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On the Dirichlet problem in elasticity for a domain exterior to an arc

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Dedicated to Professor Dr.rer.nat. Erhard Meister on the occasion of his 60th birthday

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Abstract

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We consider here a Dirichlet problem for the two-dimensional linear elasticity equations in the domain exterior to an open arc in the plane. It is shown that the problem can be reduced to a system of boundary integral equations with the unknown density function being the jump of stresses across the arc. Existence, uniqueness as well as regularity results for the solution to the boundary integral equations are established in appropriate Sobolev spaces. In particular, asymptotic expansions concerning the singular behavior for the solution near the tips of the arc are obtained. By adding special singular elements to the regular splines as test and trial functions, an augmented Galerkin procedure is used for the corresponding boundary integral equations to obtain a quasi-optimal rate of convergence for the approximate solutions.

Keywords: Linear elasticity, singularities, boundary integral equations, augmented Galerkin method, crack tips, Gårding's inequality.

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1. Introduction

This paper extends the results of the work by the authors [21,29,31,32] on the crack and screen problems for the Laplace and Helmholtz equations as well as on the exterior elasticity problems with a regular smooth boundary to the Dirichlet problem for the two-dimensional Lamé system exterior to an arc. Some of the results here were presented in [18] and have been recently generalized to the Neumann problem [33]. Throughout the paper, let Γ be an open arc in the plane $\mathbb{R}^2$, i.e., a simple oriented curve joining two points which will be termed the "crack tips". We consider here the boundary value problem consisting of the linear elasticity equation for the displacement field $\boldsymbol{u}$:

$$\Delta^* \boldsymbol{u} := \mu \Delta \boldsymbol{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega_\Gamma = \mathbb{R}^2 \backslash \bar{\Gamma}, \tag{E}$$

together with the boundary condition

$$\boldsymbol{u}|_\Gamma = \boldsymbol{g}, \tag{B}$$

where $\mu > 0$ and $\lambda > -\mu$ are given Lamé constants [13], and $\boldsymbol{g}$ is a prescribed smooth function. In addition, we assume that $\boldsymbol{u} - \boldsymbol{r}$ is *regular* at infinity, where $\boldsymbol{r} = \boldsymbol{r}(x)$ denotes the rigid motion associated with $\boldsymbol{u}$. Following [21,23], by this we mean

$$\mathrm{D}^\alpha(\boldsymbol{u} - \boldsymbol{r}) = \mathrm{O}(|\boldsymbol{x}|^{-\alpha-1}), \quad \alpha = 0, 1, \text{ as } |x| \to \infty, \tag{C}$$

with $\mathrm{D} = \partial/\partial x_i$. More precisely, let us represent the rigid motion $\boldsymbol{r}(x)$ in the form

$$\boldsymbol{r}(x) = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 (x_2 \hat{e}_1 - x_1 \hat{e}_2),$$

where $\hat{e}_i$ denotes the unit vectors in $\mathbb{R}^2$ and the ω_i's are constants, which together with $\boldsymbol{u}$ are unknown. As indicated in [17,21], condition (C) implies that

$$\int_\Gamma [T(\boldsymbol{u})] \, \mathrm{d}s_y = \boldsymbol{0}, \tag{C$_1$}$$

and in order to ensure the uniqueness, we further impose the equilibrium condition of vanishing total momentum

$$\int_\Gamma (y_2 \hat{e}_1 - y_1 \hat{e}_2) \cdot [T(\boldsymbol{u})] \, \mathrm{d}s_y = 0, \tag{C$_2$}$$

which will become transparent later (see (2.12)). Here, $\mathrm{d}s$ stands for the arc-length element and y denotes the point of integration. We note that condition (C_2) will not be needed in the case when ω_3 is given [21]. Here $[T(\boldsymbol{u})]$ stands for the jump of traction $T(\boldsymbol{u})$ across Γ,

$$T(\boldsymbol{u}) := 2\mu \frac{\partial}{\partial \hat{n}} \boldsymbol{u} + \lambda \hat{n} \operatorname{div} \boldsymbol{u} + \mu \hat{n} \times \operatorname{curl} \boldsymbol{u}, \tag{1.1}$$

with $\hat{n}$ being the unit normal to Γ, and $\operatorname{curl} \boldsymbol{u} := \operatorname{curl}(u_1, u_2, 0)$.

In the following we shall refer to the problem defined by (E), (B), (C_1) and (C_2) as the crack problem (P). Such crack problems arise if, e.g., an inlet of rigid material is immersed at Γ into the elastic material occupying Ω_Γ. Note that this rigid part might also be moved against the surrounding infinite elastic material. We remark that, alternatively, we may also formulate the crack problem with (B) replaced by the two boundary conditions

$$\boldsymbol{u}|_{\Gamma_+} = \boldsymbol{u}_+ \quad \text{and} \quad \boldsymbol{u}|_{\Gamma_-} = \boldsymbol{u}_-.$$

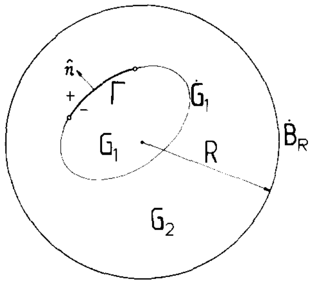

Fig. 1.

