( S \in S \). We denotes by \( \kappa_S \) the least uncountable regular cardinal such that \( S \) is \( \kappa \)-presentable and \( \lambda < \kappa \). When \( S \) is a finite sketch-category, \( \kappa_S = \aleph_1 \) for \( S \) (finite or) countable, and \( \kappa_S = (\#(S))^+ \) otherwise.

A transfinite sequence

\[
\mathcal{U} = \langle \langle U_\beta \rangle_{\beta \leq \alpha}, \langle u_{\beta\gamma}: U_\beta \to U_\gamma \rangle_{\beta \leq \gamma < \alpha} \rangle
\]

of sketches, with a system of connecting maps, is called an infinitary deduction (\( \infty \)-deduction) if it is compatible (\( u_{\beta,\gamma} \circ u_{\gamma,\Delta} = u_{\beta,\Delta} (\beta < \gamma < \Delta < \alpha) \), \( u_{\beta,\beta} = 1_{U_\beta} (\beta < \alpha) \), continuous (for each limit ordinal \( \beta < \alpha \), \( U_\beta \) is the colimit of the restricted system with indices \( \gamma < \beta \), with colimit coprojections the \( u_{\gamma,\beta} \) for \( \gamma \leq \beta \), and each \( u_{\beta,\beta+1} (\beta + 1 \leq \alpha) \) is an immediately deducible arrow; it is of length \( \alpha \); it is based on the sketch \( S \) if \( U_0 = S \).

Appropriately, a deduction is an \( \infty \)-deduction of finite length.

\( \infty \)-deduction can be composed. If \( \mathcal{U} \) is as above, and

\[
\mathcal{V} = \langle \langle V_\nu \rangle_{\nu \leq \lambda}, \langle v_{\mu\nu}: V_\mu \to V_\nu \rangle_{\mu \leq \nu \leq \lambda} \rangle
\]

is another \( \infty \)-deduction such that \( V_0 = U_\alpha \), then \( \mathcal{V} \circ \mathcal{U} \) is the \( \infty \)-deduction

\[
\mathcal{V} \circ \mathcal{U} = \langle \langle W_\xi \rangle_{\xi \leq \alpha + \lambda}, \langle w_{\mu,\nu}: W_\mu \to W_\nu \rangle_{\mu \leq \nu \leq \xi \leq \alpha + \lambda} \rangle
\]

on length \( \alpha + \lambda \) for which \( W_\beta = U_\beta (\beta \leq \alpha) \), \( W_{\alpha+\nu} = V_\nu (\nu \leq \lambda) \), \( w_{\beta,\gamma} = u_{\beta,\gamma} (\beta \leq \gamma \leq \alpha) \), \( w_{\beta,\beta+\gamma} = v_{\nu,\gamma} u_{\beta,\gamma} (\beta \leq \alpha, \gamma \leq \lambda) \) and \( w_{\alpha+\mu,\alpha+\nu} = v_{\mu,\nu} (\mu \leq \nu \leq \lambda) \).

Let us call an immediately deducible arrow \( u: U \to U' \) proper if it results from some pushout (1'), with \( r \in \mathcal{R} \), in which \( \phi \) does not factor through \( r \). In particular, this implies that \( u \) is not an isomorphism; but in general, the latter condition is weaker than \( u \) being a proper immediately deducible arrow.

The \( \infty \)-deduction (2) is proper if each \( u_{\beta,\beta+1} (\beta + 1 \leq \alpha) \) is a proper immediately deducible arrow.

**Proposition 4.** For any sketch \( S \), all proper \( \infty \)-deductions based on \( S \) are of length \( < \kappa_S \).

