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An iterative–bijective approach to generalizations of Schur's theorem

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Abstract

We start with a bijective proof of Schur's theorem due to Alladi and Gordon and describe how a particular iteration of it leads to some very general theorems on colored partitions. These theorems imply a number of important results, including Schur's theorem, Bressoud's generalization of a theorem of Göllnitz, two of Andrews' generalizations of Schur's theorem, and the Andrews–Olsson identities.

MSC: 11P81; 05A17

1. Introduction

In 1926 Schur [16] proved that the number of partitions of m into distinct parts not divisible by 3 is equal to the number of partitions of m where parts differ by at least 3 and multiples of 3 differ by at least 6. Over the years there have been a number of proofs of this theorem (e.g. [2,4,6,9,11,14]), including a few delightfully simple combinatorial arguments [2,11,14]. In [2], Alladi and Gordon showed how to deduce Schur's theorem

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from an interpretation of the infinite product

$$\prod_{k=1}^{\infty}(1+y_1q^k)(1+y_2q^k) \tag{1.1}$$

as a generating function for partitions whose parts come in three colors:

Theorem 1.1 (*Alladi–Gordon*). *The number of pairs of partitions* (μ_1, μ_2), *where* μ_r *is a partition into* x_r *distinct parts and the sum of all of the parts is* m, *is equal to the number of partitions of* m *into distinct parts occurring in three colors (labelled* 1, 2, *and* 3*) such that* (i) *the part* 1 *does not occur in color* 3, (ii) *consecutive parts differ by at least* 2 *if the larger has color* 3 *or if the larger has color* 1 *and the smaller has color* 2, (iii) x_j *is the number of parts with color* j *plus the number of parts with color* 3.

Among their proofs of this theorem is an attractive bijective argument which adapts some ideas of Bressoud [13]. In this paper we describe a particular iteration of this bijection, which leads an interpretation of the infinite product

$$\prod_{k=1}^{\infty}(1+y_1q^k)(1+y_2q^k)\cdots(1+y_nq^k) \tag{1.2}$$

as a generating function for certain partitions where the parts come in $2^n - 1$ colors. To state the theorems, we require some notation. For t-colored partitions, we denote the t colors by the natural numbers $(1, \ldots, t)$, with the parts ordered first according to size and then according to color. We write $\omega(c)$ for the number of powers of 2 occurring in the binary representation of c, and $v(c)$ (resp. $z(c)$) for the smallest (resp. largest) power of 2 occurring in this representation. The function $\delta(c, d)$ is equal to 1 if $z(c) < v(d)$ and is 0 otherwise. Finally, we use c_i to denote the color of a part λ_i.

Theorem 1.2. *Let* $A(x_1, x_2, \ldots, x_n; m)$ *denote the number of n-tuples* $(\mu_1, \mu_2, \ldots, \mu_n)$, *where* μ_r *is a partition into* x_r *distinct parts, and the sum of all of the parts is* m. *Let* $B(x_1, x_2, \ldots, x_n; m)$ *denote the number of partitions* $\lambda_1 + \cdots + \lambda_s$ *of* m *into distinct parts occurring in* $2^n - 1$ *colors, where* (i) $\lambda_s \geq \omega(c_s)$, (ii) x_r *of the colors* c_i *have* 2^{r-1} *in their binary representations, and* (iii) $\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta(c_i, c_{i+1})$. *Then* $A(x_1, x_2, \ldots, x_n; m) = B(x_1, x_2, \ldots, x_n; m)$.

For example, using the notation $(\lambda_1, \ldots, \lambda_s)$ to represent the sum $\lambda_1 + \cdots + \lambda_s$ and the symbol ϵ to denote the empty partition, the 4-tuples of partitions counted by $A(2, 0, 1, 1; 9)$ are

$$((6,1),\epsilon,1,1),\ ((5,2),\epsilon,1,1),\ ((4,3),\epsilon,1,1),$$
$$((5,1),\epsilon,2,1),\ ((5,1),\epsilon,1,2),\ ((4,2),\epsilon,2,1),$$
$$((4,2),\epsilon,1,2),\ ((4,1),\epsilon,3,1),\ ((4,1),\epsilon,2,2),\ ((4,1),\epsilon,1,3)$$
$$((3,2),\epsilon,3,1),\ ((3,2),\epsilon,2,2),\ ((3,2),\epsilon,1,3)((3,1),\epsilon,4,1),$$
$$((3,1),\epsilon,3,2),\ ((3,1),\epsilon,2,3),\ ((3,1),\epsilon,1,4),\ ((2,1),\epsilon,5,1),$$
$$((2,1),\epsilon,4,2),\ ((2,1),\epsilon,3,3),\ ((2,1),\epsilon,2,4),\ ((2,1),\epsilon,1,5)$$

and the partitions counted by $B(2, 0, 1, 1; 9)$ are

$$(8_{13}, 1_1), (7_{13}, 2_1), (7_9, 2_5), (7_5, 2_9), (6_{13}, 3_1), (6_9, 3_5), (6_5, 3_9),$$
$$(6_1, 3_{13}), (5_1, 4_{13}), (6_9, 2_4, 1_1), (6_{12}, 2_1, 1_1), (6_5, 2_8, 1_1),$$
$$(5_9, 3_4, 1_1), (5_4, 3_9, 1_1), (5_8, 3_5, 1_1), (5_9, 3_1, 1_4), (5_1, 3_9, 1_4),$$
$$(5_5, 3_1, 1_8), (5_{12}, 3_1, 1_1), (5_1, 3_{12}, 1_1), (4_8, 3_1, 2_5), (4_4, 3_1, 2_9).$$

Actually, a closer look at the proof of Theorem 1.2 will reveal that it can be extended by letting μ_1 be either a partition into distinct parts congruent to R modulo M or a partition into parts that differ by at least M. These cases correspond to the products

$$\prod_{k=1}^{\infty}(1 + y_1 q^{(k-1)M+R})(1 + y_2 q^k) \cdots (1 + y_n q^k) \tag{1.3}$$

and

$$\sum_{k=0}^{\infty} \frac{y_1^k q^{M(k(k-1)/2)+k}}{(1-q)(1-q^2)\cdots(1-q^k)} \prod_{k=1}^{\infty}(1 + y_2 q^k) \cdots (1 + y_n q^k). \tag{1.4}$$

Let $\omega_e(c)$ denote the number of even powers of 2 in the binary representation of c.

Theorem 1.3. *Let $A_{R,M}(x_1, x_2, \ldots, x_n; m)$ denote the number of n-tuples $(\mu_1, \mu_2, \ldots, \mu_n)$, where each μ_r is a partition into x_r distinct parts, μ_1 is a partition into distinct parts congruent to R modulo M, and the sum of all of the parts is m. Let $B_{R,M}(x_1, \ldots, x_n; m)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of m counted by $B(x_1, \ldots, x_n; m)$ such that* (i) *if λ_S denotes the smallest part with odd color, then $\lambda_S \equiv R + \sum_{\ell=S}^{s} \omega_e(c_\ell) \pmod{M}$ and* (ii) *if $\lambda_i \geq \lambda_j$ are any two parts with odd color, then $\lambda_i \equiv \lambda_j + \sum_{\ell=i}^{j-1} \omega_e(\lambda_\ell) \pmod{M}$. Then $A_{R,M}(x_1, x_2, \ldots, x_n; m) = B_{R,M}(x_1, x_2, \ldots, x_n; m)$.*

Theorem 1.4. *Let $A_M(x_1, x_2, \ldots, x_n; m)$ denote the number of n-tuples $(\mu_1, \mu_2, \ldots, \mu_n)$, where each μ_r is a partition into x_r distinct parts and μ_1 is a partition into parts differing by at least M, and the sum of all of the parts is m. Let $B_M(x_1, x_2, \ldots, x_n; m)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of m counted by $B(x_1, \ldots, x_n; m)$ such that if $\lambda_i \geq \lambda_j$ are any two parts with odd color, then $\lambda_i - \lambda_j \geq M + \sum_{\ell=i}^{j-1} \omega_e(\lambda_\ell)$. Then $A_M(x_1, x_2, \ldots, x_n; m) = B_M(x_1, x_2, \ldots, x_n; m)$.*

Theorems 1.2–1.4 are closely related to a number of important results in the theory of partitions. For example, by appropriately defining a conjugation on the partitions counted by $B(x_1, x_2, \ldots, x_n; m)$ we will arrive at Theorem 1.5 below, which is a generalization of the Andrews–Olsson identities [10] and which was proven by Bessenrodt [12, Theorem 2.4, $\mathcal{C}' = \emptyset$] and stated by Alladi [1, Theorem 15]. Here we shall use uncolored parts as well as colored parts, assuming that an uncolored part of a given size occurs before all other parts of that size.

