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GREEDY LINEAR EXTENSIONS WITH CONSTRAINTS

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Loosely speaking, a greedy linear extension of an ordered set is a linear extension obtained by following the rule: "climb as high as you can". Given an ordered set P and a partial extension P' of P is there a greedy linear extension of P which satisfies all of the inequalities of P'? We consider special instances of this question. In particular, we impose conditions bearing on the diagram of an ordered set. Our results have applications, to the 'jump number scheduling problem' and to the 'greedy dimension'.

1. Introduction

It is a well-known and often used fact that every ordered set has a linear extension. Linear extensions arise in scheduling problems, for example to construct 'optimal' linear extensions. This article is inspired by a particular scheduling problem: a single machine performs a set of jobs one at a time; precedence constraints prohibit the start of certain jobs until some others are already completed; a job which is performed immediately after a job which is not constrained to precede it requires a 'setup" or "jump"—entailing some fixed additional cost. The scheduling problem is to

construct a schedule to minimize the number of jumps.

It is commonly known as the 'jump number problem'.

In the language of ordered sets, the jobs and their precedence constraints are commonly rendered as an ordered set. A schedule is a linear extension of this ordered set. A linear extension L of an ordered set P can be viewed as a total order $L = \{x_1 < x_2 < \cdots\}$ of P, that is, $x_i < x_j(P)$ implies $i < j$. The linear extension can also be viewed as a decoposition of P into chains $L = C_1 \oplus C_2 \oplus \cdots$, where each C_k is a chain of P and $\sup C_k \nleq \inf C_{k+1}(P)$.

Obviously, for each i, x_{i+1} is minimal in $P - \{x_1, x_2, \ldots, x_i\}$. We say that L is a *greedy linear extension* of P if, for each i, if there is a minimal element x in $P - \{x_1, x_2, \ldots, x_i\}$ satisfying $x > x_i$, then $x_{i+1} > x_i$. (See Fig. 1.) Algorithmically a greedy linear extension is obtained by following the rule: "climb as high as you can". Equivalently, L is greedy if, for every $k < l$ for which $\inf C_l > \sup C_k(P)$, then there exists $x \in P - \bigcup_{r \leq k} C_r$ such that $x < \inf C_l(P)$. For several important classes of ordered sets, such greedy linear extensions are 'optimal' for the jump number scheduling problem (cf. Cogis and Habib [2], Rival [5], Rival and Zaguia [6]).

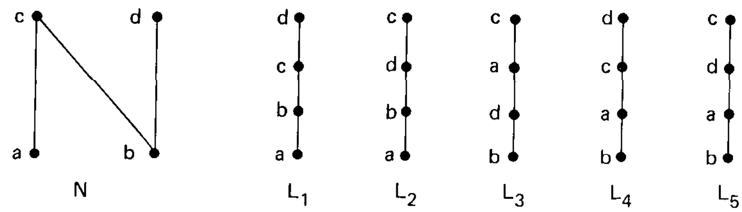

Fig. 1. L_1, L_2 and L_3 are the only greedy linear extensions of N.

Let a, b be elements of an ordered set P and suppose that $a \ngeq b$. It is well known that there is a linear extension L of P in which $a < b$. Indeed, there is even a greedy linear extension in which $a < b$ (see El-Zahar and Rival [3]). More generally, if a, b, c are elements of P satisfying $a \ngeq b$, $b \ngeq c$, and $a \ngeq c$, then there is again a linear extension L of P in which $a < b < c(L)$. Nevertheless (and this has been observed by several authors) it is not always possible to choose this linear extension to be greedy. The simplest example is the ordered set W (see Fig. 2). And, without any "subdiagram" W, there is no impediment at all to such greedy linear extensions.

Theorem 1. *Let P be a W-free ordered set. Then for every antichain $\{a_1, a_2, \ldots, a_n\}$, $n \geq 2$, in P, there is a greedy linear extension L of P such that $a_1 < a_2 < \cdots < a_n(L)$.*

Nonetheless, there must be more to the story. The converse of Theorem 1 cannot hold. The ordered set M illustrated in Fig. 3 has W as subdiagram yet, for every antichain $\{a_1, a_2, a_3\}$ there is a greedy linear extension in which $a_1 < a_2 < a_3$. If, in addition, the ordered set does not even contain a subdiagram N (that is, P is N-*free*) then there is a greedy linear extension $L = C_1 \oplus C_2 \oplus \cdots$ in which the sequence $a_1, a_2, \ldots, a_n$ is distributed consecutively in a sequence of these chains, that is, there is an integer $j \geq 1$ such that $a_i \in C_{j+i}$, for each $i = 1, 2, \ldots, n$. This property is not characteristic just of N-free ordered sets though (see El-Zahar and Rival [3]).

