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# Cylindrical lattice walks and the Loehr–Warrington $10^n$ conjecture

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**Abstract**

The following special case of a conjecture by Loehr and Warrington was proved recently by Ekhad, Vatter, and Zeilberger:

There are $10^n$ zero-sum words of length $5n$ in the alphabet $\{+3, -2\}$ such that no zero-sum consecutive subword that starts with $+3$ may be followed immediately by $-2$.

We give a simple bijective proof of the conjecture in its original and more general setting. To do this we reformulate the problem in terms of cylindrical lattice walks.
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## 1. Introduction

Let $a$ and $b$ be positive integers. Given a word $w$ in the alphabet $\{+a, -b\}$, a zero-sum consecutive subword of $w$ is said to be *illegal* if it starts with $+a$, and $-b$ comes immediately after the subword in $w$. Example where $a = 3$ and $b = 2$:

$$w = -2 + 3 \underbrace{+\mathbf{3} - 2 - 2 - 2 + 3}_{\text{illegal subword}} \mathbf{-2} - 2 + 3.$$

We will prove the following:

**Theorem 1.** *If $a$ and $b$ are relatively prime, there are $\binom{a+b}{a}^n$ zero-sum words of length $(a+b)n$ in the alphabet $\{+a, -b\}$ without illegal subwords.*

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**Example.** If $a = 2$, $b = 1$ and $n = 2$, the $\binom{2+1}{2}^2 = 9$ words counted in the theorem are

$$+2-1-1+2-1-1$$
$$-1+2-1+2-1-1$$
$$-1-1+2+2-1-1$$
$$-1-1-1+2+2-1$$
$$-1+2-1-1+2-1$$
$$-1-1+2-1+2-1$$
$$-1-1-1+2-1+2$$
$$-1-1-1-1+2+2$$
$$-1-1+2-1-1+2$$

The theorem was conjectured by Nick Loehr and Greg Warrington. (In fact they conjectured a slightly stronger statement equivalent to our Proposition 2.) In a recent paper [1] Ekhad et al. proved the special case $a = 3$, $b = 2$, using a computer. Inspired by their proof, Loehr and Warrington, together with Sagan [4], found a computer-free proof of the more general case when $b = 2$ and $a$ is any odd positive integer. (In fact they proved our Proposition 2 for $b = 2$ and any positive $a$.) We greatly admire the automatic approach of Ekhad, but we feel that a beautiful problem like this ought to have a beautiful solution. And indeed it has!

First we will present a geometrical construction where the words in the alphabet $\{+a, -b\}$ are interpreted as walks on a cylinder graph. Then we will give bijections between these walks, certain weight functions on the edges of the graph, and ordered sequences of closed laps. The latter ones are easy to count. Finally, in the last section we discuss some generalizations; in particular we examine what happens if $a$ and $b$ have a common factor.

On the basis of a preprint of the present paper, Zeilberger [5] has written a popular presentation of our algorithms and even implemented them in Maple.

## 2. The geometrical construction

In the following we let $a$ and $b$ be any positive integers.

After thinking about the Loehr–Warrington conjecture for a while, most people will probably discover the following natural reformulation:

You live in a skyscraper $\mathbb{Z}$. In the morning you get your exercises by climbing out through the window, following $(a + b)n$ one-way ladders, and climbing into your apartment again. At each level there is one ladder going $a$ levels up and another ladder going $b$ levels down. Once you have climbed up from a level you never climb down from that level again that morning. In how many ways can you perform your exercises?

Now here is the key observation: Since $a \equiv -b \pmod{a + b}$, after $x$ ladders we are at a level $y$ such that $y \equiv ax \pmod{a + b}$. We define a directed graph $G_{a,b}$ whose vertex set is the subset of the infinite cylinder $\mathbb{Z}_{a+b} \times \mathbb{Z}$ consisting of all points $(x, y)$ such that $y \equiv ax \pmod{a + b}$. From every vertex point $(x, y)$ there is an up-edge $(x, y) \to (x + 1, y + a)$ and a down-edge $(x, y) \to (x + 1, y - b)$. If $a$ and $b$ are relatively prime, no two points in $G_{a,b}$ have the same $y$-coordinate.[1] We have mapped the ladders to the cylinder such that no ladders intersect!

[1] This is the only time we use the assumption in Theorem 1 that $a$ and $b$ are relatively prime.