**Proof.** Suppose that, contrary to the assertion, there is a proper \( \infty \)-deduction

\[
\langle \langle U_\beta \rangle_{\beta < \kappa}, \langle u_{\beta,\alpha}: U_\beta \to U_\alpha \rangle_{\beta < \kappa < \alpha} \rangle
\]

with \( U_0 = S \), \( \kappa = \kappa_S \). For every ordinal \( \alpha < \kappa \), there
is \( r_\alpha \in \mathcal{R} \) such that \( u_{\alpha, \alpha+1} \) is a pushout of \( r_\alpha \):

\[
\begin{array}{ccc}
R_\alpha & \xrightarrow{r_\alpha} & R'_\alpha \\
\downarrow \varphi_\alpha & & \downarrow \varphi'_\alpha \\
U_\alpha & \xrightarrow{u_{\alpha, \alpha+1}} & U_{\alpha+1}
\end{array}
\]

and such that

(3) \( \varphi_\alpha \) does not factor through \( r_\alpha \).

Furthermore, if \( \alpha \) is a limit ordinal,

(4) \( U_\alpha = \operatorname{colim}(\langle u_{\gamma, \beta} : U_\gamma \to U_\beta \rangle_{\gamma < \beta < \alpha}) \) with coprojections \( \langle u_{\beta, \alpha} \rangle_{\beta < \alpha} \).

By induction on \( \alpha < \kappa \), we see that \( U_\alpha \) is \( \kappa \)-presentable; the reason is that a colimit of a diagram of size \( < \kappa \) of \( \kappa \)-presentable objects is \( \kappa \)-presentable. Since also \( \lambda < \kappa \), it follows by remarks in Section 2 that for all \( \text{fp } R \) and \( \alpha < \kappa \),

(5) \( \operatorname{hom}(R, U_\alpha) \) has cardinality \( < \kappa \).

Let \( \alpha < \kappa \) be any limit ordinal. Since \( R_\alpha \) is \( \text{fp} \), by (4) there is \( \beta_\alpha < \alpha \) such that \( \varphi_\alpha \) factors through \( u_{\beta_\alpha, \alpha} \): we have

\[
\begin{array}{ccc}
R_\alpha & \xrightarrow{\psi} & R' \\
\downarrow \varphi_\alpha & & \downarrow \varphi'_\alpha \\
U_{\beta_\alpha} & \xrightarrow{u_{\beta_\alpha, \alpha}} & U_\alpha
\end{array}
\]

Since \( \kappa \) is regular and uncountable, by the regressive function theorem (Fodor's lemma, [15]), there is \( \beta < \kappa \) and an unbounded(-below-\( \kappa \)) set \( A \) of limit ordinals \( < \kappa \) such that for all \( \alpha \in A \), \( \beta_\alpha = \beta \). Since \( \# \mathcal{R} \leq \lambda < \kappa \), and \( \kappa \) is regular, there are \( (R \to R') \in \mathcal{R} \) and an unbound subset \( A' \) of \( A \) such that for all \( \alpha \in A' \), \( r_\alpha = r \), in particular, \( R_\alpha = R \). By (5), \( \operatorname{hom}(R, U_\beta) \) has cardinality \( < \kappa \). Therefore, there are \( \psi : R \to U_\beta \) and an unbound subset \( A'' \) of \( A' \) such that for all \( \alpha \in A'' \), \( \psi_\alpha = \psi \). Let \( \alpha_1 < \alpha_2 \) be two distinct elements of \( A'' \). Applying (5) to \( \alpha = \alpha_1 \) and \( \alpha_2 \), we get that in

\[
\begin{array}{ccc}
R & \xrightarrow{r} & R' \\
\downarrow \psi_{\alpha_1} & & \downarrow \psi' \phi_{\alpha_1} \\
U_{\alpha_1} & \xrightarrow{u_{\alpha_1, \alpha_2}} & U_{\alpha_2}
\end{array}
\]

the right-hand triangle commutes. Consider

\[
\begin{array}{ccc}
R & \xrightarrow{r} & R' \\
\downarrow \phi_{\alpha_1} & & \downarrow \psi' \phi_{\alpha_1} \\
U_{\alpha_1} & \xrightarrow{u_{\alpha_1, \alpha_2}} & U_{\alpha_2}
\end{array}
\]
in which $\varphi' = u_{z_1 + 1, z_2} \circ \varphi'$. We obtain that

$$\varphi' \circ r = u_{z_1 + 1, z_2} \circ u_{z_1, z_2 + 1} \circ \varphi_{z_1} = u_{z_1, z_2} \circ \varphi_{z_1} = \varphi_{z_2}$$