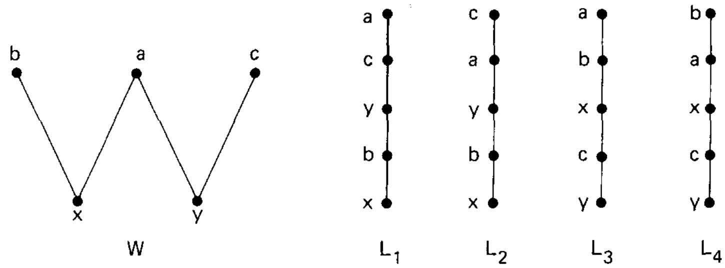

Fig. 2. L_1, L_2, L_3, L_4 are all of the greedy linear extensions of W. None satisfies $a < b < c$.

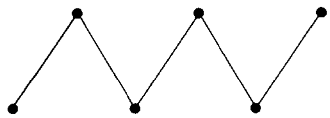

Fig. 3.

Here is a pleasant, although unexpected application of these techniques to the dimension of an ordered set. The *dimension* of an ordered set P, denoted by $\dim P$, is the minimum number of linear extensions of P whose intersection is P. The *greedy dimension* of P, denoted by $\dim_g P$, is the minimum number of greedy linear extensions of P whose intersection is P. This concept was introduced by Bouchitté, Habib and Jégou [1] who proved that $\dim_g P = \dim P$ for every N-free ordered set P.

Theorem 2. *If P is a* W*-free ordered set, then* $\dim_g P = \dim P$.

All of these considerations have led us to this apparently fundamental problem.

The greedy linear extension problem. *Let P be an ordered set and let P' be a partial extension. Is there a linear extension of P' which is itself a greedy linear extension of P? Is there an effective procedure to decide?*

For instance, if P' is itself a linear extension of P, then either it is a greedy linear extension of P, or it is not. In either case we can decide effectively. If $a \not> b$ in P and P' is the partial extension of P in which $a < b$ then we can construct a greedy linear extension of P which satisfies the 'constraint' $a < b$. The general problem though, seems to us difficult—even for ordered sets of length one. While subdiagrams W should be central their precise role remains yet to be delineated. We have at this writing some framents of a theory. In particular, *there is an inductive procedure to decide, whether or not there exists, for an antichain $\{a_1, a_2, b_1, b_2, \ldots, b_n\}$ of an ordered set P, a greedy linear extension L satisfying $b_j < a_i(L)$ for each $i = 1, 2$ and for each $j = 1, 2, \ldots, n$.*

If P has length one and if there are greedy linear extensions $L_1, L_2, \ldots$ such that

$$b_j < a_1(L_j) \quad \text{and} \quad b_j < a_2(L_j),$$

for every $j = 1, 2, \ldots$, then, as we shall see, there is a greedy linear extension L of P such that

$$b_j < a_1(L) \quad \text{and} \quad b_j < a_2(L),$$

for every $j = 1, 2, \ldots$. (This will follow from Theorem 8.) As a matter of fact, there is a simple procedure to recognize whether or not there is a greedy linear

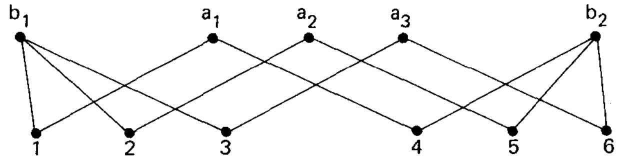

Fig. 4.

extension in which $b_j < a_1$ and $b_j < a_2$. This procedure fails, even for ordered sets of length one, if there are more a_i's involved. For example, for the ordered set illustrated in Fig. 4, there exist greedy linear extensions L_1 and L_2 such that

$$b_j < a_1(L_j), \qquad b_j < a_2(L_j) \quad \text{and} \quad b_j < a_3(L_j),$$

for every $j = 1, 2$ yet, there is no greedy linear extension L for which

$$b_j < a_1(L), \qquad b_j < a_2(L) \quad \text{and} \quad b_j < a_3(L),$$

for every $j = 1, 2$.

2. W-free ordered sets

Say that *a covers b* in *P* if, $a > b$ and if for each c satisfying $a > c \geq b$, then $c = b$. We write $a \succ b$ (or $b \prec a$) and call a an *upper cover* of b (or b a *lower cover* of a). Say that an ordered set S is a subdiagram of P if there is a cover-preserving subset of P isomorphic to S. Say P is S-free if P contains no subdiagram S at all (see Fig. 5).