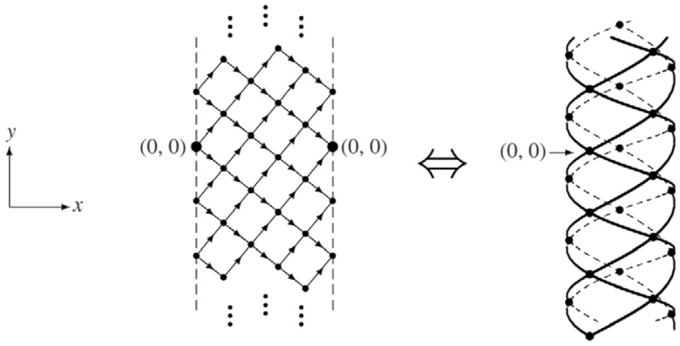


Fig. 1. The graph $G_{3,2}$ represented as a vertical strip to the left. (We have stretched the $x$-axis by a factor $\sqrt{ab}$ to make the lattice rectangular, but this is merely cosmetic.) When the borders are welded together the result is the cylinder to the right.

Fig. 1 shows a graphical representation of $G_{3,2}$. It is an infinite vertical strip whose borders are welded together. The points with $x = 0$ constitute the *weld* and are called *weld points*.

As far as we know, no one has studied this graph before. The closest related research we could find is two papers about nonintersecting lattice walks on the cylinder, one by Forrester [2] and one by Fulmek [3]. Curiously, their walks essentially go along the axis of the cylinder while ours essentially go around it.

Let us fix the following graph terminology: A *walk* is an ordered sequence of vertices $v_0v_1, \ldots, v_m$ such that there is an edge from $v_{i-1}$ to $v_i$ for $i = 1, 2, \ldots, m$. (Repeated vertices and edges are allowed.) The integer $m$ is the *length* of the walk. If $v_0 = v_m$ the walk is *closed*.

A walk on $G_{a,b}$ may leave a certain vertex several times, sometimes going down, sometimes going up. If for each vertex all downs come before all ups, the walk is said to be *downs-first*. Now Theorem 1 can be reformulated:

**Proposition 2.** *There are $\binom{a+b}{a}^n$ closed downs-first walks on $G_{a,b}$ of length $(a+b)n$ starting at the origin.*

We will prove this proposition for any positive integers $a$ and $b$. (Note that this does not imply that Theorem 1 is true if $a$ and $b$ have a common factor; see footnote 1.)

A walk of length $a + b$ that starts and ends on the weld is called a *lap*. Obviously, there are $\binom{a+b}{a}$ different closed laps starting at the origin. Our proof will be a bijection that maps closed downs-first walks to a sequence of closed laps. The following lemma is crucial.

**Lemma 3.** *A closed downs-first walk beginning at a weld point never visits higher weld points.*

**Proof.** Suppose the closed downs-first walk $C$, starting at a weld point $p$, visits a weld point $q$ higher than $p$. Let $pq$ be the walk along $C$ from its starting point $p$ to $q$ (if $C$ visits $q$ several times, choose any visit), and let $qp$ be the remaining walk along $C$ from $q$ to the finish point $p$. The walk $pq$ must make at least one lap starting at $p$ or below and ending at a weld point above $p$. Similarly, $qp$ must make at least one lap starting at a weld point above $p$ and ending at $p$ or below. Clearly these laps must intersect at a point where $pq$ goes up and $qp$ goes down. But this contradicts the assumption that $C$ is downs-first. $\square$

We conclude that the weld points above the origin, and hence the points above the highest closed lap from the origin, can never be reached by the closed downs-first walks counted in

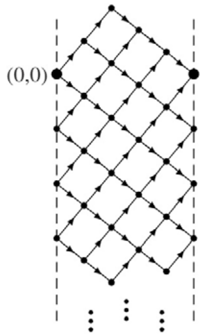


Fig. 2. The semi-infinite cylinder graph $H_{3,2}$.

Proposition 2. Let $H_{a,b}$ be the resulting graph when these points are removed from $G_{a,b}$. Fig. 2 shows an example.

Now Proposition 2 can be slightly reformulated:

**Proposition 4.** *For any positive integers a, b, there are* $\binom{a+b}{a}^n$ *closed downs-first walks on* $H_{a,b}$ *of length* $(a+b)n$ *starting at the origin.*

Before proving this proposition, and hence our main theorem, we need some more definitions.

A *weight function* on $H_{a,b}$ is an assignment of a nonnegative integer to every edge in $H_{a,b}$. The *in-weight* (resp. the *out-weight*) of a vertex in $H_{a,b}$ is the sum of the weights of the edges going into (resp. out of) the vertex. A weight function is said to be *balanced* if at each vertex the in-weight and out-weight are equal. A walk in $H_{a,b}$ is said to be *covered* by the weight function if every edge is used by the walk at most as many times as its weight. The weight function is *origin-connected* if for every vertex with positive out-weight there is a covered walk from the origin to the vertex.