Theorem 1.5. *Let $C(x_1, x_2, \ldots, x_n; m)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of m into distinct parts occurring either in n colors or uncolored, where* (i) *the smallest part is colored,* (ii) *x_r of the parts have color r, and* (iii) *$\lambda_i - \lambda_{i+1} \leq 1$, with strict inequality* *if* $c_{i+1} < c_i$ *or* λ_i *is uncolored. Then* $A(x_1, x_2, \ldots, x_n; m) = B(x_1, x_2, \ldots, x_n; m) = C(x_1, x_2, \ldots, x_n; m)$.

Theorem 1.5 and some of the many other partition theorems contained in Theorems 1.2–1.4 are discussed in more detail in Section 5. In the following section we review the Alladi–Gordon bijective proof of Theorem 1.1 and explain our proof of Theorem 1.2 in the case $n = 3$. In Section 3 we undertake the proof of Theorem 1.2 in full generality and in Section 4 we prove the extensions, Theorems 1.3 and 1.4.

2. Two basic cases

Although the basic idea behind the proofs of Theorems 1.2–1.4 is a simple one, the amount of notation required may obscure this fact. Therefore, we shall present the cases $n = 2$ and 3 in detail. First, we review the Alladi–Gordon bijective proof of Theorem 1.1, which is the case $A(x_1, x_2; m) = B(x_1, x_2; m)$ of Theorem 1.2. We begin with a partition λ into x_1 distinct parts colored by 1 and a partition τ into x_2 distinct parts colored by 2.

• **Step 1.** For each part k of τ that is less than or equal to the number of parts of λ, we add 1 to the first k parts of λ and 2 to the color of λ_k. We then have the difference conditions

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i),$$

for $c_i, c_{i+1} \neq 2$. Notice that all parts with color 3 are bigger than 1.

• **Step 2.** Now write the unused parts of τ in decreasing order to the left of the parts from λ. Remove a staircase, i.e., subtract 0 from the smallest part, 1 from the next smallest, and so on. We therefore get the difference conditions

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i) - 1$$

for $c_i, c_{i+1} \neq 2$.

• **Step 3.** Each part τ_j of color 2 remaining in τ is inserted in λ after the smallest part that is bigger than τ_j. We now have the difference conditions

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} \omega(c_i) - 1, & c_i, c_{i+1} = 1, 3, \\ 0, & c_i = 2, \\ 1, & c_i = 1, 3,\ c_{i+1} = 2. \end{cases}$$

• **Step 4.** In each case above, the minimum difference is exactly $\omega(c_i) + \delta(c_i, c_{i+1}) - 1$. We add back the staircase removed in Step 2 and we have

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta(c_i, c_{i+1}).$$

This is condition (ii) of Theorem 1.1. Conditions (i) and (iii) are straightforward.

For example, starting with $\lambda = (8_1, 3_1, 2_1, 1_1)$ and $\tau = (10_2, 5_2, 3_2, 2_2)$, we perform the steps of the bijection:

$$\begin{aligned} (\tau, \lambda) &\iff ((10_2, 5_2), (10_1, 5_3, 3_3, 1_1)) \quad \text{(Step 1)} \\ &\iff ((5_2, 1_2), (7_1, 3_3, 2_3, 1_1)) \quad \text{(Step 2)} \\ &\iff (\epsilon, (7_1, 5_2, 3_3, 2_3, 1_2, 1_1)) \quad \text{(Step 3)} \\ &\iff (12_1, 9_2, 6_3, 4_3, 2_2, 1_1) \quad \text{(Step 4).} \end{aligned}$$

Now, since the result of the above process is another partition into distinct parts, it is natural to attempt to apply the bijection again, starting with the partition λ into distinct parts having three colors satisfying $\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta(c_i, c_{i+1})$ and $\lambda_i \geq \omega(c_i)$ and a new partition, τ, into distinct parts occurring in the color 4. Let us see what happens when we try to repeat the steps above.

• **Step 1.** For each part k of τ that is less than or equal to the number of parts of λ, we add 1 to the first k parts of λ and 4 to the color of λ_k. We then have the difference conditions

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta^*(c_i, c_{i+1}),$$

and $\lambda_i \geq \omega(c_i)$ for $c_i, c_{i+1} \neq 4$. Here $\delta^*(c_i, c_{i+1}) = \delta(c_i, c_{i+1})$ if $c_i < 4$ and $\delta(c_i - 4, c_{i+1})$ otherwise.

• **Step 2.** Now write the unused parts of τ to the left of the parts from λ. Remove a staircase, i.e. subtract 0 from the smallest part, 1 from the next smallest, and so on. We get

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta^*(c_i, c_{i+1}) - 1,$$

for $c_i, c_{i+1} \neq 4$.

• **Step 3.** Now we take the largest part τ_1 of τ and insert it into λ after the smallest part that is bigger than τ_1. It is easy to check that after such an insertion, we have

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} \omega(c_i) - \delta^*(c_i, c_{i+1}) - 1, & c_i, c_{i+1} \neq 4 \\ 0 = \omega(c_i) - \delta(c_i, c_{i+1}) - 1, & c_i = 4 \\ 1 = \omega(c_i) - \delta(c_i, c_{i+1}) - u(c_i) - 1, & c_i \neq 4,\ c_{i+1} = 4. \end{cases} \tag{2.5}$$

Here $u(j) = 1$ if $i = 3$ or 7, and $u(j) = 0$ otherwise.

Unlike the situation for $n = 2$, we do not always have the condition

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta(c_i, c_{i+1}) - 1. \tag{2.6}$$

There are four cases where (2.5) and (2.6) do not match up: (i) if $c_i = 7$ and $c_{i+1} = 4$, the minimal difference between λ_i and λ_{i+1} is $1 = \omega(c_i) + \delta(c_i, c_{i+1}) - 2$; (ii) when $c_i = 5$ and $c_{i+1} = 6$, this difference is $2 = \omega(c_i) + \delta(c_i, c_{i+1})$; (iii) for $c_i = 3$ and $c_{i+1} = 4$, it is $1 = \omega(c_i) + \delta(c_i, c_{i+1}) - 2$; and (iv) for $c_i = 5$ and $c_{i+1} = 2$, it is $2 = \omega(c_i) + \delta(c_i, c_{i+1})$.

In the second and fourth cases, the guaranteed minimum difference between λ_i and λ_{i+1} is too large. In the first and third cases, it is too small. For the latter cases, we shall remedy the problem by redistributing the powers of 2 occurring in the colors c_i and c_{i+1}. Specifically, if $c_i = 7$, $c_{i+1} = 4$, and $\lambda_i - \lambda_{i+1} = 1$, then let $c_i = 5$ and $c_{i+1} = 6$. If $c_i = 3$, $c_{i+1} = 4$, and $\lambda_i - \lambda_{i+1} = 1$, then let $c_i = 5$ and $c_{i+1} = 2$. Notice that the new colors correspond exactly to those in cases (ii) and (iv) above, with the difference between λ_i and λ_{i+1} exactly one less than the minimum difference in (2.5) corresponding to these two cases. This is a double bonus. First, it makes the change of colors described above bijective, and second, it makes the minimum difference what we want for the theorem.

We repeat Step 3 with the largest part remaining in τ and continue until all the parts of color 4 are inserted in λ. We then have (2.6) for all i.

• **Step 4.** Finally, we can add back the staircase and we have

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta(c_i, c_{i+1}).$$

This is condition (iii) of Theorem 1.2. Conditions (i) and (ii) are again straightforward.