To prove Theorem 1 we shall use this.

Lemma 3. *Let P be an ordered set which contains no* W *as subdiagram. Then for every element a in P there exists a greedy linear extension L of P such that* $x \leq a(L)$ *if and only if* $x \leq a(P)$.

Proof. We call an element u *accessible in P* if $D(u) = \{x \in P \mid x \leq u\}$ is a chain, and call u *maximally accessible* if u is accessible and y is not, for every $y > u$. Let $u_1, \ldots, u_k$ be the set of maximally accessible elements in $D(a)$. If there is j such that u_j is maximally accessible in P then $D(u_j)$ can be a first (greedy) chain in

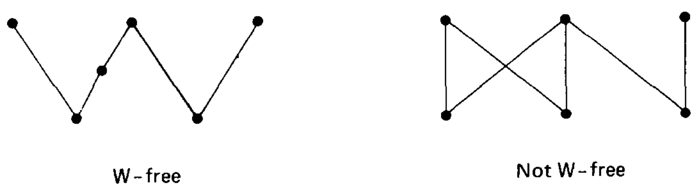

Fig. 5.

some greedy linear extension of P and, by induction, the proof is done. Assume that for every $j \in \{1, \ldots, k\}$, there exists v_j such that $v_j > u_j$, $v_j \nleq a$ and v_j is accessible in P. If there is $x \leq a$ such that x covers u_i and u_j then $\{u_i, u_j, v_i, x, v_j\} \cong \mathsf{W}$, is a subdiagram. Otherwise, let x be a minimal element with this property: $x \leq a$ and $x > u_i$ for some i. Since u_i is a maximally accessible element in $D(a)$, x is not accessible. By the minimality of x all lower covers of x are accessible so, there must exist $y < X$ and $y < u_j$, for some j. Let y_1 be an upper cover of y such that $y < y_1 \leq u_j$. Then $\{u_i, y, v_i, x, y_1\}$ is a subdiagram of P, ismorphic to W. That completes the proof. □

Proof of Theorem 1. We proceed by induction on n. Assume that P does not contain W as subdiagram and let $\{x_1, \ldots, x_n\}$, $n \geq 2$, be an antichain of P. By Lemma 3 there exists a greedy linear extension $L = C_1 \oplus C_2 \oplus \cdots$ of P such that $x_1 < x_i(L)$ for every $i \in \{2, \ldots, n\}$. Suppose that x_1 is in C_j; thus, $\{x_2, \ldots, x_n\} \subseteq \bigcup_{i>j} C_i$. By induction, there exists a greedy linear extension $L_1 = C'_1 \oplus C'_2 \oplus \cdots$ of $\bigcup_{i>j} C_i$ such that $x_2 < \cdots < x_n(L_2)$. Therefore

$$L' = C_1 \oplus \cdots \oplus C_j \oplus C'_1 \oplus C'_2 \oplus \cdots$$

is a greedy linear extension of P in which $x_1 < x_2 < \cdots < x_n$. □

Before we proceed to the proof of Theorem 2, we recall some further terminology. A chain C in P is called a *greedy chain* in P if $C = \{x \mid x \leq u\}$ for some maximally accessible element u in P. A noncomparable pair (a, b) in P is called a *critical pair* if $x < b$ implies $x < a$ and $x > a$ implies $x > b$. We say that a partial extension P' of P contains the critical pair (a, b) if $a < b(P')$. We denote by Crit P the set of all critical pairs of P.

For every noncomparable pair (x, y) in P there exist $a \geq x$ and $b \leq y$ such that (a, b) is a critical pair. For instance, choose a maximal element a with respect to the condition $a \ngtr y$ and then choose b a minimal element with respect to the condition $b \nleq a$.

If P is not a chain then dim P is also the minimum number of linear extensions of P whose union contains Crit P (see Kelly and Trotter [4]).

