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Note

Edge-connectivity augmentation of graphs over symmetric parity families

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Abstract

In this note we solve the edge-connectivity augmentation problem over symmetric parity families. It provides a solution for the minimum T-cut augmentation problem. We also extend a recent result of Zhang [C.Q. Zhang, Circular flows of nearly eulerian graphs and vertex splitting, J. Graph Theory 40 (2002) 147–161].

Keywords: Parity families; Edge-connectivity augmentation problem; Minimum T-cut augmentation problem

1. Introduction

Parity families were introduced in Goemans and Ramakrishnan [2]. In this note we consider only *symmetric* parity families. The definition and some examples for such families can be found in Section 2. Our definition is equivalent to that of Goemans and Ramakrishnan by Lemma 9 in [2].

The main purpose of this paper is to solve the minimum T-cut augmentation problem, namely for a given connected undirected graph G, a subset T of vertices of G of even cardinality and an integer k, what is the minimum number of new edges whose addition results in a graph where the minimum cardinality of a T-cut is at least k. In fact we will solve a more general problem, namely the minimum $\mathcal{F}$-cut augmentation problem where $\mathcal{F}$ is a symmetric parity family. Our main result (Theorem 5) also contains as a special case the min–max theorem of Watanabe and Nakamura [7] for the global edge-connectivity augmentation problem.

This paper is organized as follows. After the necessary definitions we give some easy properties of symmetric parity families and $\mathcal{F}$-joins. Then, in Section 4, we present a min–max theorem on the minimum value of a symmetric submodular function over a symmetric parity family. We need this result to prove, in Section 5, the existence of a splitting off that maintains the minimum $\mathcal{F}$-cut in a graph G for a symmetric parity family $\mathcal{F}$. This splitting off theorem will be applied, in Section 6, to solve the global edge-connectivity augmentation problem over a symmetric parity family. In Section 5 we also provide a common generalization of the weak orientation theorem of Nash-Williams [4] and a result of Rizzi [5]. In Section 7, we generalize a splitting off result of Zhang [8].

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2. Definitions

Let $G = (V, E)$ be an undirected graph. For a vertex s of G we denote by $\boldsymbol{\Gamma_G(s)}$ the set of neighbours of s. For $X, Y \subseteq V$, $\boldsymbol{\delta_G(X, Y)}$ denotes the set of edges between $X - Y$ and $Y - X$, $\boldsymbol{\delta_G(X)} = \delta_G(X, V - X)$, $\boldsymbol{d_G(X)} = |\delta_G(X)|$, $\boldsymbol{d_G(X, Y)} = |\delta_G(X, Y)|$ and $\boldsymbol{\overline{d}_G(X, Y)} = d_G(X \cap Y, V - (X \cup Y))$. The subgraph of G induced by X is denoted by $\boldsymbol{G[X]}$. For all $X, Y \subseteq V$,

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y), \tag{1}$$

$$d_G(X) + d_G(Y) = d_G(X - Y) + d_G(Y - X) + 2\overline{d}_G(X, Y). \tag{2}$$

For two vertices u and v of G, the *local edge-connectivity* between u and v is defined as $\boldsymbol{\lambda_G(u, v)} = \min\{d_G(X) : X \subset V, u \in X, v \in V - X\}$. Let $D = (V, A)$ be a directed graph. For a set $X \subseteq V$, the number of arcs leaving X is denoted by $\boldsymbol{d^+_D(X)}$. For two vertices u and v of D, the *local edge-connectivity* from u to v is defined $\boldsymbol{\lambda_D(u, v)} = \min\{d^+_D(X) : X \subset V, u \in X, v \in V - X\}$. More generally, for a function f on V and for $u, v \in V$, we define the *local f-connectivity* as $\boldsymbol{\lambda_f(x, y)} := \min\{f(X) : X \subset V, x \in X, y \in V - X\}$.