For example, we start with $\lambda = (12_1, 9_2, 6_3, 4_3, 2_2, 1_1)$ and $\tau = (17_4, 11_4, 8_4, 6_4, 3_4, 1_4)$, following the steps of the bijection:

$$\begin{aligned}
(\tau, \lambda) &\iff ((17_4, 11_4, 8_4), (15_5, 11_2, 8_7, 5_3, 3_2, 2_5)) \quad \text{(Step 1)} \\
&\iff ((9_4, 4_4, 2_4), (10_5, 7_2, 5_7, 3_3, 2_2, 2_5)) \quad \text{(Step 2)} \\
&\iff ((4_4, 2_4), (10_5, 9_4, 7_2, 5_7, 3_3, 2_2, 2_5)) \quad \text{(Step 3)} \\
&\iff ((2_4), (10_5, 9_4, 7_2, \underbrace{5_7, 4_4}, 3_3, 2_2, 2_5)) \\
&\iff ((2_4), (10_5, 9_4, 7_2, 5_5, 4_6, 3_3, 2_2, 2_5)) \\
&\iff (\epsilon, (10_5, 9_4, 7_2, 5_5, 4_6, \underbrace{3_3, 2_4}, 2_2, 2_5)) \\
&\iff (\epsilon, (10_5, 9_4, 7_2, 5_5, 4_6, 3_5, 2_2, 2_2, 2_5)) \\
&\iff (18_5, 16_4, 13_2, 10_5, 8_6, 6_5, 4_2, 3_2, 2_5) \quad \text{(Step 4).}
\end{aligned}$$

Here the underbraces indicate where a reassignment of colors needs to take place.

The proof of Theorem 1.2 is an iteration of the above process, described for a general n in the following section.

3. Proof of Theorem 1.2

The proof is by induction. For $n = 1$ the theorem is a tautology. Now suppose that it is true for a given natural number $n - 1$ and that we can map any partition counted by $A(x_1, x_2, \ldots, x_{n-1}; m)$ to a partition counted by $B(x_1, x_2, \ldots, x_{n-1}; m)$. Take a partition counted by $A(x_1, x_2, \ldots, x_n; m)$ and break it into two partitions: the first is a partition counted by $A(x_1, x_2, \ldots, x_{n-1}; m')$ and the second is a partition of $m - m'$ into x_n distinct parts. We apply the map to the first partition to get a partition λ counted by $B(x_1, x_2, \ldots, x_{n-1}; m')$. We call the second partition τ and color its parts with the color 2^{n-1}.

• **Step 1.** First, we change λ in the following way: for each part k of τ that is less than or equal to the number of parts of λ, add 1 to each of the first k parts of λ and then add 2^{n-1} to the *color* of the kth part. Here we record that we have

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} \omega(c_i) + \delta(c_i, c_{i+1}), & \text{if } c_i < 2^{n-1}, \\ \omega(c_i) + \delta(c_i - 2^{n-1}, c_{i+1}), & \text{if } c_i > 2^{n-1}. \end{cases}$$

and that $\lambda_i \geq \omega(c_i)$ for $c_i \neq 2^{n-1}$. To be concise, we shall write $\delta^*(c_i, c_{i+1})$ to mean $\delta(c_i, c_{i+1})$ when $c_i < 2^{n-1}$ and $\delta(c_i - 2^{n-1}, c_{i+1})$ when $c_i > 2^{n-1}$.

• **Step 2.** Now write the unused parts from τ in descending order to the left of the parts from λ. Remove a staircase, i.e., subtract 0 from the smallest part, 1 from the next smallest, and so on. Here we record that we have

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta^*(c_i, c_{i+1}) - 1 \tag{3.1}$$

for $c_i \neq 2^{n-1}$.

• **Step 3.** Starting from the largest part k with color 2^{n-1}, we insert k into the partition λ as the part λ_i so that $\lambda_i - \lambda_{i+1} \geq 0$ with i minimal. Since $\omega(c_i) = \omega(2^{n-1}) = 1$ and $\delta(2^{n-1}, c_{i+1}) = 0$, this condition is the same as $\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta(c_i, c_{i+1}) - 1$. The minimality of i guarantees that $\lambda_{i-1} - \lambda_i \geq 1$. Hence it is possible that

$$\lambda_{i-1} - \lambda_i < \omega(c_{i-1}) + \delta(c_{i-1}, c_i) - 1, \tag{3.2}$$

and in this case we shall execute a redistribution of colors between λ_i and λ_{i-1}. Specifically, if $\lambda_{i-1} - \lambda_i = j$, where $j \geq 1$ and $\omega(c_{i-1}) > 1 + j - \delta(c_{i-1}, c_i)$, we form two new parts $\tilde{\lambda}_{i-1}$ and $\tilde{\lambda}_i$ with colors $\tilde{c}_{i-1}$ and $\tilde{c}_i$ by taking the first j smallest powers of two from the color c_{i-1}, adding them to the color 2^{n-1} to get the color $\tilde{c}_{i-1}$, and letting $\tilde{c}_i$ be what is left of c_{i-1}.

Some comments on this change of colors are in order. First, note that $\tilde{c}_i \neq 2^{n-1}$, $z(\tilde{c}_i) = z(c_{i-1})$, $v(\tilde{c}_{i-1}) = v(c_{i-1})$, and $\delta(\tilde{c}_{i-1} - 2^{n-1}, \tilde{c}_i) = 1$. Second,

$$\begin{aligned}\tilde{\lambda}_{i-1} - \tilde{\lambda}_i &= j \\ &= \omega(\tilde{c}_{i-1}) + \delta(\tilde{c}_{i-1}, \tilde{c}_i) - 1,\end{aligned}$$

since the fact that $\tilde{c}_{i-1} > 2^{n-1}$ implies that $\delta(\tilde{c}_{i-1}, \tilde{c}_i) = 0$. However, as $\delta(\tilde{c}_{i-1} - 2^{n-1}, \tilde{c}_i) = 1$, this difference j between $\tilde{\lambda}_{i-1}$ and $\tilde{\lambda}_i$ is one less than the minimum difference guaranteed by the second case of (3.1), which makes the change of colors bijective—one can always identify when it has taken place. We are also guaranteed that any further occurrences of k in color 2^{n-1} may now be inserted without any problem, as $\tilde{c}_{i-1} > 2^{n-1}$.

Next, since we were in the case of (3.2) after having inserted the part k as λ_i, we may observe that c_{i-1} is not 2^{n-1}. Moreover $c_{i+1} = 2^{n-1}$ would contradict the fact that we start with the largest part k. So we had $\lambda_{i-1} - \lambda_{i+1} \geq \omega(c_{i-1}) + \delta^*(c_{i-1}, c_{i+1}) - 1$ according to (3.1). Hence we may deduce that

$$\begin{aligned}\tilde{\lambda}_i - \lambda_{i+1} &= \tilde{\lambda}_i - \lambda_{i-1} + \lambda_{i-1} - \lambda_{i+1} \\ &\geq \omega(c_{i-1}) - j + \delta^*(c_{i-1}, c_{i+1}) - 1 \\ &= \omega(\tilde{c}_i) + \delta^*(\tilde{c}_i, c_{i+1}) - 1,\end{aligned}$$

as $z(\tilde{c}_i) = z(c_{i-1})$ and $\tilde{c}_i \neq 2^{n-1}$.

Note also that there is no change in the required difference between λ_{i-2} and $\tilde{\lambda}_{i-1}$ because $\tilde{\lambda}_{i-1} = \lambda_{i-1}$ and $v(\tilde{c}_{i-1}) = v(c_{i-1})$ implies that $\delta(c_{i-2}, c_{i-1}) = \delta(c_{i-2}, \tilde{c}_{i-1})$.

We continue this procedure until all the parts of color 2^{n-1} are inserted in λ.