Proof of Theorem 2. Let P be an ordered set which contains no W as subdiagram and let L be a linear extension of P which contains the critical pairs (a_i, b_i), for $i = 1, \ldots, n$. It is enough to prove that there exists a greedy linear extension of P which contains the same critical pairs. We proceed by induction on n. Let $j \in \{1, \ldots, n\}$ such that $a_j < a_k$ in L for all $k \neq j$. Then the down set $D(a_j) = \{x \in P \mid x \leq a_j \text{ in } P\}$ does not contain any b_i, for $i = 1, \ldots, n$. Indeed, if $b_i < a_j$ then $a_i < b_1 < a_j$ in L, which contradicts the choice of j. Now, by Lemma 3, there exists a greedy linear extension $L' = C_1 \oplus C_2 \oplus \cdots$ of P such that $x \leq a_j(L')$ if and only if $x \leq a_j(P)$. Assume that $a_j \in C_k$ and, for every $i \neq j$, $a_1 \nleq a_j$ in P. If $x > a_j$ in C_k then $x > a_j$ in P and thus $x > b_j$ in P. However, $b_j \notin C_1 \cup \cdots \cup C_k$

which is a contradiction. Therefore $a_j = \sup C_k$. By induction, there exists a greedy linear extension L'' of $P - (C_1 \cup \cdots \cup C_k)$,

$$L'' = C_1' \oplus C_2' \oplus \cdots$$

such that $a_i < b_i(L'')$ for every $i \neq j$. Thus the greedy linear extension

$$C_1 \oplus \cdots \oplus C_k \oplus C_1' \oplus C_2' \oplus \cdots$$

of P contains all the critical pairs (a_i, b_i) for $i = 1, \ldots, n$. This completes the proof. □

3. The greedy linear extension problem

For the sequel we consider the question whether there exists a greedy linear extension of P in which $a_i < b_j$ for $i = 1, 2$ and $j = 1, 2, \ldots, n$, where $\{a_1, a_2, b_1, b_2, \ldots, b_n\}$ is an antichain of P. To this end we establish several lemmas.

Lemma 4 (El-Zahar and Rival [3]). *Let P be an ordered set. For every element a in P there exists a greedy linear extension L of P such that $x < a(L)$ for every element x noncomparable to a in P.*

In particular, if $\{a_1, a_2, \ldots, a_n\}$ is an antichain in P then there exists a greedy linear extension L of P such that $a_i < a_1(L)$, for every $i > 1$.

Lemma 5. *Let $\{a_1, a_2\}$ be an antichain in an ordered set P and let Q be a subset of P. If P contains an element u noncomparable with every element in Q and $u < a_i$, $i = 1, 2$, then there exists a greedy linear extension L of P such that $x < a_i$ for $i = 1, 2$ and for every x in Q.*

Proof. According to Lemma 4, there exists a greedy linear extension L of P such that $x < u(L)$ for every x noncomparable with u. Thus $x < a_i(L)$. □

Lemma 6. *Let $\{a_1, a_2\}$ be an antichain in an ordered set P. If P contains an element u such that $u < a_1$ and $u < a_2$ then there exists a greedy linear extension L of P such that $x < a_i(L)$, $i = 1, 2$, for every x noncomparable with a_1 and a_2.*

Proof. Let $L_1 = C_1 \oplus C_2 \oplus \cdots$ be a greedy linear extension of P such that $x < u(L)$ for every x noncomparable with u. Assume that $u \in C_i$ and set $P_1 = P - \bigcup_{j<i} C_j$. Every element in P_1 is comparable with u, the only minimal element in P_1. Thus, in $P_2 = P_1 - \{x \mid x \geqslant a_1\}$ every greedy chain contains at least two elements and is a greedy chain in P_1 since $a_1 > u$. If all the elements noncomparable with a_1 and a_2 are in $P - P_1$ then L_1 is the required greedy linear extension. Otherwise, and according to Lemma 4, there is a greedy chain K in P_2 which does not contain a_2. Thus in $P_1 - K$, a_1 is minimal. Let $L_2 = C'_1 \oplus C'_2 \oplus \cdots$ be a greedy linear extension of $P_2 - K$ such that $x < a_2(L)$ for every x noncomparable to a_2 in P_2. Let $L_3 = C''_1 \oplus C''_2 \oplus \cdots$ be a greedy linear extension of $\{x \mid x \geqslant a_1\}$. Thus in the greedy linear extension

$$L = C_1 \oplus \cdots \oplus C_{i-1} \oplus K \oplus C'_1 \oplus C'_2 \oplus \cdots \oplus C''_1 \oplus C''_2 \oplus \cdots,$$

of P, $x < a_i(L)$ for every x noncomparable with a_1 and a_2. □

Let $\{a_1, a_2, b_1, \ldots, b_n\}$ be an antichain of an ordered set P. We say that $\{a_1, a_2\}$ is separable relative to $\{b_1, \ldots, b_n\}$ if there exists a greedy linear extension $L = C_1 \oplus \cdots \oplus C_m$ of P such that $b_j < a_i(L)$ for each $i = 1, 2$ and for every $j = 1, \ldots, n$.