(the last equality by (6')), that is, $\varphi_{z_2}$ factors through $r = r_{z_2}$, contrary to (3) (properness). This contradiction completes the proof. \(\square\)

The $\infty$-deduction (2) is an extension of another one, say $\mathcal{U}'$, if, for some $\alpha' < \alpha$, $\mathcal{U}' = \mathcal{U} \upharpoonright \alpha' = (\langle u_{\beta} \rangle_{\beta \leq \alpha'}, \langle u_{\gamma} \gamma : U_{\beta} \rightarrow U_{\gamma} \gamma \leq \gamma \leq \alpha')$; $\mathcal{U}$ is a proper extension of $\mathcal{U}'$ if, in addition, $\alpha' < \alpha$. A proper $\infty$-deduction is maximal if it does not have any proper extension which is a proper $\infty$-deduction. Note that all these concepts are relative to a given set $\mathcal{R}$ of sketch-axioms.

It is immediately seen that $\mathcal{U} \upharpoonright \alpha$ as in (2) is maximal if and only if $\mathcal{U} \upharpoonright \alpha$ satisfies every $r \in \mathcal{R}$, i.e., if and only if $\Gamma_{\mathcal{U} \upharpoonright \alpha} \in S_{\mathcal{R}}$.

**Corollary 5.** Any proper $\infty$-deduction can be extended to a maximal one.

**Proof.** Consider the class $C$ of proper $\infty$-deductions based on a fixed sketch $S$. Each $\mathcal{U} \in C$ has a length $\kappa = \kappa_{\alpha}$, as noted in the previous proof, every object in $\mathcal{U}$ is $\kappa$-presentable. Let $B$ be a small set of $\kappa$-presentable objects such that every $\kappa$-presentable object is isomorphic to one in $B$. Let $C'$ be the set of $\mathcal{U} \in C$ such that all the objects of $\mathcal{U}$ come from $B$. Clearly, $C'$ is small. A chain $X$ in $C'$ is a subset of $C'$ such that for any two elements of $X$, one is an extension of the other. The union of any chain in $C'$ is clearly again an element of $C'$. By Zorn's lemma, for any $\mathcal{U} \in C'$, there is $\mathcal{U}' \in C'$ extending $\mathcal{U}$ which is maximal in $C'$. Clearly, if $\mathcal{U} \in C'$ is not maximal, then it is not maximal in $C'$ either, that is, it has a proper extension in $C'$. Hence, any $\mathcal{U} \in C'$ has $\mathcal{U}' \in C'$ extending $\mathcal{U}$ such that $\mathcal{U}'$ is maximal in $C'$, that is, maximal. \(\square\)

We call any sketch of the form $\Gamma_{\mathcal{U} \upharpoonright \alpha}$, with $\mathcal{U}$ a maximal proper $\infty$-deduction based on $S$, a doctrinal hull of $S$; the corresponding canonical map $\varphi_{\mathcal{U}} : S \rightarrow \mathcal{U}$ is $\nu_{0\alpha}$ when $\mathcal{U}$ is as in (2). Each doctrinal hull belongs to the doctrine $S_{\mathcal{R}}$.