The orientation of Γ defines the direction of the normal vector $\hat{n}$ (see Fig. 1). Here $\Gamma_{\pm}$ denote the positive and negative sides of Γ, correspondingly. The jump, $[\boldsymbol{u}] := \boldsymbol{u}_- - \boldsymbol{u}_+$ vanishes at the tips of Γ. As usual, $\boldsymbol{u}|_{\Gamma_+}$ and $\boldsymbol{u}|_{\Gamma_-}$ denote the traces of $\boldsymbol{u}$ on Γ from the + and − sides, respectively. Then our approach below can be easily adapted with $\boldsymbol{g}$ replaced appropriately (see, e.g., [15]).

Our aim is to develop a solution procedure for (P) by making use of an integral equation method which allows us to obtain the explicit singular behavior of the "stress" near the tips of Γ. Following [21,32], we reduce the problem (P) to a system of boundary integral equations of the first kind [12,16,35] with the jump of traction across Γ as the unknown. These boundary integral equations are derived by the "direct approach" based on the Betti formula. By using the method of local Mellin transform as in [4–8,10,22] and the calculus of pseudodifferential operators [11,28], we establish existence, uniqueness and regularity results for the solution of our boundary integral equations. In particular, we are able to obtain appropriate asymptotic expansions for the jump of tractions near the tips of Γ. The latter provides us useful information concerning numerical treatment such as the Galerkin scheme for our boundary integral equations. In fact, in our augmented boundary element method, we use, as in [24,32,34,36] in addition to the regular finite elements, appropriate singular elements concentrated near the tips and improve significantly the asymptotic convergence rates of our approximate solutions.

It should be emphasized that since our boundary integral equations are derived directly from the Betti formula, the boundary charges are precisely the jumps of tractions across Γ. From our boundary element method using augmented test and trial function spaces with the appropriate singular elements, we are able to compute both approximate boundary charges and the stress intensity factors simultaneously. Hence, our asymptotic error estimates include explicit estimates for the stress intensity factors, as well.

2. Integral representation

We begin with the variational formulation for the problem (P). We then derive the integral representation for the variational solution by the direct method based on the Betti formula. In order to characterize the variational solution of (P), we introduce the vector-valued function space $\boldsymbol{H}_c^1(\Omega_\Gamma)$, the completion of all C^∞-functions $\boldsymbol{f}(x)$ of the form

$$\boldsymbol{f}(x) = \boldsymbol{f}_0(x) + \boldsymbol{r}(x), \tag{2.1}$$

with respect to the norm $\|\cdot\|_{1,c}$ defined by

$$\|\boldsymbol{f}\|_{1,c} := \left\{ \int_{\Omega_\Gamma} \mathscr{E}(\boldsymbol{f}, \boldsymbol{f})\,\mathrm{d}x + \int_\Gamma |\boldsymbol{f}|^2\,\mathrm{d}s \right\}^{1/2}, \tag{2.2}$$

where f_0 is regular at infinity and r denotes a rigid motion of the form

$$r(x) = \omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3(x_2\hat{e}_1 - x_1\hat{e}_2) =: M(x)\vec{\omega}, \tag{2.3}$$

with $\vec{\omega} = (\omega_1, \omega_2, \omega_3)^{\mathrm{T}}$ and $M(x)$ the corresponding 2×3 matrix. Here

$$\begin{aligned}\mathscr{E}(f, g) := {} & (\lambda + \mu) \operatorname{div} f \operatorname{div} g + \frac{1}{2}\mu \sum_{j \neq k = 1}^{2} \left(\frac{\partial f_j}{\partial x_k} + \frac{\partial f_k}{\partial x_j}\right)\left(\frac{\partial g_j}{\partial x_k} + \frac{\partial g_k}{\partial x_j}\right) \\ & + \frac{1}{2}\mu \sum_{j,k=1}^{2} \left(\frac{\partial f_j}{\partial x_k} - \frac{\partial f_k}{\partial x_j}\right)\left(\frac{\partial g_j}{\partial x_k} - \frac{\partial g_k}{\partial x_j}\right)\end{aligned} \tag{2.4}$$

is a bilinear form for the derivatives of f and g (see [23]). We denote by $\mathring{H}_c^1(\Omega_\Gamma)$ the closed subspace of $H_c^1(\Omega_\Gamma)$ such that

$$\mathring{H}_c^1(\Omega_\Gamma) = \{ f \in H_c^1(\Omega_\Gamma) \colon f|_\Gamma = \mathbf{0} \}.$$

For the variational formulation of the crack problem (P), let $h \in H^1(\mathbb{R}^2)$ with compact support be an extension of the boundary values $g \in H^{1/2}(\Gamma)$ with $h|_\Gamma = g$. Hence, $h \in H_c^1(\Omega_\Gamma)$ and it is regular at infinity. By a simple modification of the results in [21], one can show that the variational formulation for problem (P) reads: *For given $h \in H_c^1(\Omega_\Gamma)$, find a function $u \in \mathring{H}_c^1(\Omega_\Gamma)$ such that $u - h \in \mathring{H}_c^1(\Omega_\Gamma)$ and*

$$B(u, \Phi) := \int_{\Omega_\Gamma} \mathscr{E}(u, \Phi) \, \mathrm{d}x = 0 \tag{2.5}$$

for all $\Phi \in \mathring{H}_c^1(\Omega_\Gamma)$. Alternatively, with $v := u - h$, (2.5) is the same as to find $v \in \mathring{H}_c^1(\Omega_\Gamma)$ such that

$$B(v, \Phi) = -\int_{\Omega_\Gamma} \mathscr{E}(h, \Phi) \, \mathrm{d}x.$$

By definition, $\mathring{H}_c^1(\Omega_\Gamma)$ is a Hilbert space, and the right-hand side is a bounded linear functional on $\mathring{H}_c^1(\Omega_\Gamma)$. Therefore, by the Riesz–Fréchet representation theorem, there exists exactly one solution $v \in \mathring{H}_c^1(\Omega_\Gamma)$ and hence, exactly one solution $u \in H_c^1(\Omega_\Gamma)$ of (2.5) with $u - h \in \mathring{H}_c^1(\Omega_\Gamma)$ which, actually, is independent of the special choice of h.