Proposition 7. *Let $\{a_1, a_2, b_1, \ldots, b_n\}$ be an antichain of the ordered set P. Then $\{a_1, a_2\}$ is separable relative to $\{b_1, \ldots, b_n\}$ if and only if there exists a greedy linear extension $L = C_1 \oplus \cdots \oplus C_m$ of P and $i \in \{1, \ldots, m\}$, such that $\{a_1, a_2\} \subseteq \bigcup_{k \geqslant i} C_k$ and $\bigcup_{k \geqslant i} C_k$ contains no subdiagram isomorphic to $\{c, d, a_1, a_2, x\} \cong \mathsf{W}$ in which $c < a_1$, $c < x$, $d < a_2$, $d < x$ and $x < b_j$ for some $j \in \{1, \ldots, n\}$.*

Proof. Assume that there exists a greedy linear extension $L = C_1 \oplus \cdots \oplus C_m$ such that $b_j < a_i(L)$ for every $i = 1, 2$ and $j = 1, \ldots, n$. Let k be the least index such that $\{b_1, \ldots, b_n\} \subseteq \bigcup_{i \leqslant k} C_i$. Thus $\{a_1, a_2\} \subseteq \bigcup_{i > k} C_i$ and $x \nleqslant b_j$ for every $x \in \bigcup_{i > k} C_i$ and for every $j = 1, 2, \ldots, n$.

Conversely, assume that $\{a_1, a_2\} \subseteq \bigcup_{i \geqslant k} C_i$ for some greedy linear extension $L = C_1 \oplus \cdots \oplus C_m$ of P, and, for every $x \leqslant b_j$ and x noncomparable with both a_1 and a_2, then $\{a_1, x, a_2\}$ is not in any subdiagram W in $\bigcup_{i \geqslant k} C_i$. Let $B = \{b_j \mid a_i < b_j(L)$ for $i = 1$ or $2\}$. If $B = \emptyset$ then the proof is done. Suppose that $a_1 < a_2(L)$ and $a_1 \in C_s$. In $\bigcup_{i \geqslant s} C_i$, the element a_1 is accessible. According to Lemma 4, there exists a greedy linear extension $L' = C'_1 \oplus \cdots \oplus C'_r$ of $\bigcup_{i \geqslant s} C_i$ such that $a_2 < a_1(L')$ and $b < a_1(L')$ for every $b \in B$. If $b < a_2$ for every $b \in B$ then $L_1 = C_1 \oplus \cdots \oplus C_{s-1} \oplus C'_1 \oplus \cdots \oplus C'_r$ is a greedy linear extension of P and $b_j < a_i(L)$ for every $i = 1, 2$ and $j = 1, \ldots, n$. Suppose that $a_2 \in C'_t$ and let $B_1 = \{b_j \mid a_2 < b_j(L')\}$. Thus in $\bigcup_{i \geqslant t} C'_i$, the elements a_1 and a_2 are both accessible. Without loss of generality, we can assume that every accessible element is comparable with either a_1 or a_2. Let u in $\bigcup_{i \geqslant t} C'_i$ such that u is noncomparable with $\{a_1, a_2\}$ and $u \leqslant b$, for some b in B_1. We choose such u minimal. Then u is not accessible and every lower cover of u is comparable with a_1 or a_2. Thus every lower cover of u is accessible and u has two lower covers v_1 and v_2 such that $v_1 < a_1$ and $v_2 < a_2$. The subdiagram $\{v_1, v_2, a_1, u, a_2\}$ of $\bigcup_{i \geqslant t} C'_i$ is isomorphic to W. This is a contradiction, which completes the proof. □

Let $\{a_1, a_2, b_1, \ldots, b_n\}$ be an antichain in an ordered set P. We set

$A_i = \{x \mid x < a_i$ and $x < b_j$ for some $j \in \{1, \ldots, n\}\}$ and $A'_i = \{x \mid x < a_i$ and x is noncomparable with b_j for every $j \in \{1, \ldots, n\}$ and for $i = 1, 2\}$. Let $A = \bigcup_{i=1,2} (A_i \cup A'_i)$. The next theorem gives an inductive procedure to recognize whether there is a greedy linear extension of P in which $b_j < a_i$ for every $i = 1, 2$ and $j = 1, \ldots, n$.