**Corollary 6.** Any sketch has at least one doctrinal hull.

An $\infty$-deduction $\mathcal{U}$ as in (2) based on $S$ gives rise to the $\infty$-directly deducible arrow $\varphi_{\mathcal{U}} : S \rightarrow \Gamma_{\mathcal{U} \upharpoonright \alpha} = U_2$, $\varphi_{\mathcal{U}} = u_{0\alpha}$. An arrow $s : S \rightarrow T$ is $\infty$-deducible if an extension of it is $\infty$-directly deducible: for some $t : T \rightarrow T'$, $t \circ s$ is $\infty$-directly deducible. Note that to say that $s : S \rightarrow T$ is $\infty$-deducible is the same as to say that there exists a doctrinal hull $\Gamma_{\mathcal{U} \upharpoonright \alpha}$ of $S$ such that $\Gamma_{\mathcal{U} \upharpoonright \alpha} \models s$.

The length of an $\infty$-directly deducible arrow $s$ is the least ordinal $\alpha$ for which there is an $\infty$-deduction $\mathcal{U}$ of length $\alpha$ with $\varphi_{\mathcal{U}} = s$.

**Lemma 7.** The length of an $\infty$-directly deducible arrow is either finite, or a limit ordinal.
Proof. First, we show that if \( \mathcal{U} = (\langle U_\beta \rangle_{\beta \leq \alpha + 1}, u_{\beta \gamma} : U_\beta \to U_\gamma, \beta \leq \gamma \leq \alpha + 1) \) is an \( \omega \)-deduction of length \( \alpha + 1 \), with \( \alpha \) a limit ordinal, then there is an \( \omega \)-deduction \( \mathcal{U}^* \) of length \( \alpha \) such that \( \varphi_\gamma = \varphi_\gamma^* \). A consequence will be that the length of an \( \omega \)-directly deducible arrow cannot be of the form \( \alpha + 1 \), with \( \alpha \) a limit ordinal.

Let \( \mathcal{U} \) be as stated. By assumption, there is a pushout square

\[
\begin{array}{ccc}
R & \xrightarrow{\psi_\alpha} & U_{\alpha+1} \\
\downarrow r & & \downarrow u_{\alpha,\alpha+1} \\
R' & \xrightarrow{\psi'_\alpha} & U_{\alpha+1}
\end{array}
\]

(7)

with \( r \in \mathbb{R} \). Since \( R \) is fp, and \( \alpha \) is a limit ordinal, there are \( \beta < \alpha \) and \( \psi_\beta : R \to U_\beta \) such that \( \psi_\alpha = u_{\alpha,\alpha+1} \circ \psi_\beta \). Let, for \( \gamma \) with \( \beta \leq \gamma \leq \alpha \), \( \psi_\gamma = u_{\beta,\gamma} \circ \psi_\beta \) (this is compatible with the previous notation for \( \gamma = \beta \) and \( \gamma = \alpha \)). Consider, for \( \beta \leq \gamma \leq \alpha \), the pushout

\[
\begin{array}{ccc}
R & \xrightarrow{\psi_\gamma} & U_{\gamma} \\
\downarrow r & & \downarrow u_{\beta,\gamma} \\
R' & \xrightarrow{\psi'_\gamma} & U_{\gamma}
\end{array}
\]

(8)

for \( \gamma = \alpha \), define (8) to be the already available (7). For \( \beta \leq \gamma \leq \delta \leq \alpha \), we have

\[
\begin{array}{ccc}
R & \xrightarrow{\psi_\gamma} & U_{\gamma} & \xrightarrow{u_{\beta,\gamma}} & U_{\delta} & \xrightarrow{u_{\delta,\alpha}} & U_{\alpha} \\
\downarrow r & & \downarrow v_\gamma & & \downarrow v_\delta & & \downarrow v_\alpha \\
R' & \xrightarrow{\psi'_\gamma} & U_{\gamma} & \xrightarrow{u_{\beta,\gamma}} & U_{\delta} & \xrightarrow{u_{\delta,\alpha}} & U_{\alpha}
\end{array}
\]

(9)

namely, the universal property of (8) gives \( u_{\beta,\gamma} \) to make the middle square (and the right-hand square) commute, and then it follows that the middle square (and the right-hand square) are pushouts, by the principle