In order to derive an integral representation of the variational solution of the problem (P), as in [18,32] we extend Γ to an arbitrary smooth simple closed curve $\dot{G}_1$. We use the notation $[v]$ for the jump $v_- - v_+$ of a function v across $\dot{G}_1$. Here the subscripts $-$, $+$ denote the limits taken from G_1 and $\mathbb{R}^2 \setminus \overline{G}_1$, respectively. For later use, let B_R be a circle with radius R sufficiently large enough to enclose $\overline{G}_1$. The domain bounded by $\dot{G}_1$ and the boundary $\dot{B}_R$ of B_R will be denoted by G_2. The boundary $\dot{G}_2$ of G_2 consists of Γ together with $\dot{G}_1 \setminus \Gamma$ and $\dot{B}_R$ (see Fig. 1).

In what follows, let $H^s(\dot{G}_1)$ be defined as the trace of $H^{s+1/2}(\mathbb{R}^2)$ for $s > 0$, as $L^2(\dot{G}_1)$ for $s = 0$, and as the dual space of $H^{-s}(\dot{G}_1)$ for $s < 0$. For $s \geqslant 0$, $H^s(\Gamma)$ denotes the usual trace space of $H^s(\dot{G}_1)$ on Γ and $\tilde{H}^s(\Gamma)$ is defined by

$$\tilde{H}^s(\Gamma) := \{ f = f^*|_\Gamma \colon f^* \in H^s(\dot{G}_1),\ f^*|_{\dot{G}_1 \setminus \Gamma} = \mathbf{0} \}$$

equipped with the topology of $H^s(\dot{G}_1)$. For $s < 0$, we define

$$H^s(\Gamma) := (\tilde{H}^{-s}(\Gamma))' \quad \text{and} \quad \tilde{H}^s(\Gamma) := (H^{-s}(\Gamma))'$$

by duality with respect to the $L^2(\Gamma)$ scalar product. It is clear that from the definition, for $s>0$

$$\tilde{H}^{-s}(\Gamma)=\{f\in H^{-s}(\dot{G}_1):\ \mathrm{supp}(f)\subset\bar{\Gamma}\},$$

which is also the completion of $C_0^\infty(\Gamma)$ with respect to the norm of $H^{-s}(\dot{G}_1)$ (see [1], [14, Theorem 2.5.1, p.55ff.]).

We now state some properties concerning the solution $u\in H_c^1(\Omega_\Gamma)$ of (2.5).

Lemma 2.1. *Let $u\in H_c^1(\Omega_\Gamma)$ be the solution of (2.5). Then $u|_{\dot{G}_1}\in H^{1/2}(\dot{G}_1)$ and $u|_\Gamma\in H^{1/2}(\Gamma)$. For the traction $T(u)$ we have*

$$T(u)|_{\dot{G}_1}\in H^{-1/2}(\dot{G}_1)\quad \textit{and}\quad T(u)|_\Gamma\in\tilde{H}^{1/2}(\Gamma)';$$

moreover, if we denote by $[T(u)]=T(u)_- - T(u)_+$ the jump of the traction across Γ, then we have

$$[T(u)]|_\Gamma\in\tilde{H}^{-1/2}(\Gamma).$$

Here in the definition of $T(u)$ (see (1.1)) the exterior normal derivative to $\dot{G}_1$ is used.

Proof. The proof is based on the trace theorem [25], the first Korn inequality [13,26] and on Weyl's lemma. First, we show for the variational solution $u\in H_c^1(\Omega_\Gamma)$ that $u|_{G_i}\in H^1(G_i)$, the standard Sobolev space associated with G_i, $i=1, 2$. Then, let $v:=u-h\in\mathring{H}_c^1(\Omega_\Gamma)$, and the first Korn inequality (see [26, Eq. (1.4.28)]) implies

$$\begin{aligned}\|u\|_{H^1(G_i)}&\leqslant\|v\|_{H^1(G_i)}+\|h\|_{H^1(G_i)}\leqslant c\left\{\int_{G_i}\mathscr{E}(v, v)\,\mathrm{d}x\right\}^{1/2}+\|h\|_{H^1(G_i)}\\&\leqslant c\{\|u\|_{H_c^1(\Omega_\Gamma)}+\|h\|_{H_c^1(\Omega_\Gamma)}\}+\|h\|_{H^1(G_i)}.\end{aligned}$$

Thus, $u|_{G_i}\in H^1(G_i)$. Moreover, since u is the solution of (2.5), by standard arguments, $\Delta^*u=0$ in Ω_Γ. Hence, by Weyl's lemma, $u\in C^\infty(\Omega_\Gamma)$. Now, take $w\in C^\infty$ with supp $w\subset B_R$. Then the first Betti formula [23, p.9] applied to G_i for $i=1, 2$, respectively, yields

$$\begin{aligned}\int_{\dot{G}_1}w\cdot[T(u)]\,\mathrm{d}s&=\int_{G_1\cup G_2}\mathscr{E}(w, u)\,\mathrm{d}x\\&\leqslant C_R(\|w\|_{H^1(G_1)}+\|w\|_{H^1(G_2)})\|u\|_{H_c^1(\Omega_\Gamma)}.\end{aligned}$$