• **Step 4.** Now all the required differences are

$$\lambda_i - \lambda_{i+1} \geq \omega(c_i) + \delta(c_i, c_{i+1}) - 1, \tag{3.3}$$

and we can add back the staircase $0, 1, 2, \ldots$ so that these difference conditions become those of condition (iii) in the theorem. Conditions (i) and (ii) are straightforward. This establishes that $A(x_1, \ldots, x_n; m) = B(x_1, \ldots, x_n; m)$. □

Remark. It will prove useful to take advantage of the symmetry in the partitions counted by $A(x_1, \ldots, x_n; m)$ and $B(x_1, \ldots, x_n; m)$ to slightly extend Theorem 1.2. Given a permutation $\sigma \in S_n$, take a partition λ counted by $B(x_1, \ldots, x_n; m)$ and create a new partition $\tilde{\lambda}$ by setting $\tilde{\lambda}_i = \lambda_i$ and changing c_i to $\sigma(c_i)$. Here $\sigma(c_i)$ is defined in the obvious way, by permuting the powers of 2 occurring in the binary representation of c_i. Now $\omega(c_i)$ has not changed so the new difference condition on $\tilde{\lambda}$ is

$$\tilde{\lambda}_i - \tilde{\lambda}_{i+1} \geq \omega(\tilde{c}_i) + \delta(\sigma^{-1}(\tilde{c}_i), \sigma^{-1}(\tilde{c}_{i+1})). \tag{3.4}$$

The mapping $\lambda \rightarrow \tilde{\lambda}$ is easily reversible so we have $B(x_1, \ldots, x_n; m) = B_\sigma(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; m)$, where $B_\sigma(x_1, \ldots, x_n; m)$ denotes the number of partitions λ of m into parts that come in $2^n - 1$ colors and satisfy conditions (i) and (ii) of Theorem 1.2 as well as (3.4). From the definition of $A(x_1, \ldots, x_n; m)$, it is obvious that for any permutation $\tau = (\tau(1), \ldots, \tau(n))$ in S_n,

$$A(x_1, \ldots, x_n; m) = A(x_{\tau(1)}, \ldots, x_{\tau(n)}; m).$$

Hence we have:

Corollary 3.1. $B_\sigma(x_{\tau(1)}, \ldots, x_{\tau(n)}; m) = A(x_1, \ldots, x_n; m)$.

To conclude this section we provide yet another example of the proof that $A(x_1, \ldots, x_n; m) = B(x_1, \ldots, x_n; m)$, this time in the case $n = 4$. We start with a partition $\lambda = (16_3, 14_7, 11_5, 8_1, 6_2, 5_7, 1_1)$ counted by $B(6, 4, 3; 61)$ and a partition $\tau = (22_8, 19_8, 18_8, 11_8, 7_8, 4_8, 2_8, 1_8)$ of 84 into 8 distinct parts of color 8. Then we follow the steps of the bijection:

$$\begin{aligned}
(\tau, \lambda) &\iff ((22_8, 19_8, 18_8, 11_8), (20_{11}, 17_{15}, 13_5, 10_9, 7_2, 6_7, 2_9)) \quad \text{(Step 1)}\\
&\iff ((12_8, 10_8, 10_8, 4_8), (14_{11}, 12_{15}, 9_5, 7_9, 5_2, 5_7, 2_9)) \quad \text{(Step 2)}\\
&\iff ((10_8, 10_8, 4_8), (14_{11}, 12_8, 12_{15}, 9_5, 7_9, 5_2, 5_7, 2_9)) \quad \text{(Step 3)}\\
&\iff ((10_8, 4_8), (14_{11}, 12_8, \underbrace{12_{15}, 10_8}, 9_5, 7_9, 5_2, 5_7, 2_9))\\
&\iff ((10_8, 4_8), (14_{11}, 12_8, 12_{11}, 10_{12}, 9_5, 7_9, 5_2, 5_7, 2_9))\\
&\iff ((4_8), (14_{11}, 12_8, 12_{11}, 10_8, 10_{12}, 9_5, 7_9, 5_2, 5_7, 2_9))\\
&\iff (\epsilon, (14_{11}, 12_8, 12_{11}, 10_8, 10_{12}, 9_5, 7_9, 5_2, \underbrace{5_7, 4_8}, 2_9))\\
&\iff (\epsilon, (14_{11}, 12_8, 12_{11}, 10_8, 10_{12}, 9_5, 7_9, 5_2, 5_9, 4_6, 2_9))\\
&\iff (24_{11}, 21_8, 20_{11}, 17_8, 16_{12}, 14_5, 11_9, 8_2, 7_9, 5_6, 2_9) \quad \text{(Step 4)}.
\end{aligned}$$

Notice that the result is a partition counted by $B(6, 4, 3, 8; 145)$, as expected.

4. Proofs of Theorems 1.3 and 1.4

The proofs of these theorems follow exactly the same steps as the proof of Theorem 1.2, but here we will pay special attention to the behavior of the parts with odd color. We begin with Theorem 1.3. When $n = 1$, we just have a partition μ_1 into parts having color 1 that are congruent to r modulo M, and the conditions (i) and (ii) of the theorem are trivial. Now suppose that the theorem is true for $n - 1$, let λ be a partition counted by $B_{r,M}(x_1, \ldots, x_{n-1}; m')$, and let τ be a partition of $m - m'$ into distinct parts, all with the color 2^{n-1}. As we apply the bijection, the part λ_S will increase by 1 each time its color or the color of a smaller part increases by 2^{n-1} in Step 1. It will also ultimately increase by 1 in Step 4 if a part of color 2^{n-1} is inserted as a part λ_{S+k} in Step 3. But in these cases

$\sum_{\ell=S}^{s} \omega_e(c_\ell)$ also increases by 1. If a redistribution of colors should take place between λ_S and λ_{S+1}, then $\tilde{\lambda}_S$ remains the smallest part with odd color, and $\sum_{\ell=S}^{s} \omega_e(c_\ell)$ does not change. Hence we have condition (i) of Theorem 1.3.

For condition (ii), the difference between two parts λ_i and λ_j with odd color will increase by 1 every time in Step 1 that the color of λ_k increases by 2^{n-1} for $i \leq k < j$. This difference will also ultimately increase by 1 in Step 4 for each time that a part with color 2^{n-1} is inserted between λ_i and λ_j in Step 3. But in these cases $\sum_{\ell=i}^{j-1} \omega_e(\lambda_\ell)$ will also increase by 1. If any part λ_i with odd color is affected by a rearrangement of colors, $\tilde{\lambda}_i$ remains odd. If λ_j has odd color with $j > i$, then $\sum_{\ell=i}^{j-1} \omega_e(\lambda_\ell)$ is not affected by this rearrangement, and if $j < i$, neither is $\sum_{\ell=j}^{i-1} \omega_e(\lambda_\ell)$. This guarantees condition (ii) and completes the proof of this part of the theorem. □

We turn to Theorem 1.4. When $n = 1$, we just have a partition μ_1 into parts having color 1 that differ by at least M, and the extra condition of the theorem is trivial. Now suppose that the theorem is true for $n - 1$, let λ be a partition counted by $B_M(x_1, \ldots, x_{n-1}; m')$, and let τ be a partition of $m - m'$ into distinct parts, all with the color 2^{n-1}. As we apply the steps of the bijection, the difference between any two parts with odd color λ_i and λ_j increases by 1 for every time in Step 1 that the color of λ_k increases by 2^{n-1} for $i \leq k < j$. This difference will also ultimately increase by 1 in Step 4 for each time that a part with color 2^{n-1} is inserted between λ_i and λ_j in Step 3. But these are precisely the cases where $\sum_{\ell=i}^{j-1} \omega_e(c_\ell)$ increases by 1. If any part λ_i with odd color is affected by a rearrangement of colors, $\tilde{\lambda}_i$ remains odd. If λ_j has odd color with $j < i$, then $\lambda_j - \tilde{\lambda}_i$ is not affected by this rearrangement, and neither is $\sum_{\ell=j}^{i-1} \omega_e(c_\ell)$. If $j > i$, then both $\tilde{\lambda}_i - \lambda_j$ and $\sum_{\ell=i}^{j-1} \omega_e(c_\ell)$ are unchanged. This guarantees the extra condition and completes the proof of this part of the theorem. □

5. Partition identities

We begin our discussion of partition identities by proving Theorem 1.5.

Proof of Theorem 1.5. For each part λ_i of a partition counted by $B(x_1, x_2, \ldots, x_n; m)$ with color $c_i = 2^{j_1} + \cdots + 2^{j_k}$ with $j_1 < \cdots < j_k$, we draw its Ferrers diagram, that is we write a row of λ_i boxes and we add subscripts on the last k boxes. Specifically, the last box has subscript $j_1 + 1$, the next to last has $j_2 + 1$, and so on. Conjugating this diagram and interpreting a subscript as the color of a column gives a partition counted by $C(x_1, x_2, \ldots, x_n; m)$. □

Example. Let us take $n = 3$ and $\lambda = (10_7, 6_5, 4_1, 1_2)$ in $B(3, 2, 2; 21)$. The Ferrers diagram with the subscripts is

$$
\begin{array}{llllllllll}
\square & \square & \square & \square & \square & \square & \square & \square_3 & \square_2 & \square_1 \\
\square & \square & \square & \square & \square_3 & \square_1 \\
\square & \square & \square & \square_1 \\
\square_2
\end{array}
$$

Now we read the columns and get $(4_2, 3, 3, 3_1, 2_3, 2_1, 1, 1_3, 1_2, 1_1)$ which is in $C(3, 2, 2; 21)$.

