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Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions☆

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ABSTRACT

Estimates on the initial coefficients are obtained for normalized analytic functions f in the open unit disk with f and its inverse $g = f^{-1}$ satisfying the conditions that $zf'(z)/f(z)$ and $zg'(z)/g(z)$ are both subordinate to a univalent function whose range is symmetric with respect to the real axis. Several related classes of functions are also considered, and connections to earlier known results are made.

1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. The Koebe one-quarter theorem [1] ensures that the image of $\mathbb{D}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius 1/4. Thus every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z)) = z, \ (z \in \mathbb{D})$ and

$$f(f^{-1}(w)) = w, \quad (|w| < r_0(f),\ r_0(f) \geq 1/4).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{D}$ if both f and f^{-1} are univalent in $\mathbb{D}$. Let σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{D}$. A domain $D \subset \mathbb{C}$ is *convex* if the line segment joining any two points in D lies entirely in D, while a domain is *starlike* with respect to a point $w_0 \in D$ if the line segment joining any point of D to w_0 lies inside D. A function $f \in \mathcal{A}$ is starlike if $f(\mathbb{D})$ is a starlike domain with respect to the origin, and convex if $f(\mathbb{D})$ is convex. Analytically, $f \in \mathcal{A}$ is starlike if and only if $\operatorname{Re} zf'(z)/f(z) > 0$, whereas $f \in \mathcal{A}$ is convex if and only if $1 + \operatorname{Re} zf''(z)/f'(z) > 0$. The classes consisting of starlike and convex functions are denoted by $\mathcal{ST}$ and $\mathcal{CV}$ respectively. The classes $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ of starlike and convex functions of order α, $0 \leq \alpha < 1$, are respectively characterized by $\operatorname{Re} zf'(z)/f(z) > \alpha$ and $1 + \operatorname{Re} zf''(z)/f'(z) > \alpha$. Various subclasses of starlike and convex functions are often investigated. These functions are typically characterized by the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying in a certain domain starlike with respect to 1 in the right-half plane. Subordination is useful to unify these subclasses.

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An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [2] unified various subclasses of starlike and convex functions for which either of the quantity $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk $\mathbb{D}$, $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) \prec \varphi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $1 + zf''(z)/f'(z) \prec \varphi(z)$. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{ST}_\sigma(\varphi)$ and $\mathcal{CV}_\sigma(\varphi)$.

Lewin [3] investigated the class σ of bi-univalent functions and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [4,5]). Brannan and Taha [6] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and obtained estimates on the initial coefficients. Recently, Srivastava et al. [7] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Bounds for the initial coefficients of several classes of functions were also investigated in [8–12].

In this paper, estimates on the initial coefficients for bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type functions are obtained. Several related classes are also considered, and a connection to earlier known results are made. The classes introduced in this paper are motivated by the corresponding classes investigated in [9].

2. Coefficient estimates

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0) = 1$, $\varphi'(0) > 0$, and $\varphi(\mathbb{D})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{2.1}$$

A function $f \in \mathcal{A}$ with $\operatorname{Re}(f'(z)) > 0$ is known to be univalent. This motivates the following class of functions. A function $f \in \sigma$ is said to be in the class $\mathcal{H}_\sigma(\varphi)$ if the following subordinations hold:

$$f'(z) \prec \varphi(z) \quad \text{and} \quad g'(w) \prec \varphi(w), \qquad g(w) := f^{-1}(w).$$

For functions in the class $\mathcal{H}_\sigma(\varphi)$, the following result is obtained.

Theorem 2.1. *If $f \in \mathcal{H}_\sigma(\varphi)$ is given by*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2.2}$$

then

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|3B_1^2 - 4B_2 + 4B_1|}} \quad \text{and} \quad |a_3| \le \left(\frac{1}{3} + \frac{B_1}{4}\right) B_1. \tag{2.3}$$

Proof. Let $f \in \mathcal{H}_\sigma(\varphi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$f'(z) = \varphi(u(z)) \quad \text{and} \quad g'(w) = \varphi(v(w)). \tag{2.4}$$

Define the functions p_1 and p_2 by

$$p_1(z) := \frac{1+u(z)}{1-u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad \text{and} \quad p_2(z) := \frac{1+v(z)}{1-v(z)} = 1 + b_1 z + b_2 z^2 + \cdots,$$

or, equivalently,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \cdots\right) \tag{2.5}$$

and

$$v(z) = \frac{p_2(z)-1}{p_2(z)+1} = \frac{1}{2}\left(b_1 z + \left(b_2 - \frac{b_1^2}{2}\right) z^2 + \cdots\right). \tag{2.6}$$