Suppose G is connected and let $T \subseteq V$ with $|T|$ even. The pair (G, T) is called *graft*. A subset X of V is called T-odd if $|X \cap T|$ is odd and then the cut $\delta(X)$ is called a *T-cut*. An edge set F is called a *T-join* if $T = \{v \in V : d_F(v) \text{ is odd}\}$. A *$T$-pairing* is a perfect matching of the complete graph on T.

A family $\mathcal{F}$ of subsets of V is called *symmetric parity family* if (i)–(iii) are satisfied: (i) $\emptyset, V \notin \mathcal{F}$, (ii) if $A \in \mathcal{F}$, then $V - A \in \mathcal{F}$, (iii) if $A, B \notin \mathcal{F}$ and $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{F}$. The most important examples for symmetric parity families are the following: $\mathcal{F} := 2^V - \{\emptyset, V\}$ and $\mathcal{F} := \{X \subset V : |X \cap T| \text{ is odd}\}$ where $T \subseteq V$ with $|T|$ even; for others see [2].

Let $\mathcal{F}$ be a symmetric parity family on V. Let $H = (V, E')$ be a tree. For each edge e of H, let $V_e \subset V$ so that $\delta_H(V_e, V - V_e) = \{e\}$. Let $\boldsymbol{J_{\mathcal{F}}(H)} := \{e \in E' : V_e \in \mathcal{F}\}$. An edge set $F \subseteq V \times V$ is called ***$\mathcal{F}$-join*** if there exists a tree H on V so that $F = J_{\mathcal{F}}(H)$. For a symmetric function f on V, let us define the value of an $\mathcal{F}$-join F by $\boldsymbol{val_f(F)} := \min\{\lambda_f(x, y) : xy \in F\}$. If $G = (V, E)$ is a graph and $X \in \mathcal{F}$, then $\delta_G(X)$ is called an ***$\mathcal{F}$-cut***. Let $\boldsymbol{\lambda^G_{\mathcal{F}}}$ denote the minimum size of an $\mathcal{F}$-cut, that is $\lambda^G_{\mathcal{F}} = \min\{d_G(X) : X \in \mathcal{F}\}$.

Let $G = (U, E)$ be a graph and $s \in U$. For two edges sr, st, the graph obtained from G by *splitting off* sr, st is denoted by $\boldsymbol{G_{r,t}} := G - \{sr, st\} + rt$. Let $\mathcal{F}$ be a symmetric parity family on $V \subseteq U$. The pair $\{sr, st\}$ is called ***$\lambda_{\mathcal{F}}$**-admissible* if after splitting off this pair, the minimum $\mathcal{F}$-cut does not decrease that is if $\lambda^{G_{r,t}}_{\mathcal{F}} \geq \lambda^G_{\mathcal{F}}$.

A function f on 2^V is called *symmetric* and *submodular* if for all $X, Y \subseteq V$, $f(X) = f(V - X)$ and $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$. Note that the degree function of an undirected graph is symmetric and, by (1), it is submodular. A function p on 2^V is called *skew-supermodular* if at least one of (3) and (4) holds for all $X, Y \subseteq V$.

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \tag{3}$$

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{4}$$

3. Preliminaries

An easy observation on T-odd sets, namely if we have two T-odd sets then either their intersection and union or their differences are T-odd sets, can be generalized for symmetric parity families as follows.

Claim 1. *Let $\mathcal{F}$ be a symmetric parity family. Then, for all $X, Y \in \mathcal{F}$, either $X \cap Y, X \cup Y \in \mathcal{F}$ or $X - Y, Y - X \in \mathcal{F}$.*

Proof. Suppose that $X \cap Y \notin \mathcal{F}$. Then, by $X \in \mathcal{F}$ and (iii), $X - Y \in \mathcal{F}$. Similarly, by $Y \in \mathcal{F}$ and (iii), $Y - X \in \mathcal{F}$. Now suppose that $X \cup Y \notin \mathcal{F}$, so by (ii), $V - (X \cup Y) \notin \mathcal{F}$. Since $X, Y \in \mathcal{F}$, we have, by (ii), $V - X, V - Y \in \mathcal{F}$. Then, by $V - X \in \mathcal{F}$ and (iii), $Y - X \in \mathcal{F}$. Similarly, by $V - Y \in \mathcal{F}$ and (iii), $X - Y \in \mathcal{F}$. □

Now we generalize the fact that a T-join and a T-cut always have an edge in common.