We note, as was done in [1, p. 25], that making the substitutions $q \to q^N$ and $y_j \to q^{a_j - N}$ in (1.2) and applying Theorem 1.5 gives the Andrews–Olsson identities referred to in the introduction:

Theorem 5.1 (*Andrews–Olsson*). *Let N be a positive integer and let $A = \{a_1, a_2, \ldots, a_n\}$ be a set of distinct positive integers arranged in increasing order with $a_n < N$. Let $P_1(A; N; m)$ denote the number of partitions of m into distinct parts each congruent to some a_i (mod N). Let $P_2(A; N; m)$ denote the number of partitions of m into parts $\equiv 0$ or some a_i modulo N such that only parts divisible by N may repeat, the smallest part is less than N, and the difference between parts is $\leq N$, with strict inequality if either part is divisible by N. Then $P_1(A; N; m) = P_2(A; N; m)$.*

We also note, before continuing, that there are conjugate versions of Theorems 1.3 and 1.4 as well, the extra conditions on the differences between parts of odd color translating under conjugation to conditions on the number of parts occurring between two parts of color 1.

Next we discuss how a theorem of Bressoud [13], which generalizes some results of Göllnitz [15], is contained in Theorem 1.3.

Theorem 5.2 (*Bressoud*). *Given positive integers n, k, and r satisfying $1 \leq r < 2k$ and $r \neq k$, let $G_{r,k}(n)$ denote the number of partitions of n into distinct parts congruent to r, k, or $2k$ modulo $2k$, and let $H_{r,k}(n)$ denote the number of partitions of n into parts congruent to r or k modulo k with minimal difference k, minimal difference $2k$ between parts congruent to r modulo k, and, if $r > k$, with the smallest part greater than or equal to k. Then $G_{r,k}(n) = H_{r,k}(n)$.*

Proof. This theorem is a special case of Theorem 1.3 when $n, M = 2$. We supply the details for $r < k$; the other case is similar. To begin, let λ be a partition counted by $B_{1,2}(x_1, x_2; m)$. The important observation is that the extra conditions (i) and (ii) in Theorem 1.3 ensure that we can "drop" the color 2 from the subscripts without losing any information. The color 1 remains 1, the color 2 is dropped, and the color 3 becomes 1. This operation corresponds to setting $a_2 = 1$ in the product $(-a_1 q; q^2)_\infty(-a_2 q; q)_\infty$. For example, the partition $(13_3, 10_1, 8_2, 7_1, 5_3, 3_2)$ becomes $(13_1, 10_1, 8, 7_1, 5_1, 3)$. One verifies that the difference conditions on parts of λ become $\lambda_i - \lambda_{i+1} \geq 2$, if $c_i = 1$, and $\lambda_i - \lambda_{i+1} \geq 1$ if c_i is uncolored. To finish, we replace q by q^k and a_1 by q^{r-k} in the product $(-a_1 q; q^2)_\infty(-q; q)_\infty$, which corresponds to replacing a part j_1 by $(j-1)k + r$ and a part j by kj in λ. The difference conditions are now precisely those of Bressoud's theorem. □

In the late 1960's Andrews [5,7] proved two superficially similar generalizations of Schur's theorem, which we need some notation to state. Consider again a set $A = \{a_1, a_2, \ldots, a_n\}$ of n distinct positive integers, this time satisfying $\sum_{i=1}^{k-1} a_i < a_k$ for all $1 \leq k \leq n$. Fix an integer N such that $N \geq \sum_{i=1}^{n} a_i$. Denote the set of $2^n - 1$ (necessarily distinct) possible sums of distinct elements of A by A' and its elements by $\alpha_1 < \alpha_2 < \cdots < \alpha_{2^n-1}$. For any natural number $N \geq \alpha_{2^n-1}$ define A_N (resp. $A'_N, -A_N, -A'_N$) to be the set of all natural numbers congruent to some a_i (resp. $\alpha_i, -a_i, -\alpha_i$) modulo N. For $x \in A'$ let $\omega_A(x)$ denote the number of terms in the defining sum of x and let $\upsilon_A(x)$ (resp. $z_A(x)$) be the smallest (resp. largest) a_i appearing in this sum. Moreover, for $x, y \in A'$, we define $\delta_A(x, y) = 1$ if $z_A(x) < \upsilon_A(y)$ and 0 otherwise. Finally, let $\beta_N(\ell)$ be the least positive residue of ℓ modulo N.

Theorem 5.3 ([7,8]). *Let $D(A_N; x_1, \ldots, x_n; m)$ denote the number of partitions of m into distinct parts taken from A_N where there are x_r parts equivalent to a_r modulo N. Let $E(A'_N; x_1, \ldots, x_n; m)$ denote the number of partitions of m into parts taken from A'_N of the form $\lambda_1 + \cdots + \lambda_s$ such that* (i) *x_r is the number of λ_i such that $\beta_N(\lambda_i)$ uses a_r in its defining sum, and* (ii)

$$\lambda_i - \lambda_{i+1} \geq N\omega_A(\beta_N(\lambda_{i+1})) + \upsilon_A(\beta_N(\lambda_{i+1})) - \beta_N(\lambda_{i+1}).$$

Then $D(A_N; x_1, \ldots, x_n; m) = E(A'_N; x_1, \ldots, x_n; m)$.

Theorem 5.4 ([5,8]). *Let $F(-A_N; x_1, \ldots, x_n; m)$ denote the number of partitions of m into distinct parts taken from $-A_N$ where there are x_r parts equivalent to $-a_r$ modulo N. Let $G(-A'_N; x_1, \ldots, x_n; m)$ denote the number of partitions of m into parts taken from $-A'_N$ of the form $\lambda_1 + \cdots + \lambda_s$ such that* (i) *$\lambda_i \geq N(\omega_A(\beta_N(-\lambda_i)) - 1)$,* (ii) *$x_r$ is the number of λ_i such that $\beta_N(-\lambda_i)$ uses a_r in its defining sum, and* (iii)

$$\lambda_i - \lambda_{i+1} \geq N\omega_A(\beta_N(-\lambda_i)) + \upsilon_A(\beta_N(-\lambda_i)) - \beta_N(-\lambda_i).$$

Then $F(-A_N; x_1, \ldots, x_n; m) = G(-A'_N; x_1, \ldots, x_n; m)$.

In the rest of the paper we first state and then prove a number of partition theorems that extend the results of Andrews. These will correspond to the substitutions $q \to q^N$ and $y_j \to q^{a_j - N}$ in (1.2)–(1.4), and $q \to q^N$ and $y_j \to q^{-a_j}$ in (1.2).

For the first two results, we take advantage of the symmetry in (3.4). We extend any permutation $\sigma \in S_n$ to the integers in A' in the obvious way, by letting $\sigma(\alpha_i) = \sum_{j=1}^{k} a_{\sigma(i_j)}$ if $\alpha_i = \sum_{j=1}^{k} a_{i_j}$.

Theorem 5.5. *Let $E_\sigma(A'_N; x_1, \ldots, x_n; m)$ denote the number of partitions of m into parts taken from A'_N of the form $\lambda_1 + \cdots + \lambda_s$ such that* (i) *x_r is the number of λ_i such that $\beta_N(\lambda_i)$ uses a_r in its defining sum, and* (ii)

$$\lambda_i - \lambda_{i+1} \geq N\omega_A(\beta_N(\lambda_{i+1})) + N\delta_A(\sigma(\beta_N(\lambda_i)), \sigma(\beta_N(\lambda_{i+1}))) + \beta_N(\lambda_i) - \beta_N(\lambda_{i+1}).$$

Then $D(A_N; x_1, \ldots, x_n; m) = E_\sigma(A'_N; x_1, \ldots, x_n; m)$.