Then p_1 and p_2 are analytic in $\mathbb{D}$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : \mathbb{D} \to \mathbb{D}$, the functions p_1 and p_2 have a positive real part in $\mathbb{D}$, and $|b_i| \le 2$ and $|c_i| \le 2$. In view of (2.4)–(2.6), clearly

$$f'(z) = \varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \quad \text{and} \quad g'(w) = \varphi\left(\frac{p_2(w)-1}{p_2(w)+1}\right). \tag{2.7}$$

Using (2.5) and (2.6) together with (2.1), it is evident that

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \cdots \tag{2.8}$$

and

$$\varphi\left(\frac{p_2(w)-1}{p_2(w)+1}\right) = 1 + \frac{1}{2}B_1b_1w + \left(\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2\right)w^2 + \cdots. \tag{2.9}$$

Since $f \in \sigma$ has the Maclaurin series given by (2.2), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + \cdots.$$

Since

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + \cdots \quad \text{and} \quad g'(w) = 1 - 2a_2w + 3(2a_2^2 - a_3)w^2 + \cdots,$$

it follows from (2.7)–(2.9) that

$$2a_2 = \frac{1}{2}B_1c_1, \tag{2.10}$$

$$3a_3 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2, \tag{2.11}$$

$$-2a_2 = \frac{1}{2}B_1b_1 \tag{2.12}$$

and

$$3(2a_2^2 - a_3) = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2b_1^2. \tag{2.13}$$

From (2.10) and (2.12), it follows that

$$c_1 = -b_1. \tag{2.14}$$

Now (2.11)–(2.14) yield

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{4(3B_1^2 - 4B_2 + 4B_1)},$$

which, in view of the well-known inequalities $|b_2| \le 2$ and $|c_2| \le 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.3).

By subtracting (2.13) from (2.11), further computations using (2.10) and (2.14) lead to

$$a_3 = \frac{1}{12}B_1(c_2 - b_2) + \frac{1}{16}B_1^2c_1^2,$$

and this yields the estimate given in (2.3). □

Remark 2.1. For the class of strongly starlike functions, the function φ is given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2z^2 + \cdots \quad (0 < \gamma \le 1),$$

which gives $B_1 = 2\gamma$ and $B_2 = 2\gamma^2$. Hence the inequalities in (2.3) reduce to the result in [7, Theorem 1, inequality (2.4), p. 3]. In the case

$$\varphi(z) = \frac{1 + (1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \cdots,$$

then $B_1 = B_2 = 2(1-\gamma)$, and thus the inequalities in (2.3) reduce to the result in [7, Theorem 2, inequality (3.3), p. 4].

A function $f \in \sigma$ is said to be in the class $\mathcal{ST}_\sigma(\alpha, \varphi)$, $\alpha \geq 0$, if the following subordinations hold:

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} \prec \varphi(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} \prec \varphi(w), \quad g(w) := f^{-1}(w).$$

Note that $\mathcal{ST}_\sigma(\varphi) \equiv \mathcal{ST}_\sigma(0, \varphi)$. For functions in the class $\mathcal{ST}_\sigma(\alpha, \varphi)$, the following coefficient estimates are obtained.

Theorem 2.2. *Let f given by (2.2) be in the class $\mathcal{ST}_\sigma(\alpha, \varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2(1+4\alpha) + (B_1 - B_2)(1+2\alpha)^2|}}, \tag{2.15}$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{(1+4\alpha)}. \tag{2.16}$$

Proof. Let $f \in \mathcal{ST}_\sigma(\alpha, \varphi)$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = \varphi(v(w)), \quad (g = f^{-1}). \tag{2.17}$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + (2(1+3\alpha)a_3 - (1+2\alpha)a_2^2)z^2 + \cdots$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = 1 - (1+2\alpha)a_2 w + ((3+10\alpha)a_2^2 - 2(1+3\alpha)a_3)w^2 + \cdots,$$

then (2.8), (2.9) and (2.17) yield

$$a_2(1+2\alpha) = \frac{1}{2}B_1 c_1, \tag{2.18}$$

$$2(1+3\alpha)a_3 - (1+2\alpha)a_2^2 = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2, \tag{2.19}$$

$$-(1+2\alpha)a_2 = \frac{1}{2}B_1 b_1 \tag{2.20}$$

and

$$(3+10\alpha)a_2^2 - 2(1+3\alpha)a_3 = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2. \tag{2.21}$$

It follows from (2.18) and (2.20) that

$$c_1 = -b_1. \tag{2.22}$$

Eqs. (2.19)–(2.22) lead to

$$a_2^2 = \frac{B_1^3(b_2 + c_2)}{4(B_1^2(1+4\alpha) + (B_1 - B_2)(1+2\alpha)^2)},$$

which, in view of the inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, yield

$$|a_2|^2 \leq \frac{B_1^3}{|B_1^2(1+4\alpha) + (B_1 - B_2)(1+2\alpha)^2|}.$$

Since $B_1 > 0$, the last inequality gives the desired estimate on $|a_2|$ given in (2.15).