Lemma 1. *Let $\mathcal{F}$ be a symmetric parity family. If F is an $\mathcal{F}$-join and $A \in \mathcal{F}$, then $\delta_F(A) \neq \emptyset$.*

Proof. By definition there exists a tree H on V so that $F = J_{\mathcal{F}}(H)$. Let us denote by $A_1, \ldots, A_k$ the connected components of $H[A]$. Since $A \in \mathcal{F}$ and $\bigcup A_i = A$, there exists an index i so that $A_i \in \mathcal{F}$ by (iii). Let us denote by $B_1, \ldots, B_l$ the connected components of $H - A_i$. Since $V - A_i \in \mathcal{F}$ by (ii) and $\bigcup B_j = V - A_i$, there exists an index j so that $B_j \in \mathcal{F}$ by (iii). H is a tree, $H[A_i]$ and $H[B_j]$ are connected and B_j is a connected component of $H - A_i$, so there exists exactly one edge $e \in E(H)$ between $H[A_i]$ and $H[B_j]$. It follows that $e \in J_{\mathcal{F}}(H) = F$ and e enters A which has to be proved. $\square$

If $\mathcal{F} = \{T\text{-odd sets}\}$, then $\mathcal{F}$-joins can be characterized as follows.

Claim 2. *If $\mathcal{F} := \{X \subset V : |X \cap T| \text{ is odd}\}$ where $T \subseteq V$ with $|T|$ even, then the $\mathcal{F}$-joins are exactly the cycle free T-joins.*

Proof. Note that any tree H on V contains a T-join F and a set X belongs to $\mathcal{F}$ if and only if $d_F(X)$ is odd. Thus if F is a T-join and H is a tree on V containing F, then $J_{\mathcal{F}}(H) = F$ and the claim follows. $\square$

4. Min–max theorems

Let f be a symmetric submodular function on 2^V and $\mathcal{F}$ a symmetric parity family on V. It is mentioned in Goemans and Ramakrishnan [2] that there exists a cut equivalent tree H_f for f and f can be minimized over $\mathcal{F}$ using H_f, namely $\min\{f(X) : X \in \mathcal{F}\} = \mathrm{val}_f(J_{\mathcal{F}}(H_f))$. This can be presented as a min–max result.

Theorem 1. *Let f be a symmetric submodular function on 2^V and $\mathcal{F}$ be a symmetric parity family on V. Then*

$$\min\{f(X) : X \in \mathcal{F}\} = \max\{\mathrm{val}_f(F) : F \text{ is an } \mathcal{F}\text{-join}\}.$$

Proof. To prove max $\leq$ min, let $X' \in \mathcal{F}$ and let F' be an $\mathcal{F}$-join. By Lemma 1, there exists an edge $x'y' \in \delta_{F'}(X')$. Then $val_f(F') = \min\{\lambda_f(x, y) : xy \in F'\} \leq \lambda_f(x', y') = \min\{f(Y) : Y \subset V, x' \in Y, y' \in V - Y\} \leq f(X')$ and the inequality follows. As we mentioned above, $J_{\mathcal{F}}(H_f)$ provides equality, thus min = max . $\square$

If $\mathcal{F} := \{T\text{-odd sets}\}$ then the dual objects in Theorem 1 can be simplified. For $f = d_G$, the following theorem gives a result of Rizzi [5] on T-cuts.