Theorem 5.6. *Let $G_\sigma(-A'_N; x_1, \ldots, x_n; m)$ denote the number of partitions of m into parts taken from $-A'_N$ of the form $\lambda_1 + \cdots + \lambda_s$ such that* (i) *$\lambda_i \geq N(\omega_A(\beta_N(-\lambda_i)) - 1)$,* (ii) *$x_r$ is the number of λ_i such that $\beta_N(-\lambda_i)$ uses a_r in its defining sum, and* (iii)

$$\lambda_i - \lambda_{i+1} \geq N\omega_A(\beta_N(-\lambda_i)) + N\delta_A(\sigma(\beta_N(-\lambda_i)), \sigma(\beta_N(-\lambda_{i+1}))) + \beta_N(-\lambda_{i+1}) - \beta_N(-\lambda_i).$$

Then $F(-A_N; x_1, \ldots, x_n; m) = G_\sigma(-A'_N; x_1, \ldots, x_n; m)$.

Let us record some examples. Suppose that $n = 2$, $A = \{1, 2\}$ and $N = 3$. It is easy to see that taking $\sigma = (1, 2)$ in Theorem 5.5 or $\sigma = (2, 1)$ in Theorem 5.6 gives back (a refinement of) Schur's theorem. On the other hand, taking $\sigma = (2, 1)$ in Theorem 5.5 or $\sigma = (1, 2)$ in Theorem 5.6 gives

Corollary 5.7. *The number of partitions of m into distinct parts $\equiv 1, 2 \pmod 3$ with x_r parts congruent to r modulo 3 is equal to the number of partitions λ of m such that* (i) *x_r is the number of parts congruent to r or 3 modulo 3 and* (ii) *if $\lambda_i \equiv j \pmod 3$ and $\lambda_{i+1} \equiv k \pmod 3$ then*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 3, & \text{if } k = 1 \text{ and } j = 1, 3, \\ 7, & \text{if } k = 1 \text{ and } j = 2, \\ 2, & \text{if } k = 2, \\ 4, & \text{if } k = 3. \end{cases}$$

This is equivalent to the case $S_1 = S_4$ of [3, Theorem 8].

For a more complicated example, let $n = 3$, $A = \{1, 2, 4\}$ and $N = 7$. Taking $\sigma = (3, 2, 1)$ in Theorem 5.5 gives

Corollary 5.8. *The number of partitions of m into distinct parts $\equiv 1, 2, 4 \pmod 7$ with x_r parts congruent to 2^{r-1} modulo 7 is equal to the number of partitions λ of m such that* (i) *x_r is the number of λ_i such that $\lambda_i \pmod 7$ uses 2^{r-1} in its defining sum, and* (ii) *if $\lambda_i \equiv j \pmod 7$ and $\lambda_{i+1} \equiv k \pmod 7$,*

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 7, & \text{if } k = 1 \text{ and } j = 1, 3, 5, 7, \\ 13, & \text{if } k = 1 \text{ and } j = 2, 4, 6, \\ 6, & \text{if } k = 2 \text{ and } j \neq 4, \\ 16, & \text{if } k = 2 \text{ and } j = 4, \\ 12, & \text{if } k = 3 \text{ and } j \neq 4, \\ 22, & \text{if } k = 3 \text{ and } j = 4, \\ 4, & \text{if } k = 4, \\ 10, & \text{if } k = 5, \\ 9, & \text{if } k = 6, \\ 15, & \text{if } k = 7. \end{cases}$$

Although it may not yet be clear, we shall see that Theorem 5.3 is the case $\sigma = Id$ of Theorems 5.5 and 5.4 is the case $\sigma = (n, n-1, \ldots, 1)$ of Theorem 5.6. While these theorems take advantage of the symmetry in Corollary 3.1, this symmetry does not persist in Theorems 1.3 and 1.4. Hence we drop the permutation σ in the last two results. For $\ell \in A'$ let $\omega_{A,1}(\ell)$ denote the number of terms not equal to a_1 in the defining sum of ℓ.

Theorem 5.9. *Let $D_{R,M}(A_N; x_1, \ldots, x_n; m)$ denote the number of partitions counted by $D(A_N; x_1, \ldots, x_n; m)$ such that the parts that are a_1 modulo N are $((R-1)N + a_1)$ modulo MN. Let $E_{R,M}(A'_N; x_1, \ldots, x_n; m)$ denote the number of partitions of m counted by $E(A'_N; x_1, \ldots, x_n; m)$ of the form $\lambda_1 + \cdots + \lambda_s$ such that* (i) *if λ_S is the smallest part such that $\beta_N(\lambda_S)$ uses a_1 in its defining sum, then*

$$\lambda_S \equiv N(R - \omega_A(\beta_N(\lambda_S))) + \beta_N(\lambda_S) + N \sum_{\ell=S}^{s} \omega_{A,1}(\beta_N(\lambda_\ell)) \pmod{MN},$$

and (ii) *if* $i < j$ *and* $\beta_N(\lambda_i)$ *and* $\beta_N(\lambda_j)$ *use* a_1 *in their defining sums, then*

$$\lambda_i - \lambda_j \equiv N(-\omega_A(\beta_N(\lambda_i)) + \omega_A(\beta_N(\lambda_j))) + \beta_N(\lambda_i) - \beta_N(\lambda_j) + N\sum_{\ell=i}^{j-1}\omega_{A,1}(\beta_N(\lambda_\ell)) \pmod{MN}.$$

Then $D_{R,M}(A_N; x_1, \ldots, x_n; m) = E_{R,M}(A'_N; x_1, \ldots, x_n; m)$

For $R = 1$, $M = 2$, $N = 3$, and $A = \{1, 2\}$, this translates to:

Corollary 5.10. *Let* $D_{1,2}(A_3; x_1, x_2; m)$ *denote the number of partitions of* m *counted by* $D(A_3; x_1, x_2; m)$ *where the parts that are* 1 *modulo* 3 *are* 1 *modulo* 6, *and let* $E_{1,2}(A'_3; x_1, x_2; m)$ *denote the number of partitions of* m *counted by* $E(A'_3; x_1, x_2; m)$ *of the form* $\lambda_1 + \cdots + \lambda_s$ *such that if* λ_S *is the smallest part that is* $\equiv 1, 3 \pmod 3$ *then* (i) *if* $\lambda_S \equiv 1 \pmod 3$ *then* $\lambda_S \equiv 1 + 3\sum_{\ell=S}^{s}\omega_{A,1}(\beta_3(\lambda_\ell)) \pmod 6$, (ii) *if* $\lambda_S \equiv 3 \pmod 3$ *then* $\lambda_S \equiv 3\sum_{\ell=S}^{s}\omega_{A,1}(\beta_3(\lambda_\ell)) \pmod 6$, *and* (iii) *if* $i < j$ *and* $\lambda_i, \lambda_j \equiv 1, 3 \pmod 3$, *then*

$$\lambda_i - \lambda_j \equiv \frac{\beta_3(\lambda_i) - \beta_3(\lambda_j)}{2} + 3\sum_{\ell=i}^{j-1}\omega_{A,1}(\beta_3(\lambda_\ell)) \pmod 6.$$

Then $D_{1,2}(A_3; x_1, x_2; m) = E_{1,2}(A'_3; x_1, x_2; m)$.

Example. For $m = 18$, we have $D(A_3; 1, 1; 18) = 6$, the relevant partitions being $(17, 1)$, $(14, 4)$, $(11, 7)$, $(10, 8)$, $(13, 5)$ and $(16, 2)$. But only three of them satisfy the conditions of the previous corollary, i.e., that the parts that are 1 modulo 3 are 1 modulo 6. These partitions are $(17, 1)$, $(11, 7)$ and $(13, 5)$. Therefore $D_{1,2}(A_3; 1, 1; 18) = 3$. Now $E(A'_3; 1, 1; 18) = 6$, the relevant partitions being (18), $(17, 1)$, $(16, 2)$, $(14, 4)$, $(13, 5)$ and $(11, 7)$. But three of them violate the condition on λ_S, namely (18) violates (ii) , and $(14, 4)$ and $(13, 5)$ violate (i) . So, $E_{1,2}(A'_3, 1, 1; 18) = 3 = D_{1,2}(A_3; 1, 1; 18)$.