I claim that \( \mathcal{U}' = (\langle U_{\gamma} \rangle_{\beta \leq \gamma \leq \alpha+1}, u_{\gamma,\delta} : U_\gamma \to U_\delta, \beta \leq \gamma \leq \delta \leq \alpha+1) \) is an \( \omega \)-deduction. It is clear that \( \mathcal{U}' \) is compatible. The fact that it is continuous follows from the fact that pushouts commute with filtered colimits. Finally, each \( u_{\gamma,\gamma+1} \) (\( \beta \leq \gamma \leq \gamma + 1 \leq \alpha \)) is immediately deducible, since it is (by (9)) a pushout of the immediately deducible arrow \( u_{\gamma,\gamma+1} \).

Consider the composite \( \omega \)-deduction \( \mathcal{U}^* = \mathcal{U}' \circ (v_\beta) \circ \mathcal{U} \mid \beta \); notice that \( v_\beta : U_\beta \to U_\beta \) gives a one-element deduction by (8) (for \( \gamma = \beta \)). \( \varphi_\gamma = u_{\beta,\gamma} \circ u_\beta = u_{\alpha,\alpha+1} \circ u_{\beta,\gamma} \circ u_\beta = \varphi_\gamma^* \), and the length of \( \mathcal{U}^* \) is \( \beta + 1 + (\alpha - \beta) = \beta + (1 + (\alpha - \beta)) = \beta + (\alpha - \beta) = \alpha \). We have shown our assertion.
Let \( x \) be the length of some \( \infty \)-directly deducible arrow. We show by induction on \( x \) that \( x \) is either finite, or a limit ordinal. If \( x = 0 \) or \( x \) is limit, there is nothing to show. Assume \( x = \beta + 1 \); let \( \vartheta \) be an \( \infty \)-deduction of length \( \beta + 1 \). If there were an \( \infty \)-deduction \( \gamma \) of length \( < \beta \) such that \( \varphi_{\gamma} = \varphi_{\vartheta} \), then \( \vartheta = \langle u_{\beta+1}, \gamma \rangle \) would be an \( \infty \)-deduction of length \( < \beta + 1 \) for which \( \varphi_{\gamma} = \varphi_{\vartheta} \), contradicting the minimality of \( x \). We obtained that \( \beta \) is the length of \( \varphi_{\vartheta} \). By induction hypothesis, therefore \( \beta \) is either finite or limit. In the first case, \( x = \beta + 1 \) is finite. In the second case, \( \beta + 1 \) is a length, and \( \beta \) is limit, in contradiction to our first-proved assertion. 

**Lemma 8.** Any relatively finite \( \infty \)-deducible entailment is deducible.

**Proof.** Suppose \( s: S \to T \) is an rf entailment. By induction on the ordinal \( x \), we show that if an \( \infty \)-directly deducible arrow of length \( x \) and with domain \( S \) factors through \( s \), then \( s \) is (finitely) deducible. By Lemma 7, \( x \) is either finite, or a limit ordinal. If \( x \) is finite, there is nothing to prove. If \( x \) is limit, \( \vartheta \) is an \( \infty \)-deduction as in (2), \( U_0 = S \), and

\[
\begin{array}{ccc}
S & \xrightarrow{u_{0x}} & U_x \\
\downarrow & \searrow \circ & \downarrow \tau \\
S & \xrightarrow{u_{00}} & U_0 \\
\end{array}
\]

then, since, in \( S \downarrow S \), the object \( (u_{00}: S \to U_0) \) is the (filtered) colimit of the objects \( (u_{0}\beta: S \to U_\beta)(\beta < x) \), with connecting arrows the \( u_{1}\beta \), and \( s \) as an object in \( S \downarrow S \) is fp, there is \( \beta < \gamma \) such that the arrow

\[
\tau: (s: S \to T) \to (u_{00}: S \to U_0)
\]

factors through

\[
u_{0}\beta: (u_{0}\beta: S \to U_\beta) \to (u_{00}: S \to U_0);
\]

there is \( \tau': (S \to T) \to (u_{0}: S \to U_\beta) \) with \( \tau = u_{0}\beta \cdot \tau' \). We can ignore the last equality; even without that, we have that

\[
\begin{array}{ccc}
S & \xrightarrow{u_{0}\beta} & U_\beta \\
\downarrow & \searrow \circ & \downarrow \tau' \\
S & \xrightarrow{u_{0}} & U_0 \\
\end{array}
\]

\( \vartheta \downarrow \beta \) is an \( \infty \)-deduction of length \( \leq \beta < x \); by the induction hypothesis, \( s \) is deducible.