Hence, with the same arguments as in [3, p.377] involving the trace theorem, we see that the left-hand side defines a bounded linear functional on $w|_{\dot{G}_1}\in H^{1/2}(\dot{G}_1)$. Consequently, $[T(u)]\in H^{-1/2}(\dot{G}_1)$, the dual space of $H^{1/2}(\dot{G}_1)$. Since $[T(u)]=0$ on $\dot{G}_1\setminus\bar{\Gamma}$, we have $[T(u)]\in\tilde{H}^{-1/2}(\Gamma)$. □

For the integral representation of the solution u, we need the fundamental solution of (E), the Kelvin matrix (see, e.g., [2]):

$$\gamma(y, x)=\frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}\left\{\log\frac{1}{|x-y|}I+\frac{\lambda+\mu}{\lambda+3\mu}\frac{(x-y)(x-y)^{\mathrm{T}}}{|x-y|^2}\right\},\tag{2.6}$$

and the corresponding stress matrix on $\dot{G}_j$ [18]:

$$\gamma_1(y, x) := (T_y \gamma(y, x))^{\mathrm{T}}$$
$$= \frac{\mu}{2\pi(\lambda + 2\mu)} \left\{ \left(I + \frac{2(\lambda+\mu)}{\mu |x-y|^2} (x-y)(x-y)^{\mathrm{T}} \right) \frac{\partial}{\partial n_y} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial s_y} \right\}$$
$$\times \log \frac{1}{|x-y|}, \tag{2.7}$$

where T stands for the transposed and $\partial/\partial s_y$ for differentiation with respect to arc length at $y \in \dot{G}_j$. By applying the Betti formula [23, p.43] to the variational solution $\boldsymbol{u}$ in G_j, $j = 1, 2$, we obtain

$$a_j \boldsymbol{u}(x) = -\int_{\dot{G}_j} \gamma_1(y, x) \boldsymbol{u}(y) \, \mathrm{d}s_y + \int_{\dot{G}_j} \gamma(y, x) T(\boldsymbol{u})(y) \, \mathrm{d}s_y \tag{2.8}$$

for fixed $x \in G_1$ with $a_1 = 1$ and $a_2 = 0$. These representations hold for $\boldsymbol{u} \in \boldsymbol{H}_{\mathrm{c}}^1(\Omega_\Gamma)$, since from Lemma 2.1 we have $\boldsymbol{u}|_{\dot{G}_j} \in \boldsymbol{H}^{1/2}(\dot{G}_j)$ and $T(\boldsymbol{u})|_{\dot{G}_j} \in \boldsymbol{H}^{-1/2}(\dot{G}_j)$. Now from (2.8) it follows that

$$\boldsymbol{u}(x) = -\int_{\dot{B}_R} \gamma_1(y, x) \boldsymbol{u}(y) \, \mathrm{d}s_y + \int_{\dot{B}_R} \gamma(y, x) T(\boldsymbol{u})(y) \, \mathrm{d}s_y$$
$$- \int_\Gamma \gamma_1(y, x) [\boldsymbol{u}](y) \, \mathrm{d}s_y + \int_\Gamma \gamma(y, x) [T(\boldsymbol{u})](y) \, \mathrm{d}s_y \tag{2.9}$$

for fixed $x \in G_1$. We note that $[T(\boldsymbol{u})]|_{\dot{G}_1 \setminus \Gamma} = \boldsymbol{0}$ and, for the boundary condition (B), $[\boldsymbol{u}]|_\Gamma = \boldsymbol{0}$.

If $\boldsymbol{u} - M(x)\vec{\omega}$ is regular at infinity, one can show that the first two terms tend to $M(x)\vec{\omega}$ as $R \to \infty$ [21]. Thus, we arrive at the modified Betti representation formula

$$\boldsymbol{u}(x) = \int_\Gamma \gamma(y, x) [T(\boldsymbol{u})](y) \, \mathrm{d}s_y + M(x)\vec{\omega}, \quad x \in G_1. \tag{2.10}$$

Clearly, in a similar manner, one can show that the same representation holds for $x \in G_2$ with arbitrary R.

We summarize the foregoing results in the following theorem.

Theorem 2.2. *Suppose $\boldsymbol{u} \in \boldsymbol{H}_{\mathrm{c}}^1(\Omega_\Gamma)$ is the variational solution of* (P). *Then $\boldsymbol{u}$ admits the integral representation*

$$\boldsymbol{u}(x) = \int_\Gamma \gamma(y, x) [T(\boldsymbol{u})] \, \mathrm{d}s_y + M(x)\vec{\omega}, \quad x \in \mathbb{R}_2 \setminus \bar{\Gamma}, \tag{2.11}$$

where $[T(\boldsymbol{u})]|_\Gamma \in \tilde{\boldsymbol{H}}^{-1/2}(\Gamma)$ is the jump of traction across Γ, satisfying the condition (C_1), *and $M(x)\vec{\omega}$ corresponds to the rigid motion of $\boldsymbol{u}$ at infinity.*

We remark that from the representation (2.11), as in [21], one may derive the generalized formula for the virtual work of displacement fields including rigid motions at infinity:

$$\int_{\Omega_\Gamma} \mathscr{E}(\boldsymbol{w}, \boldsymbol{u}) \, \mathrm{d}x = -\int_\Gamma \boldsymbol{w} \cdot [T(\boldsymbol{u})] \, \mathrm{d}s + \int_\Gamma (M(y)\Omega(\boldsymbol{w})) \cdot [T(\boldsymbol{u})] \, \mathrm{d}s_y \tag{2.12}$$

for $w \in H_c^1(\Omega_\Gamma)$, where $u \in H_c^1(\Omega_\Gamma)$ is a variational solution of (P) and $M(y)\Omega(w)$ corresponds to the rigid motion of w at infinity according to (2.1), (2.3). This formula indicates that one indeed needs the additional condition such as (C_2) in order to ensure the uniqueness of the solution of the problem (P).