Theorem 5.11. *Let* $D_M(A_N; x_1, \ldots, x_n; m)$ *denote the number of partitions counted by* $D(A_N; x_1, \ldots, x_n; m)$ *such that the parts equivalent to* a_1 *modulo* N *differ at least by* MN. *Let* $E_M(A'_N; x_1, \ldots, x_n; m)$ *denote the number of partitions of* m *counted by* $E(A'_N; x_1, \ldots, x_n; m)$ *of the form* $\lambda_1 + \cdots + \lambda_s$ *such that if* $i < j$ *and* $\beta_N(\lambda_i)$ *and* $\beta_N(\lambda_j)$ *use* a_1 *in their defining sums, then*

$$\lambda_i - \lambda_j \geq MN + N(-\omega_A(\beta_N(\lambda_i)) + \omega_A(\beta_N(\lambda_j))) + \beta_N(\lambda_i) - \beta_N(\lambda_j) + N\sum_{\ell=i}^{j-1}\omega_{A,1}(\beta_N(\lambda_\ell)).$$

Then $D_M(A_N; x_1, \ldots, x_n; m) = E_M(A'_N; x_1, \ldots, x_n; m)$.

Let $M = 2$, $N = 3$, and $A = \{1, 2\}$:

Corollary 5.12. *Let* $D_2(A_3; x_1, x_2; m)$ *denote the number of partitions of* m *counted by* $D(A_3; x_1, x_2; m)$ *such that the parts equivalent to* 1 *modulo* 3 *differ by at least 6 and let* $E_2(A'_3; x_1, x_2; m)$ *denote the number of partitions of* m *counted by* $E(A'_3; x_1, x_2; m)$ *such* *that if* $i < j$ *and* $\lambda_i, \lambda_j \equiv 1, 3 \pmod 3$ *then* $\lambda_i - \lambda_j \geq 5 + 3\sum_{\ell=i}^{j-1} \omega_{A,1}(\beta_3(\lambda_\ell))$. *Then* $D_2(A_3; x_1, x_2; m) = E_2(A_3'; x_1, x_2; m)$.

Example. For $m = 22$, we have $D_2(A_3; 2, 1; 22) = 8$, with the relevant partitions being $(16, 4, 2)$, $(16, 5, 1)$, $(13, 7, 2)$, $(13, 5, 4)$ $(13, 8, 1)$, $(14, 7, 1)$, $(10, 8, 4)$ and $(11, 10, 1)$. Also $E_2(A_3', 2, 1; 22) = 8$, with the relevant partitions being $(18, 4)$, $(19, 3)$, $(15, 7)$, $(16, 6)$, $(16, 5, 1)$, $(14, 7, 1)$, $(13, 7, 2)$ and $(13, 8, 1)$.

We now turn to the proofs of all of these theorems.

Proof of Theorem 5.5. Fix a permutation $\sigma \in S_n$. In a partition counted by $A(x_1, x_2, \ldots, x_n; m)$, we replace a part of size k from μ_j by $N(k-1) + a_j$. Then each λ_i in the corresponding partition λ counted by $B_{\sigma^{-1}}(x_1, \ldots, x_n; m)$ is replaced by $\tilde{\lambda}_i = N(\lambda_i - \omega_A(\alpha_{c_i})) + \alpha_{c_i}$. This corresponds to replacing q by q^N and y_j by $q^{a_j - N}$ in (1.2). Suppose that before this replacement, we had $\lambda_{i+1} = \lambda_i - \omega(c_i) - \delta(\sigma(c_i), \sigma(c_{i+1})) - d$, where $d \geq 0$. Then after the replacement we have

$$\tilde{\lambda}_i = N(\lambda_i - \omega_A(\alpha_{c_i})) + \alpha_{c_i}$$

and

$$\tilde{\lambda}_{i+1} = N(\lambda_i - \omega_A(\alpha_{(c_i)}) - \delta(\sigma(c_i), \sigma(c_{i+1})) - d - \omega_A(\alpha_{c_{i+1}})) + \alpha_{c_{i+1}}.$$

Since $\delta(\sigma(c_i), \sigma(c_{i+1})) = \delta_A(\alpha_{\sigma(c_i)}, \alpha_{\sigma(c_{i+1})})$, we get

$$\tilde{\lambda}_i - \tilde{\lambda}_{i+1} = N(d + \delta_A(\alpha_{\sigma(c_i)}, \alpha_{\sigma(c_{i+1})}) + \omega_A(\alpha_{c_{i+1}})) + \alpha_{c_i} - \alpha_{c_{i+1}}.$$

Note that $\beta_N(\tilde{\lambda_i}) = \alpha_{c_i}$ and that $\sigma(\beta_N(\tilde{\lambda_i})) = \alpha_{\sigma(c_i)}$ for all i and the theorem follows. □

Proof of Theorem 5.6. In a partition counted by $A(x_1, x_2, \ldots, x_n; m)$, we replace a part of size k from μ_j by $Nk - a_j$. Each λ_i in the corresponding partition λ counted by $B_{\sigma^{-1}}(x_1, \ldots, x_n; m)$ is replaced by $\tilde{\lambda} = N\lambda_i - \alpha_{c_i}$. This corresponds to replacing q by q^N and y_j by q^{-a_j} in (1.2). Suppose that before this replacement, we had $\lambda_{i+1} = \lambda_i - \omega(c_i) - \delta(\sigma(c_i), \sigma(c_{i+1})) - d$, where $d \geq 0$. Then after the replacement we have

$$\tilde{\lambda}_i = N\lambda_i - \alpha_{c_i}$$

and

$$\tilde{\lambda}_{i+1} = N(\lambda_i - \omega_A(\alpha c_i) - \delta(\sigma(c_i), \sigma(c_{i+1})) - d) - \alpha_{c_{i+1}}.$$

Since $\delta(\sigma(c_i), \sigma(c_{i+1})) = \delta_A(\alpha_{\sigma(c_i)}, \alpha_{\sigma(c_{i+1})})$, we get

$$\tilde{\lambda}_i - \tilde{\lambda}_{i+1} = N(d + \delta_A(\alpha_{\sigma(c_i)}, \alpha_{\sigma(c_{i+1})}) + \omega_A(\alpha_{c_i})) + \alpha_{c_i} - \alpha_{c_{i+1}}.$$

Note that $\beta_N(-\tilde{\lambda_i}) = \alpha_{c_i}$ and that $\sigma(\beta_N(-\tilde{\lambda_i})) = \alpha_{\sigma(c_i)}$ for all i and the theorem follows. □

For general n, it may not be clear that Theorems 5.5 and 5.6 are indeed generalizations of Theorems 5.3 and 5.4. But it becomes clear with the following lemma.

Lemma 5.13. *For* $x, y \in A'$,

$$N + \upsilon_A(y) > N\delta_A(x, y) + x \geq \upsilon_A(y).$$

Proof. We consider two cases. First, if $\delta_A(x, y) = 0$, then $z_A(x) \geq v_A(y)$. The first inequality is trivial as $N > x$. The second follows from the fact that $x \geq z_A(x) \geq v_A(y)$. On the other hand, if $\delta(x, y) = 1$, then $z_A(x) < v_A(y)$. The second inequality is trivial as $N > v_A(y)$. The first follows from the fact that $v_A(y) > x$ if $v_A(y) > z_A(x)$. □

Now we can prove

Corollary 5.14. *Theorem* 5.3 *is Theorem* 5.5 *with* $\sigma = (1, 2, \ldots, n)$.

Proof. As $\sigma = (1, 2, \ldots, n)$, we have $\delta_A(\sigma(\beta_N(\lambda_i)), \sigma(\beta_N(\lambda_{i+1}))) = \delta_A(\beta_N(\lambda_i)), \beta_N(\lambda_{i+1}))$. Lemma 5.13 gives

$$N + v_A(\beta_N(\lambda_{i+1})) > N\delta_A(\beta_N(\lambda_i), \beta_N(\lambda_{i+1})) + \beta_N(\lambda_i) \geq v_A(\beta_N(\lambda_{i+1})).$$

This shows that the minimal difference in Theorem 5.5 is at least the one claimed in Theorem 5.3 but not greater. □

Corollary 5.15. *Theorem* 5.4 *is Theorem* 5.6 *with* $\sigma = (n, n-1, \ldots, 1)$.