Now, further computations from (2.19)–(2.22) lead to

$$a_3 = \frac{(B_1/2)((3+10\alpha)c_2 + (1+2\alpha)b_2) + b_1^2(1+3\alpha)(B_2 - B_1)}{4(1+3\alpha)(1+4\alpha)}.$$

Using the inequalities $|b_1| \le 2$, $|b_2| \le 2$ and $|c_2| \le 2$ for functions with positive real part yield

$$|a_3| \le \frac{(B_1/2)(2(3+10\alpha)+2(1+2\alpha))+4(1+3\alpha)(B_2-B_1)}{4(1+3\alpha)(1+4\alpha)} = \frac{B_1+|B_2-B_1|}{(1+4\alpha)}.$$

This is precisely the estimate in (2.16). $\square$

For $\alpha = 0$, Theorem 2.2 readily yields the following coefficient estimates for Ma-Minda bi-starlike functions.

Corollary 2.1. *Let f given by (2.2) be in the class $\mathcal{ST}_\sigma(\varphi)$. Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2+B_1-B_2|}} \quad \text{and} \quad |a_3| \le B_1 + |B_2 - B_1|.$$

Remark 2.2. For the class of strongly starlike functions, the function φ is given by

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \le 1),$$

and so $B_1 = 2\gamma$ and $B_2 = 2\gamma^2$. Hence, when $\alpha = 0$ (bi-starlike function), the inequality in (2.15) reduces to the estimates in [6, Theorem 2.1]. On the other hand, when $\alpha = 0$ and

$$\varphi(z) = \frac{1+(1-2\gamma)z}{1-z} = 1 + 2(1-\gamma)z + 2(1-\gamma)z^2 + \cdots,$$

then $B_1 = B_2 = 2(1-\gamma)$ and thus the inequalities in (2.15) and (2.16) reduce to the estimates in [6, Theorem 3.1].

Next, a function $f \in \sigma$ belongs to the class $\mathcal{M}_\sigma(\alpha, \varphi)$, $\alpha \ge 0$, if the following subordinations hold:

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z)$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w),$$

$g(w) := f^{-1}(w)$. A function in the class $\mathcal{M}_\sigma(\alpha, \varphi)$ is called bi-Mocanu-convex function of Ma-Minda type. This class unifies the classes $\mathcal{ST}_\sigma(\varphi)$ and $\mathcal{CV}_\sigma(\varphi)$.

For functions in the class $\mathcal{M}_\sigma(\alpha, \varphi)$, the following coefficient estimates hold.

Theorem 2.3. *Let f given by (2.1) be in the class $\mathcal{M}_\sigma(\alpha, \varphi)$. Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{(1+\alpha)|B_1^2 + (1+\alpha)(B_1 - B_2)|}} \tag{2.23}$$

and

$$|a_3| \le \frac{B_1 + |B_2 - B_1|}{1+\alpha}. \tag{2.24}$$

Proof. If $f \in \mathcal{M}_\sigma(\alpha, \varphi)$, then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(u(z)) \tag{2.25}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = \varphi(v(w)). \tag{2.26}$$

Since

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1+\alpha)a_2 z + (2(1+2\alpha)a_3 - (1+3\alpha)a_2^2)z^2 + \cdots$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)}+\alpha\left(1+\frac{wg''(w)}{g'(w)}\right)=1-(1+\alpha)a_2w+((3+5\alpha)a_2^2-2(1+2\alpha)a_3)w^2+\cdots,$$

from (2.8), (2.9), (2.25) and (2.26), it follows that

$$(1+\alpha)a_2=\frac{1}{2}B_1c_1, \tag{2.27}$$

$$2(1+2\alpha)a_3-(1+3\alpha)a_2^2=\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{2.28}$$

$$-(1+\alpha)a_2=\frac{1}{2}B_1b_1 \tag{2.29}$$

and

$$(3+5\alpha)a_2^2-2(1+2\alpha)a_3=\frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2. \tag{2.30}$$

Eqs. (2.27) and (2.29) yield

$$c_1=-b_1. \tag{2.31}$$

From (2.28), (2.30) and (2.31), it follows that

$$a_2^2=\frac{B_1^3(b_2+c_2)}{4(1+\alpha)(B_1^2+(1+\alpha)(B_1-B_2))},$$

which yields the desired estimate on $|a_2|$ as described in (2.23).

As in the earlier proofs, use of (2.28)–(2.31) shows that

$$a_3=\frac{(B_1/2)((1+3\alpha)b_2+(3+5\alpha)c_2)+b_1^2(1+2\alpha)(B_2-B_1)}{4(1+\alpha)(1+2\alpha)},$$

which yields the estimate (2.24). □

For $\alpha=0$, Theorem 2.3 gives the coefficient estimates for Ma-Minda bi-starlike functions, while for $\alpha=1$, it gives the following estimates for Ma-Minda bi-convex functions.