Theorem 8. *Let $\{a_1, a_2, b_1, \ldots, b_n\}$ be an antichain in an ordered set P. Then $\{a_1, a_2\}$ is not separable relative to $\{b_1, \ldots, b_n\}$ if and only if*

(i) *A is an antichain, A_1, A'_1, A_2, A'_2 are pairwise disjoint subsets and $A_i \cup A'_i \neq \emptyset$ for $i = 1, 2$,*

(ii) *for every u in A'_i, $\{u, a_j\}$ is not separable relative to $\{b_1, \ldots, b_n\}$ $(i \neq j)$, and*

(iii) *for u in $A_i \cup A'_i$ and $w > u$, if $v \in A_j \cup A'_j$, $i \neq j$, and $w \not\geq v$, then $\{u, v\}$ is not separable relative to* $\max(A \cup \{x \mid x < w\}) - \{u, v\}$.

Proof. Assume that $\{a_1, a_2\}$ is not separable relative to $\{b_1, \ldots, b_n\}$. Let x_1 and x_2 be lower covers of a_1 and a_2, respectively. Suppose that $x_1 \leq x_2$. Then $x_1 < a_1$ and $x_1 < a_2$. According to Lemma 6, there exists a greedy linear extension in which $b_j < a_i$ for each $i = 1, 2$ and every $j = 1, \ldots, n$, which contradicts the hypothesis that $\{a_1, a_2\}$ is not separable relative to $\{b_1, \ldots, b_n\}$. Thus A_1, A'_1, A_2 and A'_2 are disjoint subsets of P and $A = A_1 \cup A'_1 \cup A_2 \cup A'_2$ is an antichain of P. Now, if $A_i \cup A'_i = \emptyset$ for some $i \in \{1, 2\}$ then a_i is a minimal element in P. Thus there is no subdiagram W of P which contains a lower cover of a_i. According to Proposition 7, $\{a_1, a_2\}$ must be separable relative to $\{b_1, \ldots, b_n\}$ which is a contradiction. This proves the statement (i). To prove (ii), suppose that $\{x, a_j\}$ is separable relative to $\{b_1, \ldots, b_n\}$ for some element x in A'_i, where $i \neq j$. Thus there exists a greedy linear extension L of P such that $b_k < \{x, a_j\}(L)$ for every $k \in \{1, \ldots, n\}$; thus $b_k < \{a_1, a_2\}(L)$, which is a contradiction. Now, let $w > u$ for some element u in $A_i \cup A'_i$ and let $v \in A_j \cup A'_j$ such that $i \neq j$ and w is noncomparable with v. Suppose that $\{u, v\}$ is separable relative to the antichain $B = \max(A \cup \{x \mid x \leq w\}) - \{u, v\}$, (see Fig. 6). Thus, there exists a greedy linear extension $L = C_1 \oplus \cdots \oplus C_m$ of P such that $x < u(L)$ and $x < v(L)$ for every x in B. Suppose that $u < v(L)$ and $u \in C_k$. Since all the lower covers of w

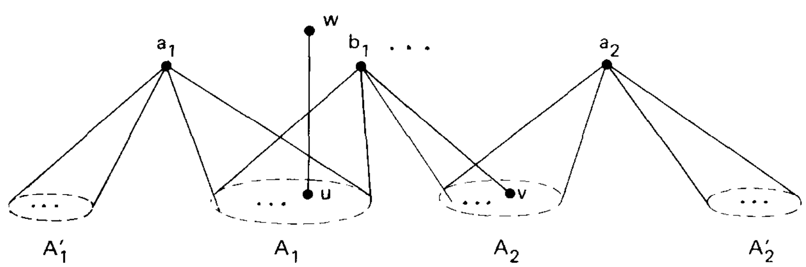

Fig. 6.

different from u are in $\bigcup_{i<k} C_i$, we consider a greedy chain C'_k in $\bigcup_{i\geq k} C_i$ which contains u and w. Thus, in $(\bigcup_{i\geq k} C_i) - C'_k$, there are no W's which contain a_i, since a_i is minimal in $(\bigcup_{i\geq k} C_i) - C'_k$. Therefore, and according to Proposition 7, $\{a_1, a_2\}$ is separable relative to $\{b_1, \ldots, b_n\}$. Now, if $v < u(L)$ and $v \in C_k$, then let $L' = C'_k \oplus \cdots$ be a greedy linear extension of $\bigcup_{i\geq k} C_i$ such that $u < v(L')$ (by using Lemma 4). Consider the greedy linear extension $L_1 = C_1 \oplus \cdots \oplus C_{k-1} \oplus C'_k \oplus \cdots$ of P. Obviously, $x < u(L_1)$ and $x < v(L_1)$ for every x in B, and $u < v(L_1)$. Applying the argument above to L_1, leads to the same contradiction, and this proves the statement (iii).