3. Boundary integral equations

We now reduce the variational boundary value problem of Section 2 to equivalent boundary integral equations for the jump of traction $[T(u)]$ across Γ. This can be achieved from the integral representation (2.11) by letting x tend to Γ. In fact, we have the following result.

Theorem 3.1. *Let $g \in H^{1/2}(\Gamma)$ be given. Then $u \in H_c^1(\Omega_\Gamma)$ is the variational solution of* (P) *with $\Phi = [T(u)]|_\Gamma$ if and only if $\Phi \in \tilde{H}^{-1/2}(\Gamma)$, $\vec{\omega} \in \mathbb{R}^3$ solve the integral equations*

$$\int_\Gamma \gamma(y, x)\Phi(y)\, ds_y + M(x)\vec{\omega} = g, \qquad \int_\Gamma \Phi\, ds = 0 \quad \textit{and} \quad \int_\Gamma m_3(y)\cdot\Phi\, ds = 0$$
$$\textit{for } x \in \Gamma, \tag{3.1}$$

where $m_3(y) := y_2\hat{e}_1 - y_1\hat{e}_2$.

Proof. For convenience, let us denote by V_Γ the boundary integral operator defined by

$$V_\Gamma\Phi(x) := \int_\Gamma \gamma(y, x)\Phi(y)\, ds_y, \quad x \in \Gamma. \tag{3.2}$$

For $\Phi \in \tilde{H}^s(\Gamma)$, V_Γ can be identified with

$$V_\Gamma\Phi(x) = \int_{\dot{G}_1} \gamma(x, y)\Phi^*(y)\, ds_y =: V_{\dot{G}_1}\Phi^*(x),$$

where Φ^* is the extension of Φ by zero onto $\dot{G}_1 \setminus \Gamma$. As is known from [21], $V_{\dot{G}_1}$ is a pseudodifferential operator with the symbol

$$\sigma(x, \xi) = \frac{\lambda + 3\mu}{4\mu(\lambda + 2\mu)}|\xi|^{-1}I + \sigma_1 \quad \text{for } |\xi| \geq 1,$$

where σ_1 is the symbol of a smoothing operator [11]. Hence, $V_\Gamma: \tilde{H}^s(\Gamma) \to H^{s+1}(\Gamma)$ is continuous for every real s (see also [19]).

Now the necessity in Theorem 3.1 follows clearly from the derivation of the integral representation (2.11) by density of $C^\infty(\Gamma)$ in $\tilde{H}^{-1/2}(\Gamma)$, since for continuous Φ, the potential $V_\Gamma\Phi$ is continuous in $\mathbb{R}^2$. Hence, any variational solution satisfies the first equation of (3.1) with $\Phi = [T(u)]|_\Gamma \in \tilde{H}^{-1/2}(\Gamma)$. By taking the limit $x \to \infty$ in the representation formula (2.11), we easily find the three last conditions in (3.1) since u is the variational solution by definition and $u - M(x)\vec{\omega}$ is regular at infinity.

For the sufficiency, we proceed as follows. Let $\Phi \in \tilde{H}^{-1/2}(\Gamma)$, $\vec{\omega} \in \mathbb{R}^3$ solve (3.1). Then the potential

$$u(x) = \int_\Gamma \gamma(y, x)\, \Phi(y)\, ds_y + M(x)\vec{\omega} \tag{3.3}$$

is well defined for every $x \notin \Gamma$ since $\gamma(y, x) \in \boldsymbol{H}^{1/2}(\Gamma)$ for $y \in \Gamma$; and the same holds for any derivative $(\partial/\partial x)^l(\partial/\partial y)^m \gamma(x, y)$.

Now, the single layer potential operator V_Γ defined in (3.2) is a pseudodifferential operator with the symbol given above of order $-\frac{3}{2}$ as a mapping from functions on the boundary $\dot{G}_1$ into functions on the bounded domain G_1 and/or its complement $G_2 = \mathbb{R}^2 \setminus \overline{G}_1$ (see [11, §8]). Thus for $\Phi \in \boldsymbol{H}^{-1/2}(\dot{G}_1)$ we have $V_\Gamma \Phi \in \boldsymbol{H}^1(G_j)$ where $\Phi|_{\dot{G}_j}$ is defined by extension by zero on $\dot{G}_j \setminus \overline{\Gamma}$, $j = 1, 2$.