For an example of a balanced origin-connected weight function, see Fig. 3.

## 3. The bijections

Please keep the cylinder graph $H_{a,b}$ in your mind throughout the paper.

Since the number of closed laps beginning at the origin is $\binom{a+b}{a}$, Proposition 4 follows from the result in this section:

**Theorem 5.** *There are bijections between the following three sets:*

1. *closed downs-first walks of length* $(a+b)n$ *beginning at the origin,*
2. *balanced origin-connected weight functions with total weight sum* $(a+b)n$*,*
3. *ordered sequences of* $n$ *closed laps beginning at the origin.*

**Proof.** We will define four functions, $f_{1,2} : 1 \to 2$, $f_{2,1} : 2 \to 1$, $f_{3,2} : 3 \to 2$, and $f_{2,3} : 2 \to 3$. It should be apparent from the presentation below that $f_{1,2} \circ f_{2,1}$, $f_{2,1} \circ f_{1,2}$, $f_{3,2} \circ f_{2,3}$, and $f_{2,3} \circ f_{3,2}$ are all identity functions. Fig. 3 gives an example of the bijections.

**1 → 2**: Given a closed downs-first walk beginning at the origin, to every edge of $H_{a,b}$ we assign a weight that is the number of times the edge is used by the walk. This weight function is obviously balanced and origin-connected. Furthermore, it is the only such function with total sum $(a+b)n$ that covers the walk.

**2 → 1**: Given a balanced and origin-connected weight function we construct a closed downs-first walk as follows: Start at the origin. At each point, go down if that edge has positive weight, otherwise go up. Decrease the weight of the followed edge by one. Continue until you come to a point with zero out-weight. This must be the origin so we have created a closed downs-first walk $C$.

We must show that the length of $C$ is $(a+b)n$. Suppose not. Then there remain some positive weights. Since the original weight function was origin-connected there exists a point $p$ on $C$ with positive out-weight. Since the remaining weight function is still balanced it covers some closed walk $C'$ that contains $p$. Now start at $p$ and follow $C$ and $C'$ in parallel until $C$ reaches the origin. Since the origin has no remaining in- or out-weight, $C'$ must have reached a point on the weld below the origin (the other weld points were removed when we constructed $H_{a,b}$). This implies that $C$ and $C'$ intersect at a point where $C$ goes up and $C'$ goes down. But that is impossible by the construction of $C$.

Thus $C$ has length $(a+b)n$, and it is easy to see that among all closed downs-first walks of that length starting at the origin, $C$ is the only one that is covered by the given weight function.

**3 → 2**: Given a sequence of closed laps $C_1, C_2, \ldots, C_n$ starting at the origin, translate $C_1, C_2, \ldots, C_{n-1}$ downwards so that, for $1 \le i \le n-1$, $C_i$ intersects $C_{i+1}$ in at least one point but otherwise goes below it. Observe that there is a unique way of "packing" the closed laps like that. Now let the weight of each edge in $H_{a,b}$ be the number of times the edge is used by the laps. The result is obviously a balanced origin-connected weight function.

**2 → 3**: Given a balanced origin-connected weight function, by iteration of the following procedure we construct $n$ closed laps. At the beginning of each iteration the weight function is always balanced.

Start at the lowest weld point $p$ with a positive out-weight and create a closed downs-first walk from there like this: In each step, go down if that edge has positive weight; otherwise go up. Decrease the weight of the followed edge by one. Stop as soon as you reach $p$ again. By Lemma 3 this closed downs-first walk never visits weld points other than $p$, which implies that it is a lap. (Remember that a walk must visit the weld every $(a+b)$-th step.)

After $n$ iterations we have consumed all weights and produced a packed sequence of closed laps $C_1, C_2, \ldots, C_n$. Simply translate the laps so that they all start at the origin. □

## 4. Generalizations

The condition that $a$ and $b$ should be relatively prime is not an essential restriction, as the following corollary to Theorem 1 shows.