Theorem 2. *Let f be a symmetric submodular function on 2^V and $\mathcal{F} := \{X \subset V : |X \cap T| \text{ is odd}\}$ where $T \subseteq V$ with $|T|$ even. Then*

$$\min\{f(X) : X \in \mathcal{F}\} = \max\{\mathrm{val}_f(P) : P \text{ is a } T\text{-pairing}\}.$$

Proof. Let $\alpha := \max\{\mathrm{val}_f(F) : F \text{ is an } \mathcal{F}\text{-join}\}$ and $\beta := \max\{\mathrm{val}_f(P) : P \text{ is a } T\text{-pairing}\}$. We show that $\alpha = \beta$ and then Theorem 1 implies Theorem 2. Since a perfect matching P on T is a T-join and, by Claim 2, is an $\mathcal{F}$-join, we have $\alpha \geq \beta$. Now, let F be an $\mathcal{F}$-join of value α for which $|F|$ is minimum. We prove that F is a T-pairing implying $\alpha \leq \beta$. Otherwise, F contains two adjacent edges uv and vw. Let F' be obtained from F by splitting off these edges. By Claim 2, F' is an $\mathcal{F}$-join. Since a set separating u and w separates either u and v or v and w, $val_f(F') \geq val_f(F) = \alpha$. Then F' is an optimum $\mathcal{F}$-join and $|F'| < |F|$, a contradiction. $\square$

We note that Theorem 2 is true for symmetric parity families that satisfy the following condition: if $X, Y \in \mathcal{F}$ and $X \cap Y = \emptyset$, then $X \cup Y \notin \mathcal{F}$. However, it can be shown that such a family can be defined as the T-odd sets for some $T \subseteq V$ with $|T|$ even.

5. Local edge-connectivity

Lemma 2. *Let $G = (U, E)$ be a graph, $s \in U$, $U - s \subseteq V \subseteq U$ and let $\mathcal{F}$ be a symmetric parity family on V. Then for all $A \in \mathcal{F}$ there exist $x \in A$ and $y \in V - A$ so that $\lambda_G(x, y) \geq \lambda^G_{\mathcal{F}}$.*

Proof. Let $f(X) := \min\{d_G(X), d_G(V-X)\}$ for all $X \subseteq V$. It is easy to check, by (1) and (2), that f is a symmetric submodular function on 2^V. Then, by Theorem 1, there exists an $\mathcal{F}$-join F on V with $\min\{\lambda_f(a,b) : ab \in F\} = \min\{f(X) : X \in \mathcal{F}\}$. By Lemma 1, there exists an edge $xy \in F$ with $x \in A$ and $y \in V-A$. Let $\delta_G(Y)$ be a minimum cut in G separating x and y so that $s \notin Y$. Then $\lambda_G(x,y) = d_G(Y)$ and $f(Y) \geq \lambda_f(x,y)$. Since $V = U - s$ or U, we have $Y \subseteq V$, thus $d_G(Y) \geq f(Y)$. It follows that $\lambda_G(x,y) = d_G(Y) \geq f(Y) \geq \lambda_f(x,y) \geq \min\{\lambda_f(a,b) : ab \in F\} = \min\{f(X) : X \in \mathcal{F}\} = \min\{\min\{d_G(X), d_G(V-X)\} : X \in \mathcal{F}\} \geq \lambda^G_{\mathcal{F}}$ and we are done. $\square$

We show two applications of the above lemma. The first result is on splitting off and it will be applied in Section 6 to prove the augmentation result.

Lemma 3. *Let $G = (V+s, E)$ be a graph so that $d(s) \neq 3$ and no cut edge is incident to s. Let $\mathcal{F}$ be a symmetric parity family on V. Then there exists a $\lambda_{\mathcal{F}}$-admissible pair.*