**Proof of Theorem 3.** Suppose the relatively finite entailment \( s: S \to T \) is valid. Then, for some (any) (see Corollary 6) doctrinal hull \( \vartheta \) of \( S \), \( \vartheta \models s \). Since \( \vartheta_S \) is in particular an \( \infty \)-deduction, \( s \) is \( \infty \)-deducible. By Lemma 8, it is deducible. 

For later reference, we summarize some properties of the doctrinal hull (see Corollary 6).
Proposition 9. Let $\mathcal{U}^\ast$ be a doctrinal hull of the sketch $S$, with $\varphi_y : S \to \mathcal{U}^\ast$ the canonical map. Then:

(i) for any regular $\kappa > \lambda_{S,A}$, if $S$ is $\kappa$-presentable, so is $\mathcal{U}^\ast$.

(ii) $\varphi_y$ has the following mapping property: for any $A \in S : \mathcal{R}$, any $\varphi : S \to A$ factors through $\varphi_y$; that is, there is $\psi : \mathcal{U}^\ast \to A$ such that $\varphi = \psi \circ \varphi_y$.

Proof. $\mathcal{U}^\ast$ is the colimit of a maximal proper $\infty$-deduction (2), $S = U_0$, and $\varphi_y$ is $u_{0\alpha} : U_0 \to U_z = \mathcal{U}^\ast$. Assume that $S$ is $\kappa$-presentable; $\kappa \geq \kappa_S$ (see before Proposition 4). In the proof of Theorem 3, we noted that each $U_\beta (\beta < \alpha)$ is $\kappa$-presentable ($\kappa$ loc. cit. is $\kappa_S$; but $\kappa \geq \kappa_S$). By Proposition 4, $\alpha < \kappa$; as a colimit of a diagram of size $< \kappa$ of $\kappa$-presentable objects, $\mathcal{U}^\ast$ is $\kappa$-presentable. This shows (i).

To show (ii), for a given $\varphi : S \to A$, by recursion on $\gamma \leq \alpha$, we construct $\psi_\gamma : U_\gamma \to A$ such that $\psi = \psi_0$, and $\psi_\gamma \circ u_{\beta \gamma} = \psi_\beta$ whenever $\beta < \gamma \leq \alpha$. When $\gamma$ is a limit ordinal, $\psi_\gamma$ is uniquely given from the $\psi_\beta, \beta < \gamma$, by the universal property of $U_\gamma$ as the colimit of the $U_\beta, \beta < \gamma$. When $\gamma = \beta + 1$, there is a pushout

$$
\begin{array}{ccc}
R & \xrightarrow{r} & R' \\
\downarrow \theta & & \downarrow \\
U_\gamma & \xrightarrow{u_{\beta \gamma}} & U_\gamma \\
\end{array}
$$

with $x \in \mathcal{R}$. Since $A \models r$, there is $\tau : R' \to A$ with $\psi_\beta \circ \theta = \tau \circ r$. By the universal property of the pushout $U_\gamma$, there is $\psi_\gamma : U_\gamma \to A$ with (among others) $\psi_\gamma \circ u_{\beta \gamma} = \psi_\beta$. It follows that $\psi_\gamma \circ u_{\beta \gamma} = \psi_\beta$, for all $\beta < \gamma$.

For $\gamma = \alpha$, $\psi_\alpha$ is the desired $\psi$. □

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References