Conversely, suppose that there exists a greedy linear extension $L = C_1 \oplus \cdots \oplus C_m$ of P such that $b_j < a_i(L)$ for every $i = 1, 2$ and every $j = 1, \ldots, n$, and assume that the properties (i) and (ii) are satisfied. Let us assume that the maximum element of A with respect to L, belongs to $A_2 \cup A'_2$. Thus there exists $k \in \{1, \ldots, m\}$ and $u \in A_1 \cup A'_1$ such that $u \in C_k$, $(A_1 \cap A'_1) \cap (\bigcup_{i>k} C_i) = \emptyset$ and $(A_2 \cap A'_2) \cap (\bigcup_{i>k} C_i) \neq \emptyset$. (This means that we exhaust $A_1 \cup A'_1$ by L before $A_2 \cup A'_2$, and u is the maximum element of $A_1 \cup A'_1$ with respect to L.) According to the property (ii), $\{u, a_2\}$ is not separable relative to $\{b_1, \ldots, b_n\}$. Therefore $\bigcup_{i>k} C_i$ contains some element from $\{b_1, \ldots, b_n\}$. Also, and without loss of generality, we may assume that $A_1 \cup A'_1 = \{u\}$, $A_2 \cup A'_2 \neq \emptyset$, and $u \in C_1$. (Consider $\bigcup_{i\geq k} C_i$ instead of P.) Since the only lower cover of a_1 is u and $a_1 \notin C_1$ ($a_1 \in C_1$ contradicts the fact that $b_j < a_1(L)$ for every $j = 1, \ldots, n$) there exists another accessible element $w > u$. According to Lemma 4, we can consider a greedy linear extension L' of P,

$$L' = C'_1 \oplus \cdots \oplus C'_r$$

such that $x < u(L')$ for every x noncomparable with u in P. Therefore if $A - \{u\} = \{v_1, \ldots, v_s\}$ and $v_1 < v_2 < \cdots < v_s < u$ in L', then $\{u, v_s\}$ is separable relative to $\max(A \cup \{x \mid x \leq w\}) - \{u, v_s\}$ which implies that the property (iii) is not satisfied. This completes the proof. □

By definition, if $\{a_1, a_2\}$ is separable relative to $\{b_1, \ldots, b_n\}$ then $\{a_1, a_2\}$ is separable relative to b_i, for every i. The converse is not true. For instance, there are greedy linear extensions L_1 and L_2 of the ordered set illustrated in Fig. 7,

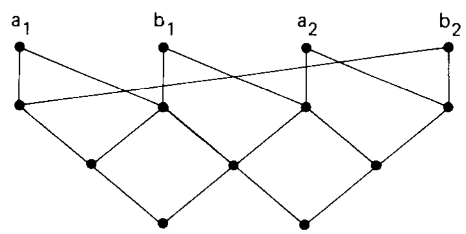

Fig. 7.

such that $b_1 < a_i(L_1)$ and $b_2 < a_i(L_2)$ for $i = 1, 2$. However there is no greedy linear extension L in which $b_1 < a_i(L)$ and $b_2 < a_i(L)$ for $i = 1, 2$.

4. The length one case

Assume that P is an ordered set of length one. Let A and B be subsets of maximal elements in P. Let D be the set $P - \max P$. We consider $B' \subseteq B$ such that for every x in B, if $y < x$ then $y < z$ for some z in A. Also, let $A' \subseteq A$ be such that, for every x in A', x has a lower cover y which is a lower cover for some element in B'. Let $D' \subseteq D$ be the set of all lower covers of B'. For brevity, we write $X < Y(L)$ for two subsets X and Y of P and a linear extension L of P, if $x < y(L)$ for every x in X and for every y in Y.

A first reduction of the problem is this.

Lemma 10. *Let P be an ordered set of length one. Let A and B be disjoint subsets of maximal elements in P such that $A \cup B = \max P$. There exists a greedy linear extension L of P such that $A < B(L)$ if and only if there exists a greedy linear extension L' of $P' = A' \cup B' \cup D'$ such that $A' < B'(L')$.*