Proof. By Mader's local splitting off theorem [3], there exists a pair of edges $\{sr, st\}$ so that $\lambda_{G_{r,t}}(x,y) = \lambda_G(x,y)$ for all $x, y \in V$. By Lemma 2, applied for $U = V + s$ and V, for all $X \in \mathcal{F}$, there exist $x \in X$ and $y \in V - X$ so that $\lambda_G(x,y) \geq \lambda^G_{\mathcal{F}}$. Then $d_{G_{r,t}}(X) \geq \lambda_{G_{r,t}}(x,y) = \lambda_G(x,y) \geq \lambda^G_{\mathcal{F}}$ so $\lambda^{G_{r,t}}_{\mathcal{F}} \geq \lambda^G_{\mathcal{F}}$ that is $\{sr, st\}$ is a $\lambda_{\mathcal{F}}$-admissible pair. $\square$

The second application of Lemma 2 is the following orientation result. It is a common generalization of the weak orientation theorem of Nash-Williams [4] (if $\mathcal{F} = 2^V - \{\emptyset, V\}$) and an orientation theorem on T-cuts of Rizzi [5] (if $\mathcal{F} = \{T\text{-odd sets}\}$).

Theorem 3. *Let $G = (V, E)$ be an undirected graph and $\mathcal{F}$ a symmetric parity family on V. Then G has an orientation $\vec{G}$ so that $d^+_{\vec{G}}(X) \geq k$ for all $X \in \mathcal{F}$ if and only if $d_G(X) \geq 2k$ for all $X \in \mathcal{F}$.*

Proof. We prove only the non-trivial part. By Nash-Williams' well-balanced orientation theorem [4], there exists an orientation $\vec{G}$ of G such that $\lambda_{\vec{G}}(x,y) \geq \lfloor \lambda_G(x,y)/2 \rfloor$ for all $(x,y) \in V^2$. We show that $\vec{G}$ will do. By Lemma 2, applied for $U = V$, for every $X \in \mathcal{F}$, there exist $x \in X$ and $y \in V - X$ so that $\lambda_G(x,y) \geq \lambda^G_{\mathcal{F}}$. Then, since, by the condition, $\lambda^G_{\mathcal{F}} \geq 2k$, we have $d^+_{\vec{G}}(X) \geq \lambda_{\vec{G}}(x,y) \geq \lfloor \lambda_G(x,y)/2 \rfloor \geq \lfloor \lambda^G_{\mathcal{F}}/2 \rfloor \geq k$. $\square$

6. Augmentation

In this section we solve the following augmentation problem: Given a graph $G = (V, E)$, a symmetric parity family $\mathcal{F}$ on V and an integer k, what is the minimum number of edges whose addition results in a graph in which each $\mathcal{F}$-cut contains at least k edges. As a special case it solves the minimum T-cut augmentation problem: how many new edges must be added to a graft so that the minimum T-cut contains at least k edges. It also contains as a special case the global edge-connectivity augmentation problem, namely how many new edges must be added to a graph to make it k-edge-connected. A general approach to solve edge-connectivity augmentation problems is summarized in the following theorem.

Theorem 4 (*Frank [1]*). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric skew-supermodular function. Then there exists a graph $(V+s, K)$ with 2γ edges, all incident to s so that $d_K(X) \geq p(X)$ for all $X \subset V$ if and only if for each subpartition $\{X_1, \ldots, X_l\}$ of V, $\sum_{i=1}^{l} p(X_i) \leq 2\gamma$.*

Theorem 4 and Lemma 3 will provide the main result of this paper. We mention that it provides a new special case of the NP-hard problem of covering symmetric skew-supermodular functions (see in [6]) that can be polynomially solved.

Theorem 5. *For a connected graph $G = (V, E)$, a symmetric parity family $\mathcal{F}$ on V and an integer $k \geq 2$, the minimum cardinality of an $\mathcal{F}$-cut can be augmented to k by adding at most γ edges if and only if for each subpartition $\{X_1, \ldots, X_l\}$ of V with $X_i \in \mathcal{F}$, $\sum_{i=1}^{l}(k - d_G(X_i)) \leq 2\gamma$.*

Proof. For $X \subset V$, let $p(X) := k - d_G(X)$ if $X \in \mathcal{F}$, and $-\infty$ otherwise. Since $\mathcal{F}$ and d are symmetric so is p. We show that p is skew-supermodular. Let $X, Y \subseteq V$. If $X \notin \mathcal{F}$ or $Y \notin \mathcal{F}$ then (3) and (4) are satisfied. If $X, Y \in \mathcal{F}$, then, by Claim 1, either $X \cap Y, X \cup Y \in \mathcal{F}$ and then (3) is satisfied by (1) or $X - Y, Y - X \in \mathcal{F}$ and then (4) is satisfied by (2).