For $x \notin \Gamma$ we use (2.10) and may interchange differentiation and integration obtaining

$$\Delta^* \boldsymbol{u} = \boldsymbol{0} \quad \text{in } \Omega_\Gamma .$$

For $|x| \geqslant R$ the potential (3.3) is analytic and with $\int_\Gamma \Phi(y)\, \mathrm{d}s_y = \boldsymbol{0}$ there holds

$$\boldsymbol{u}(x) = M(x)\vec{\omega} + \mathrm{O}\left(\frac{1}{|x|}\right) \quad \text{and} \quad \mathscr{E}(\boldsymbol{u}, \boldsymbol{u}) = \mathrm{O}(|x|^{-4}).$$

Hence, from (2.2) follows $\boldsymbol{u} \in \boldsymbol{H}^1_{\mathrm{c}}(\Omega_\Gamma)$. □

In order to guarantee that the system (3.1) is always solvable, we need some properties of the operators in (3.1). Let Λ_Γ be the functional defined by

$$\Lambda_\Gamma \Phi := \int_\Gamma M^T(y)\Phi(y)\, \mathrm{d}s_y , \tag{3.4}$$

where $M^{\mathrm{T}}(y)$ is the transposed of the matrix $M(y)$ in (2.3), that is,

$$M^{\mathrm{T}}(y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ y_2 & -y_1 \end{bmatrix}. \tag{3.5}$$

We also introduce the matrix operator A_Γ:

$$A_\Gamma(\Phi, \vec{\omega}) := \begin{pmatrix} V_\Gamma & M \\ \Lambda_\Gamma & 0 \end{pmatrix} \begin{pmatrix} \Phi \\ \vec{\omega} \end{pmatrix}. \tag{3.6}$$

In the theorem below we will show, as one expects from [32,36], that the operator A_Γ here also satisfies a Gårding inequality in the energy space $\tilde{\boldsymbol{H}}^{-1/2}(\Gamma) \times \mathbb{R}^3$. This means that A_Γ is a Fredholm operator of index zero [19,20,30] and hence, together with the uniqueness of the solution of (3.1), it implies that (3.1) will always be *uniquely solvable*.

Theorem 3.2. *The matrix operator A_Γ and its adjoint A^*_Γ with respect to the duality*

$$\langle (\Psi, \vec{k}), (\Phi, \vec{\omega}) \rangle_{L^2(\Gamma) \times \mathbb{R}^3} := (\Psi, \Phi)_{L^2(\Gamma)} + \vec{k} \cdot \vec{\omega}, \tag{3.7}$$

both are continuous and bijective mappings,

$$\tilde{\boldsymbol{H}}^s(\Gamma) \times \mathbb{R}^3 \to \boldsymbol{H}^{s+1}(\Gamma) \times \mathbb{R}^3 \quad \textit{for} -1 < s < 0. \tag{3.8}$$

Moreover, the operator A_Γ satisfies a Gårding inequality on $\tilde{\boldsymbol{H}}^{-1/2}(\Gamma) \times \mathbb{R}^3$, i.e., *there exists a constant $\gamma > 0$ and a compact mapping*

$$C_\Gamma : \tilde{\boldsymbol{H}}^{-1/2}(\Gamma) \times \mathbb{R}^3 \to \boldsymbol{H}^{1/2}(\Gamma) \times \mathbb{R}^3, \tag{3.9}$$

such that the inequality

$$\langle (A_\Gamma + C_\Gamma)(\Phi, \vec{\omega}), (\Phi, \vec{\omega})\rangle_{L^2(\Gamma)\times\mathbb{R}^3} \geqslant \gamma\{ \|\Phi\|^2_{\tilde{H}^{-1/2}(\Gamma)} + |\vec{\omega}|^2 \} \tag{3.10}$$

holds for all $(\Phi, \vec{\omega}) \in \tilde{H}^{-1/2}(\Gamma) \times \mathbb{R}^3$.

Proof. An elementary computation gives that A_Γ^* is the formal adjoint to A_Γ with respect to the L^2-duality between $\tilde{H}^{-1/2}(\Gamma) \times \mathbb{R}^3$ and $H^{1/2}(\Gamma) \times \mathbb{R}^3$. The continuity (3.8) for $s = -\frac{1}{2}$ is a direct consequence of Theorem 3.1. For the remaining s with $-1 < s < 0$, the continuity (3.8) holds too, as shown above.

Before we prove the bijectivity of the mapping (3.8), let us first show that the Gårding inequality (3.10) holds. Let us write the kernel in (2.6) as

$$\gamma(x, y) = \frac{\lambda + 3\mu}{4\pi\mu(\lambda + 2\mu)} \log\frac{1}{|x-y|} I + L, \quad L = \frac{\lambda + \mu}{4\pi\mu(\lambda + 2\mu)} \frac{(y-x)(y-x)^{\mathrm{T}}}{|x-y|^2}. \tag{3.11}$$

This decomposition induces a corresponding decomposition of the integral operator

$$V_\Gamma\Phi = \mathring{V}_\Gamma\Phi + L_\Gamma\Phi. \tag{3.12}$$

Next note that if $\Phi \in \tilde{H}^{-1/2}(\Gamma)$, then $\Phi^* \in H^{-1/2}(\dot{G}_1)$ and hence

$$\left(\mathring{V}_\Gamma\Phi, \Phi\right)_{L^2(\Gamma)} = \left(\mathring{V}_{\dot{G}_1}\Phi^*, \Phi^*\right)_{L^2(\dot{G}_1)},$$

with $\mathring{V}_{\dot{G}_1}\Phi^*$ defined via (3.11) and (3.12). Hence from [35, Lemma 2.1]

$$\begin{aligned}\left(\mathring{V}_\Gamma\Phi, \Phi\right)_{L^2(\Gamma)} &\geqslant \gamma \|\Phi^*\|^2_{H^{-1/2}(\dot{G}_1)} - (C_1\Phi^*, \Phi^*)_{L^2(\dot{G}_1)} \\ &= \|\Phi\|^2_{\tilde{H}^{-1/2}(\Gamma)} - (C_1\Phi, \Phi)_{L^2(\Gamma)},\end{aligned} \tag{3.13}$$

with $\gamma > 0$ and C_1 a compact mapping from $\tilde{H}^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. On the other hand, we note that the kernel L in (3.11) belongs to $C^m(\Gamma \times \Gamma)$ if $\Gamma \in C^{m+2}$. Hence since Γ is smooth, L_Γ is a compact operator from $\tilde{H}^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$. We obtain (3.10) with $C_\Gamma = C_1 + L_\Gamma$.