Corollary 2.2. *Let f given by (2.1) be in the class $\mathcal{CV}_\sigma(\varphi)$. Then*

$$|a_2|\le\frac{B_1\sqrt{B_1}}{\sqrt{2|B_1^2+2B_1-2B_2|}}\quad\text{and}\quad|a_3|\le\frac{1}{2}(B_1+|B_2-B_1|).$$

Remark 2.3. For φ given by

$$\varphi(z)=\frac{1+(1-2\gamma)z}{1-z}=1+2(1-\gamma)z+2(1-\gamma)z^2+\cdots,$$

evidently $B_1=B_2=2(1-\gamma)$, and thus when $\alpha=1$ (bi-convex functions), the inequalities in (2.23) and (2.24) reduce to a result in [6, Theorem 4.1].

Next, a function $f\in\sigma$ is said to be in the class $\mathcal{L}_\sigma(\alpha,\varphi)$, $\alpha\ge0$, if the following subordinations hold:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha}\prec\varphi(z)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha}\prec\varphi(w),$$

$g(w):=f^{-1}(w)$. This class also reduces to the classes of Ma-Minda bi-starlike and bi-convex functions. For functions in this class, the following coefficient estimates are obtained.

Theorem 2.4. *Let f given by (2.1) be in the class $\mathcal{L}_\sigma(\alpha,\varphi)$. Then*

$$|a_2| \leq \frac{2B_1\sqrt{B_1}}{\sqrt{|2(\alpha^2-3\alpha+4)B_1^2+4(\alpha-2)^2(B_1-B_2)|}} \tag{2.32}$$

and

$$|a_3| \leq \frac{2(3-2\alpha)\Big(B_1+|B_1-B_2|\Big)}{|(3-2\alpha)(\alpha^2-3\alpha+4)|}. \tag{2.33}$$

Proof. Let $f \in \mathcal{L}_\sigma(\alpha,\varphi)$. Then there are analytic functions $u, v : \mathbb{D} \to \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = \varphi(u(z)) \tag{2.34}$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha} = \varphi(v(w)). \tag{2.35}$$

Since

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\alpha} = 1+(2-\alpha)a_2z+\left(2(3-2\alpha)a_3+\frac{(\alpha-2)^2-3(4-3\alpha)}{2}a_2^2\right)z^2+\cdots$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\alpha}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\alpha} = 1-(2-\alpha)a_2w+\left(\left(8(1-\alpha)+\frac{1}{2}\alpha(\alpha+5)\right)a_2^2-2(3-2\alpha)a_3\right)w^2+\cdots,$$

from (2.8), (2.9), (2.34) and (2.35), it follows that

$$(2-\alpha)a_2 = \frac{1}{2}B_1c_1, \tag{2.36}$$

$$2(3-2\alpha)a_3+\left((\alpha-2)^2-3(4-3\alpha)\right)\frac{a_2^2}{2} = \frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2, \tag{2.37}$$

$$-(2-\alpha)a_2 = \frac{1}{2}B_1b_1 \tag{2.38}$$

and

$$\left(8(1-\alpha)+\frac{1}{2}\alpha(\alpha+5)\right)a_2^2-2(3-2\alpha)a_3 = \frac{1}{2}B_1\left(b_2-\frac{b_1^2}{2}\right)+\frac{1}{4}B_2b_1^2. \tag{2.39}$$

Now (2.36) and (2.38) clearly yield

$$c_1 = -b_1. \tag{2.40}$$

Eqs. (2.37), (2.39) and (2.40) lead to

$$a_2^2 = \frac{B_1^3(b_2+c_2)}{2(\alpha^2-3\alpha+4)B_1^2+4(\alpha-2)^2(B_1-B_2)},$$

which yields the desired estimate on $|a_2|$ as asserted in (2.32).

Proceeding similarly as in the earlier proof, using (2.37)–(2.40), it follows that

$$a_3 = \frac{(B_1/2)\Big((16(1-\alpha)+\alpha(\alpha+5))c_2+(3(4-3\alpha)-(\alpha-2)^2)b_2\Big)+2b_1^2(3-2\alpha)(B_1-B_2)}{4(3-2\alpha)(\alpha^2-3\alpha+4)}$$

which yields the estimate (2.33). □

Remark 2.4. Sharp estimates for the coefficients $|a_2|$, $|a_3|$ and other coefficients of functions belonging to the classes investigated in this paper are yet open problems. Indeed it would be of interest even to find estimates (not necessarily sharp) for $|a_n|$, $n \geq 4$.

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