The subpartition condition of Theorem 5 implies that the subpartition condition of Theorem 4 is satisfied, thus, by Theorem 4, there exists a graph $(V + s, K)$ with 2γ edges, all incident to s so that $d_K(X) \geq p(X)$ for all $X \subset V$. Let $L := (V + s, E \cup K)$. Then $d_L(X) = d_G(X) + d_K(X) \geq d_G(X) + p(X) = k$ for all $X \in \mathcal{F}$ that is $\lambda^L_{\mathcal{F}} \geq k$. Note that $d_L(s) = 2\gamma \neq 3$ and, since G is connected, no cut edge of L is incident to s. Then, by Lemma 3, we can split off all the edges of L incident to s by preserving $\lambda_{\mathcal{F}}$. Thus the resulting graph G' is obtained from G by adding γ edges and $\lambda^{G'}_{\mathcal{F}} = \lambda^L_{\mathcal{F}} \geq k$. $\square$

By applying Theorem 5, for $\mathcal{F} = 2^V - \{\emptyset, V\}$, we get the theorem of Watanabe and Nakamura [7], and for $\mathcal{F} = \{T\text{-odd sets}\}$, we get the following theorem on T-cuts.

Theorem 6. *For a graft (G, T), the minimum cardinality of a T-cut can be augmented to k $(k \geq 2)$ by adding at most γ edges if and only if $\sum_1^l (k - d(X_i)) \leq 2\gamma$ for each subpartition $\{X_1, \ldots, X_l\}$ of $V(G)$ into T-odd sets.* $\square$

7. Splitting off

In this section we present a generalization of a result of Zhang [8].

For a graph $G = (V, E)$, $\boldsymbol{\mathcal{F}_G} := \{X \subset V : d_G(X) \text{ is odd}\}$ is a symmetric parity family such that each $\mathcal{F}_G$-cut is odd. The odd-edge-connectivity λ_o of G is defined as $\lambda^G_{\mathcal{F}_G}$.

Theorem 7 (*Zhang [8]*). *Let G be a graph with odd-edge-connectivity λ_o. Let s be a vertex of G such that $d(s) \neq \lambda_o$ and $\neq 2$. Arbitrarily label the edges of G incident with s as $\{e_1, \ldots, e_{d(s)}\}$. Then there is an integer $i \in \{1, \ldots, d(s)\}$ such that the new graph obtained from G by splitting e_i and e_{i+1} $(\text{mod } d(s))$ off at s remains of odd-edge-connectivity λ_o.*

Let s be a vertex of a graph $G = (V, E)$ and let N be a graph on vertex set $\Gamma_G(s)$. A pair $\{sr, st\}$ of edges of G is called N-**allowed** if $rt \in E(N)$. This definition is motivated by Theorem 7 in which we are only allowed to split off consecutive pairs of edges, $e_i = sv_i$ and $e_{i+1} = sv_{i+1}$ for some $1 \leq i \leq d(s)$, that is if $N_Z = (\Gamma_G(s), \{v_i v_{i+1} : 1 \leq i \leq d(s)\})$, then we can only split off N_Z-allowed pairs. Note that $v_1v_2, v_2v_3, \ldots, v_{d(s)}v_1$ provides an eulerian walk of N_Z, and hence N_Z is connected.

In the following we generalize Theorem 7 and provide a proof that is much shorter than that of [8].