Now A_Γ is bijective from $\tilde{H}^{-1/2}(\Gamma) \times \mathbb{R}^3$ onto $H^{1/2}(\Gamma) \times \mathbb{R}^3$ since Gårding's inequality (3.10) shows that A_Γ is a Fredholm operator of index zero and the integral equation (3.1) is uniquely solvable.

For uniqueness we consider the solution Φ_0, $\vec{\omega}_0$ of the homogeneous equations

$$\int_\Gamma \gamma(x, y)\Phi_0(y)\,\mathrm{d}s_y + M(x)\vec{\omega}_0 = \mathbf{0},$$

$$\int_\Gamma \Phi_0\,\mathrm{d}s = \mathbf{0}, \qquad \int_\Gamma \boldsymbol{m}_3(y)\cdot\Phi_0\,\mathrm{d}s = 0.$$

Then the corresponding potential

$$\boldsymbol{u}_0(x) := \int_\Gamma \gamma(x, y)\Phi_0(y)\,\mathrm{d}s_y + M(x)\vec{\omega}_0$$

will satisfy $[T(\boldsymbol{u}_0)] = \Phi_0$ on Γ due to the jump relations, and $\boldsymbol{u}_{0|_\Gamma} = 0$. Moreover, (2.12) for $\boldsymbol{u}_0$ implies

$$\int_{\Omega_\Gamma} \mathscr{E}(\boldsymbol{u}_0, \boldsymbol{u}_0)\,\mathrm{d}x = -\int_\Gamma \boldsymbol{u}_0\cdot\Phi_0\,\mathrm{d}s + \int_\Gamma M(y)\vec{\omega}_0\cdot\Phi_0\,\mathrm{d}s_y = 0.$$

Hence, $\boldsymbol{u}_0(x) = M(x)\vec{\omega}_0$. Then $\Phi_0 = \mathbf{0}$ and moreover $\vec{\omega}_0 = \mathbf{0}$, since $\boldsymbol{u}_{0|_\Gamma} = \mathbf{0}$.

For $-1 < s < 0$ the operator $\mathring{V}_\Gamma$ is bijective from $\tilde{\boldsymbol{H}}^s(\Gamma)$ onto $\boldsymbol{H}^{s+1}(\Gamma)$ due to [36, Theorem 2.2] provided $\operatorname{diam}(G_1) < 1$. Hence, we easily obtain the bijectivity (3.8) of A_Γ and A_Γ^* in the general case by scaling (see [32]). □

4. Regularity of results

We now arrive at the point of our main concern—the singularity of $[T(\boldsymbol{u})]$ near the crack tips Z_i. As in the case of potential theory [28,30,31], the variational solution $\boldsymbol{u}$ of (2.5) has in general unbounded traction $[T(\boldsymbol{u})]$, even for C^∞-data. Hence, one will not be able to improve the approximation of $[T(\boldsymbol{u})]$ by the conventional constructive methods such as finite-element or boundary element methods without any modifications, since for better approximations, one generally requires higher regularity of the exact solution $[T(\boldsymbol{u})]$. For this purpose, we decompose $[T(\boldsymbol{u})]$ into special singular terms concentrated near the tips Z_i and a regular remainder. In this way, as in [8,24,29,31,32,36], one may then augment the finite-element spaces to test and trial functions with appropriate global singular elements, according to the special forms of the singular terms in $[T(\boldsymbol{u})]$, to improve the order of convergence of the approximations. A different approach for improving the approximation consists of using graded meshes associated with the singular terms [27].

Theorem 4.1. *For* $|\sigma| < \frac{1}{2}$, *let* $g \in \boldsymbol{H}^{s+\sigma}(\Gamma)$, $s = \frac{3}{2}$ *or* $\frac{5}{2}$ *be given. Then the solution* $[T(\boldsymbol{u})]|_\Gamma \in \tilde{\boldsymbol{H}}^{-1/2}(\Gamma)$ (*and* $\vec{\omega} \in \mathbb{R}^3$) *of the integral equations* (3.1) *admits the asymptotic representation near the endpoints* $Z_i \in \bar{\Gamma}$:

(a)
$$[T(\boldsymbol{u})]|_\Gamma = \sum_{i=1}^{2} \vec{\alpha}_i \rho_i^{-1/2} \chi_i + \Psi_0 \quad \textit{for } s = \tfrac{3}{2},$$
$$\textit{with } \Psi_0 \in \tilde{\boldsymbol{H}}^{1/2+\sigma}(\Gamma), \quad \vec{\alpha}_i \in \mathbb{R}^2, \ i = 1, 2;$$

(b)
$$[T(\boldsymbol{u})]|_\Gamma = \sum_{i=1}^{2} \left(\vec{\alpha}_i \rho_i^{-1/2} + \vec{\beta}_i \rho_i^{1/2}\right) \chi_i + \Psi_1 \quad \textit{for } s = \tfrac{5}{2},$$
$$\textit{with } \Psi_1 = \tilde{\boldsymbol{H}}^{3/2+\sigma}(\Gamma), \quad \vec{\alpha}_i, \ \vec{\beta}_i \in \mathbb{R}^2.$$

Here ρ_i *denotes the distance between* $x \in \Gamma$ *and* Z_i, *while* χ_i *is a* C^∞ *cut-off function such that* $0 \leqslant \chi_i \leqslant 1$, $\chi_i = 1$ *near* Z_i *and* $\chi_i = 0$ *elsewhere*, $i = 1, 2$.