Proof. As $\sigma = (n, n-1, \ldots, 1)$, we have $\delta_A(\sigma(\beta_N(-\lambda_i)), \sigma(\beta_N(-\lambda_{i+1}))) = \delta_A(\beta_N(-\lambda_{i+1}), \beta_N(-\lambda_i))$. Lemma 5.13 gives

$$N + v_A(\beta_N(-\lambda_i)) > N\delta_A(\beta_N(-\lambda_{i+1}), \beta_N(-\lambda_i)) + \beta_N(-\lambda_{i+1}) \geq v_A(\beta_N(-\lambda_i)).$$

This shows that the minimal difference in Theorem 5.6 is at least the one claimed in Theorem 5.4 but not greater. □

Proof of Theorem 5.9. In a partition counted by $A_{R,M}(x_1, x_2, \ldots, x_n; m)$, we replace a part of size k from μ_j by $N(k-1) + a_j$. In the corresponding partition λ counted by $B_{R,M}(x_1, x_2, \ldots, x_n; m)$, each λ_i is replaced by $\tilde{\lambda}_i = N(\lambda_i - \omega_A(\alpha_{c_i})) + \alpha_{c_i}$. This implies that $\tilde{\lambda} \in E(A'_N; x_1, \ldots, x_n; m)$ as in Theorem 5.3. Note that $\alpha_{c_i} = \beta_N(\tilde{\lambda}_i)$, $\omega(c_i) = \omega_A(\beta_N(\tilde{\lambda}_i))$ and $\omega_e(c_i) = \omega_{A,1}(\beta_N(\tilde{\lambda}_i))$. If λ_S was the smallest part with odd color, then $\lambda_S \equiv R + \sum_{\ell=S}^{s} \omega_e(c_\ell) \pmod{M}$. This condition is translated to: if $\tilde{\lambda}_S$ is the smallest part such that $\beta_N(\tilde{\lambda}_S)$ uses a_1 in its defining sum,

$$\tilde{\lambda}_S \equiv RN - N\omega_A(\beta_N(\tilde{\lambda}_S)) + \beta_N(\tilde{\lambda}_S) + N\sum_{\ell=S}^{s} \omega_{A,1}(\beta_N(\tilde{\lambda}_\ell)) \pmod{MN}.$$

If λ_i and λ_j were any two parts with odd color, then $\lambda_i - \lambda_j \equiv \sum_{\ell=i}^{j-1} \omega_e(\lambda_\ell) \pmod{M}$ and this gives $N\lambda_i - N\lambda_j \equiv N\sum_{\ell=i}^{j-1} \omega_e(\lambda_\ell) \pmod{MN}$. Then λ_i is changed to $\tilde{\lambda}_i = N(\lambda_i - \omega(c_i)) + \alpha_{c_i}$ and λ_j is changed to $\tilde{\lambda}_j = N(\lambda_j - \omega(c_j)) + \alpha_{c_j}$. We get that if $\beta_N(\tilde{\lambda}_i)$ and $\beta_N(\tilde{\lambda}_j)$ use a_1 in their defining sums, then

$$\begin{aligned}\tilde{\lambda}_i - \tilde{\lambda}_j \equiv{}& N(-\omega_A(\beta_N(\tilde{\lambda}_i)) + \omega_A(\beta_N(\tilde{\lambda}_j))) + \beta_N(\tilde{\lambda}_i) \\ & - \beta_N(\tilde{\lambda}_j) + N\sum_{\ell=i}^{j-1} \omega_{A,1}(\beta_N(\tilde{\lambda}_\ell)) \pmod{MN}. \quad \square\end{aligned}$$

Proof of Theorem 5.11. The proof use the same ideas as the proof of Theorem 5.9. In a partition λ counted by $A_M(x_1, x_2, \ldots, x_n; m)$, we replace a part of size k from μ_j by

$N(k-1)+a_j$. In the corresponding partition counted by $B_M(x_1, x_2, \ldots, x_n; m)$, each λ_i is replaced by $\tilde{\lambda}_i = N(\lambda_i - \omega_A(\alpha_{c_i})) + \alpha_{c_i}$. This corresponds to replacing q by q^N and y_j by $q^{a_j - N}$ in (1.3). This implies that $\tilde{\lambda} \in E(A'_N; x_1, \ldots, x_n; m)$ as in Theorem 5.3. Note that $\alpha_{c_i} = \beta_N(\tilde{\lambda}_i)$, $\omega(c_i) = \omega_A(\beta_N(\tilde{\lambda}_i))$ and $\omega_e(c_i) = \omega_{A,1}(\beta_N(\tilde{\lambda}_i))$ for all i.

If λ_i and λ_j were any two parts with odd color, then $\lambda_i - \lambda_j \geq M + \sum_{\ell=i}^{j-1} \omega_e(\lambda_\ell) \pmod M$ translates to: if $\beta_N(\tilde{\lambda}_i)$ and $\beta_N(\tilde{\lambda}_j)$ use a_1 in their defining sums, then $\tilde{\lambda}_i - \tilde{\lambda}_j \geq MN + N(-\omega_A(\beta_N(\tilde{\lambda}_i)) + \omega_A(\beta_N(\tilde{\lambda}_j))) + \beta_N(\tilde{\lambda}_i) - \beta_N(\tilde{\lambda}_j) + N\sum_{\ell=i}^{j-1} \omega_{A,1}(\tilde{\lambda}_\ell)$. $\square$

6. Concluding remarks

There are undoubtedly many more applications of the iterative methods described in this paper. To help motivate the iterative process, we have described how to reproduce and generalize some famous results by executing a rearrangement of colors at each step. However, the Alladi–Gordon bijection may also be "naively" iterated without performing this rearrangement. It would be worthwhile to investigate the theorems on colored partitions that arise in this way. Also, as observed in [3], one can write down companions to Schur's theorem by reordering the parts immediately before Step 4 of the proof of Theorem 1.1 presented in Section 2. Such reorderings could probably be applied in the case of Theorem 1.2 as well. Finally, it will be noted that the products in (1.3) and (1.4) are rather asymmetric. A more general problem of the nature considered here would be to give an interpretation of the infinite product

$$\prod_{k=1}^{\infty}(1+y_1 q^{M_1 k - R_1})(1+y_2 q^{M_2 k - R_2}) \cdots (1+y_n q^{M_n k - R_n}) \tag{6.1}$$

in terms of partitions whose parts occur in $2^n - 1$ colors and satisfy some tractable difference conditions.

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References

[1] K. Alladi, Refinements of Rogers–Ramanujan type identities, Fields Inst. Commun. 14 (1997) 1–35.
[2] K. Alladi, B. Gordon, Generalizations of Schur's partition theorem, Manuscripta Math. 79 (1993) 113–126.
[3] K. Alladi, B. Gordon, Schur's partition theorem, companions, refinements, and generalizations, Trans. Amer. Math. Soc. 347 (1995) 1591–1608.
[4] G.E. Andrews, On Schur's second partition theorem, Glasg. J. Math. 9 (1967) 127–132.
[5] G.E. Andrews, A new generalization of Schur's second partition theorem, Acta Arith. 4 (1968) 429–434.

[6] G.E. Andrews, On partition functions related to Schur's second partition theorem, Proc. Amer. Math. Soc. 19 (1968) 441–444.
[7] G.E. Andrews, A general partition theorem with difference conditions, Amer. J. Math. 191 (1969) 18–24.
[8] G.E. Andrews, The use of computers in search of identities of the Rogers–Ramanujan type, in: Computers in Number Theory, Academic Press, New York, 1971.
[9] G.E. Andrews, Schur's theorem, Capparelli's conjecture, and q-trinomial coefficients, Contemp. Math. 166 (1994) 141–151.
[10] G.E. Andrews, J. Olsson, Partition identities with an application to group representation theory, J. Reine Angew. Math. 413 (1991) 198–212.
[11] C. Bessenrodt, A combinatorial proof of a refinement of the Andrews–Olsson partition identity, Eur. J. Combin. 12 (1991) 271–276.
[12] C. Bessenrodt, Generalisations of the Andrews–Olsson partition identity and applications, Discrete Math. 141 (1995) 11–22.
[13] D.M. Bressoud, On a partition theorem of Göllnitz, J. Reine Angew. Math. 305 (1979) 215–217.
[14] D.M. Bressoud, A combinatorial proof of Schur's 1926 partition theorem, Proc. Amer. Math. Soc. 79 (1980) 338–340.
[15] H. Göllnitz, Partitionen mit Differenzenbedingungen, J. Reine Angew. Math. 225 (1967) 154–190.
[16] I. Schur, Zur additiven Zahlentheorie, in: Gessamelte Abhandlungen, vol. 2, Springer, Berlin, 1973, pp. 43–50.