Theorem 8. *Let $G = (V, E)$ be a graph, $s \in V$. Let $\mathcal{F}$ be a symmetric parity family on V such that $d_G(X) \equiv \lambda^G_{\mathcal{F}}$ (mod 2) for all $X \in \mathcal{F}$. Let $2 \leq d(s) \neq \lambda^G_{\mathcal{F}}$. Let $N = (\Gamma_G(s), M)$ be a connected graph with $M \neq \emptyset$. Then there exists an N-allowed $\lambda_{\mathcal{F}}$-admissible pair.*

Proof. We call a set $X \subseteq V - s$ **tight** if $X \in \mathcal{F}$ and $d_G(X) = \lambda^G_{\mathcal{F}}$. Since each element of $\mathcal{F}$ has the same parity as $\lambda^G_{\mathcal{F}}$, a pair $\{sr, st\}$ is not $\lambda_{\mathcal{F}}$-admissible if and only if there exists a tight set containing r and t. Let $t \in \Gamma_G(s)$. If t belongs to no tight set, then for an edge $rt \in M$ (r exists since N is connected and $M \neq \emptyset$) $\{sr, st\}$ is an N-allowed $\lambda_{\mathcal{F}}$-admissible pair. Otherwise, let Q be a maximal tight set containing t. Suppose $\Gamma_G(s) - Q = \emptyset$. If $\{s\} \in \mathcal{F}$, then $\lambda^G_{\mathcal{F}} \leq d(s)$. Since $\delta(s) \subseteq \delta(Q)$ and Q is tight, $d(s) \leq d(Q) = \lambda^G_{\mathcal{F}}$. These inequalities provide $\lambda^G_{\mathcal{F}} = d(s)$, a contradiction. If $\{s\} \notin \mathcal{F}$, then, since $V - Q \in \mathcal{F}$, $V - Q - s \in \mathcal{F}$ and then, by $d(s) \geq 2$, $\lambda^G_{\mathcal{F}} \leq d(V - Q - s) = d(Q) - d(s) < d(Q) = \lambda^G_{\mathcal{F}}$, a contradiction. Thus $\Gamma_G(s) - Q \neq \emptyset$, $t \in \Gamma_G(s) \cap Q$ and N is connected so there exists an edge $qr \in M$ such that $r \in Q, q \notin Q$. Then $\{sq, sr\}$ is N-allowed. The following claim completes the proof.

Claim 3. *$\{sq, sr\}$ is $\lambda_{\mathcal{F}}$-admissible.*

Proof. Suppose that $\{sq, sr\}$ is not $\lambda_{\mathcal{F}}$-admissible that is there is a tight set R containing q and r. By Claim 1, either $R - Q, Q - R \in \mathcal{F}$ and then, by (2) and the existence of sr, $\lambda^G_{\mathcal{F}} + \lambda^G_{\mathcal{F}} = d(R) + d(Q) = d(R - Q) + d(Q - R) + 2\overline{d}(Q, R) \geq \lambda^G_{\mathcal{F}} + \lambda^G_{\mathcal{F}} + 2$, a contradiction or $Q \cap R, Q \cup R \in \mathcal{F}$ and then, by (1) and the maximality of Q, $\lambda^G_{\mathcal{F}} + \lambda^G_{\mathcal{F}} = d(Q) + d(R) \geq d(Q \cap R) + d(Q \cup R) > \lambda^G_{\mathcal{F}} + \lambda^G_{\mathcal{F}}$, a contradiction. $\square$

Example. The following example shows the condition that each element of $\mathcal{F}$ is of the same parity cannot be omitted from Theorem 8. Let $G := (\{q, r, s, t\}, \{qr, qs, qs, rs, rt, st, st\})$. Let $\mathcal{F} := \{X : X \cap \{q, t\} = 1\}$. Then $\lambda^G_{\mathcal{F}} = 3$ and $d_G(s) = 5$. Let $N := (\{q, r, t\}, \{qr, rt\})$. Then for the N-allowed splittings, $\lambda^{G'}_{\mathcal{F}} = 2 < 3 = \lambda^G_{\mathcal{F}}$.

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