**Corollary 6.** *Let $a$ and $b$ be any positive integers, and put $c = \gcd(a, b)$. The number of zero-sum words in the alphabet $\{+a, -b\}$ of length $(a+b)n$ without illegal subwords is*

$$\binom{(a+b)/c}{a/c}^{cn}.$$

**Proof.** Let $A = a/c$ and $B = b/c$. Clearly the words counted in the corollary are in one-to-one correspondence with the zero-sum words in the alphabet $\{+A, -B\}$ of length $(A+B)cn$ without illegal subwords. According to Theorem 1 there are $\binom{A+B}{A}^{cn}$ such words. □

However, Theorem 1 can be generalized in a less trivial way:

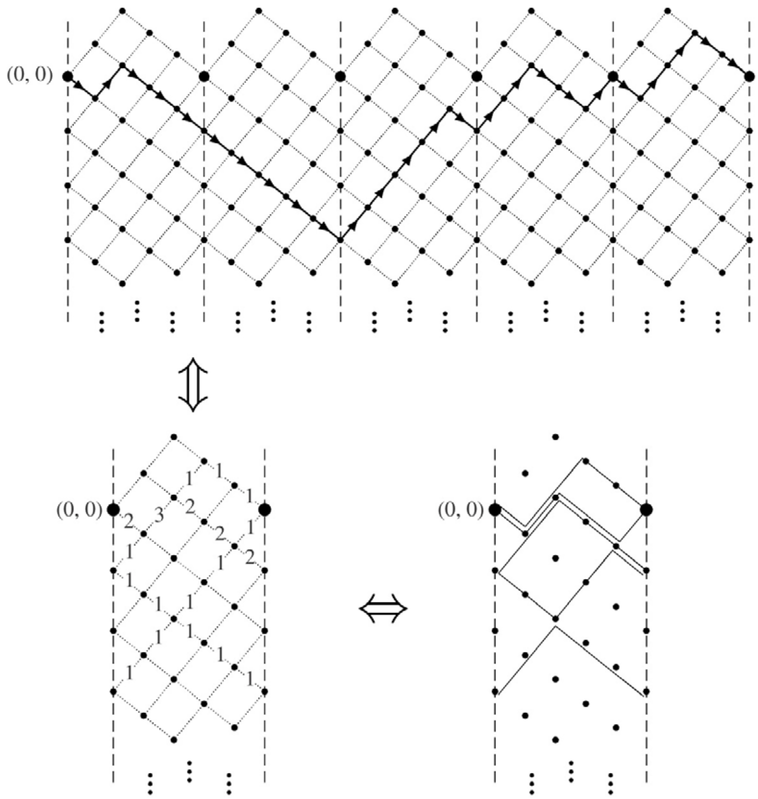


Fig. 3. An example of the bijections in Theorem 5 with $a = 3$, $b = 2$, and $n = 5$. At the top is a closed downs-first walk; below is the corresponding weight function to the left and its packed sequence of closed laps to the right.

**Theorem 7.** *Let $a$ and $b$ be any positive integers. There are $\binom{a+b}{a}^n$ zero-sum words in the alphabet $\{+a, -b\}$ of length $(a + b)n$ without illegal subwords whose length is a multiple of $a + b$.*

**Proof.** In the proof of Theorem 1 the only time we make use of the fact that $a$ and $b$ are relatively prime is when we conclude that no two points in $G_{a,b}$ have the same $y$-coordinate. We need this in order to show that the vertices in $G_{a,b}$ are in one-to-one correspondence with the levels in the skyscraper.

However, even if $a$ and $b$ have a common factor, the condition that the word should have no illegal subword whose length is a multiple of $a + b$ is equivalent to the downs-first condition on the corresponding walk on $G_{a,b}$. (It just happens that the phrase "whose length is a multiple of $a + b$" is unnecessary if $a$ and $b$ are relatively prime.) Thus we can simply bypass the skyscraper nonsense and go directly to Proposition 2. $\square$

Finally, we would like to comment on the topological ingredient in the proofs of Lemma 3 and the $2 \to 1$ part of Theorem 5. These proofs are both based on an argument which could be called a discrete version of Jordan's curve theorem. Perhaps, if the topological content were elucidated more explicitly, it could lead to a generalization of the results.

## References

[1] S.B. Ekhad, V. Vatter, D. Zeilberger, A proof of the Loehr–Warrington amazing TEN to the power $n$ conjecture, arXiv:math.CO/0509347, ver. 1, 2005.
[2] P.J. Forrester, Exact solution of the lock step model of vicious walkers, J. Phys. A 23 (1990) 1259–1273.
[3] M. Fulmek, Nonintersecting lattice paths on the cylinder, Sém. Lothar. Combin. 52 (2004) 16. Article B52b.
[4] N. Loehr, B. Sagan, G. Warrington, A human proof for a generalization of Shalosh B. Ekhad's $10^n$ lattice paths Theorem, arXiv:math.CO/0510079, ver. 1, 2005.
[5] D. Zeilberger, Another proof that Euler Missed: Jonas Sjöstrand's amazingly simple (and lovely!) proof of the no-longer-so-amazing Loehr–Warrington lattice paths conjecture, The Personal Journal of Shalosh B. Ekhad and Doron Zeilberger. Available at http://www.math.rutgers.edu/~zeilberg/pj.html, 2005.