Proof. Let $L = C_1 \oplus C_2 \oplus \cdots$ be a greedy linear extension of P such that $A < B(L)$ and let L_1 be the linear extension induced by L in $P' = A' \cup B' \cup D$. Since $A' \subseteq A$ and $B' \subseteq B$, then $A' < B'(L_1)$. Assume that $L' = C_1' \oplus C_2' \oplus \cdots$, where $C_i' = C_i \cap P'$, for $i \geq 1$. If L' is a greedy linear extension then the proof is done. Otherwise let i be the least index such that C_i' is not a greedy chain in $\bigcup_{k \geq i} C_k'$. Thus there is $x \in C_j'$ such that x is noncomparable with y for every $y \in C_k'$, $i < k < j$, and $x > z$ in P where $\{z\} = C_i'$, since the length of P is one. But $A' < B'(L')$ so, if $x \in A'$ then for every $y < x(L')$, $y \notin B'$. Therefore the linear extension

$$L_1 = C_1' \oplus \cdots \oplus C_1' \cup \{x\} \oplus \cdots \oplus C_j' - \{x\} \oplus \cdots$$

is greedy up to $C_i' \cup \{x\}$, and also $A' < B'(L_1')$. The same argument can be used if $x \in B'$ and $y \notin A'$, for every $y \in C_k'$ where $k \in \{i, \ldots, j-1\}$. Assume that $x \in B'$, and that there is y in A' such that $y \in C_k'$, $i < k < j$. Suppose that $C_i = \{z, t\}$. Thus $t > z$ and $x > z$ in P. Since $A \cup B = \max P$, $t \in A'$, which is a contradiction. So, $C_i = C_i' = \{z\}$. The linear extension L is greedy in P, so there is u in $\bigcup_{i<k<j} C_k$ such that $u \in P - P'$ and $u < x$. But $x \in B'$, thus $u \in D'$, which is a contradiction.

Conversely, assume that $L' = C_1' \oplus C_2' \oplus \cdots$ is a greedy linear extension of P' such that $A' < B'(L')$. We shall prove that L' is an initial segment of a greedy linear extension of P. Suppose that $C_i' = \{z\}$ and that there exists u in $P - P'$ such that $C_i' \cup \{u\}$ is a greedy chain in $P - \bigcup_{k<i} C_k'$. If $u \in B - B'$, then $u > u'$ where $u' \notin D'$. Therefore u cannot be accessible (before we take u'). If

$u \in A - A'$, then u cannot have a lower cover, which is, itself, a lower cover of some element in B'. But $u \succ z$ and $z \in D'$, which is a contradiction. Now, if $x \in B - B'$, since x is maximal in P, then x has a lower cover noncomparable with every element in A. So, this lower cover of x is in $D - D'$. Consider a greedy linear extension L of P which extends L' in such a way that all of the lower covers noncomparable with the elements of $A - A'$ are the last elements in D. Obviously we get $A < B(L)$. □

We shall assume that the set of maximal elements of P is $A \cup B$, and every minimal element is comparable to an element in A and an element in B. Let $D = \{1, \ldots, n\}$ be the set of minimal elements of P. For every a_i in A and for every $b_i \in B$ we define $A_i = \{x \in D \mid x < a_i\}$ and $B_i = \{x \in D \mid x < b_i\}$. Thus A induces a partition $A_1, \ldots, A_k$ of $\{1, \ldots, n)$, $k = |A|$, and B another partition $B_1, \ldots, B_r$ of $\{1, \ldots, n\}$, $r = |B|$. The problem to decide whether there is a greedy linear extension L such that $A < B(L)$ can then be rendered as follows:

Given partitions $(A_i)_{i \leqslant k}$ and $(B_i)_{i \leqslant r}$ of $\{1, \ldots, n\}$, is there a permutation σ of $\{1, \ldots, n\}$ such that if

$$B_i - \bigcup_{l<j} \sigma(l) = \sigma(j),$$

then there exists s such that

$$A_s - \bigcup_{l<j} \sigma(l) = \sigma(j).$$

For example, there is no greedy linear extension L of the ordered set illustrated in Fig. 4 for which $a_i < b_j$ for $i = 1, 2$ and $j = 1, 2, 3$. Moreover if we consider the two partitions $\{A_1 = \{1, 2, 3), A_2 = \{4, 5, 6\}\}$ and $\{B_1 = \{1, 4\}, B_2 = \{2, 5\}, B_3 = \{3, 6\}\}$ of $\{1, \ldots, 6\}$ defined by $\{a_1, a_2\}$ and $\{b_1, b_2, b_3\}$, we observe that for every permutation σ of $\{1, \ldots, 6\}$, either for some B_i, $B_i - \{\sigma(1)\} = \sigma(2)$ and $A_j - \sigma(1) \neq \sigma(2)$ for $j = 1, 2$, or $B_i - \{\sigma(1), \sigma(2), \sigma(3)\} = \sigma(4)$ and $A_j - \{\sigma(1), \sigma(2), \sigma(3)\} \neq \sigma(4)$ for $j = 1, 2$.

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