We remark that the coefficients $\vec{\alpha}_i$ and $\vec{\beta}_i$ in the above theorem are indeed the *stress intensity factors* as in the crack problem, since $T(\boldsymbol{u})$ is the traction. We further comment that the regularity of the remainder term in $[T(\boldsymbol{u})]|_\Gamma$ may be improved as the given datum $\boldsymbol{g}$ becomes smoother; however, in general the solution $[T(\boldsymbol{u})]|_\Gamma$ possesses singularities and is then always *unbounded* at the tips.

Proof of Theorem 4.1. Let us assume that $z = z(s)$ is the regular parameter representation of the arc Γ, where s is the parameter of arc length measured from the endpoint Z_1. The kernel (3.11) of the integral operator V_Γ in (3.3) has the form

$$\gamma(z(s), z(t)) = \frac{-(\lambda + 3\mu)}{4\pi\mu(\lambda + 2\mu)} \log|s - t| I + \mathrm{O}(s^2 + t^2), \tag{4.1}$$

as can be seen easily from Taylor's expansion since Γ is sufficiently smooth (see [5, (2.19)] and [19, (A.6)]). This yields near Z_1 the decomposition

$$V_\Gamma = \mathring{V}_\Gamma + L_\Gamma \quad \text{with } \mathring{V}_\Gamma = \mathring{V}_0 + R_1,\ \ L_\Gamma = L_0 + R_2, \tag{4.2}$$

where $\mathring{V}_0$ and L_0 are given by

$$\begin{aligned} \mathring{V}_0\Psi(t) &= -\frac{(\lambda+3\mu)}{4\pi\mu(\lambda+2\mu)}\int_0^\infty \log\left|1-\frac{t}{s}\right|\Psi(s)\,\mathrm{d}s, \\ L_0\Psi(t) &= \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\int_0^\infty \Psi(s)\,\mathrm{d}s \end{aligned} \tag{4.3}$$

for $\Psi \in C_0^\infty(0, \infty)$. Here R_1, R_2 and L_0 are smoothing operators (see [4,5,10]). The application of the Mellin transform yields

$$(\mathring{V}_0\Psi)^\wedge(\zeta) = \hat{\mathring{V}}_0(\zeta)\hat{\Psi}(\zeta - \mathrm{i}) \quad \text{for } \operatorname{Im}\zeta \in (0, 1), \tag{4.4}$$

with the Mellin symbol

$$\hat{\mathring{V}}_0(\zeta) = -\frac{(\lambda+3\mu)}{4\pi\mu(\lambda+2\mu)}\frac{1}{\zeta}\frac{\cosh\pi\zeta}{\sinh\pi\zeta}. \tag{4.5}$$

The Mellin transform $\hat{\Psi}(\zeta)$ for $\Psi \in C_0^\infty(0, \infty)$ is an entire analytic function defined by

$$\hat{\Psi}(\zeta) = \int_0^\infty x^{\mathrm{i}\zeta-1}\Psi(x)\,\mathrm{d}x. \tag{4.6}$$

The inversion formula is

$$\Psi_h(x) = \frac{1}{2\pi}\int_{\operatorname{Im}\zeta=h} x^{-\mathrm{i}\zeta}\hat{\Psi}(\zeta)\,\mathrm{d}\zeta, \tag{4.7}$$

where $\Psi_h = \Psi$ for all $h \in \mathbb{R}$ if $\Psi \in C_0^\infty(0, \infty)$. (For vector-valued functions the expressions (4.6), (4.7) are understood componentwise.) If $\Psi \in \tilde{H}^s(\mathbb{R}_+)$ has compact support, then $\hat{\Psi}(\zeta)$ exists and is holomorphic for $\operatorname{Im}\zeta < s - \frac{1}{2}$, and $\Psi = \Psi_h$ for $h \leqslant k_1 = s - \frac{1}{2}$. Having performed the Mellin transform of the local version of the integral equation (3.1), one can find by means of Kondratiev's method [22] the asymptotic expansions (a), (b) of the solution. Since one can find from (4.5) the explicit decaying behavior of $\hat{\mathring{V}}_0(\zeta)$ for $\operatorname{Im}\zeta = s - \frac{1}{2}$ with $|\zeta| \to \infty$, one obtains the mapping properties of A_Γ by taking into account possible poles of the Mellin transformed equations using the Cauchy residue theorem.

In detail we proceed as follows. For simplicity let us assume $\vec{\omega} = \mathbf{0}$. We take a partition of unity on Γ and we assume that the unknown function in the integral equation has its support in a neighborhood of the origin being the only endpoint of Γ considered. Then we may rewrite the first equation of (3.1) in the form $\mathring{V}_0\Psi = f$ where $f := g - (R_1 + L_0 + R_2)\Phi$. By continuity arguments one can assume that f is smooth. Then the Mellin transformed equation

$$\hat{\mathring{V}}_0(\zeta)\hat{\Psi}(\zeta - \mathrm{i}) = \hat{f}(\zeta) \tag{4.8}$$

has a solution

$$\hat{\Psi}(\zeta - \mathrm{i}) = -\frac{4\pi\mu(\lambda+2\mu)}{(\lambda+3\mu)}\frac{\zeta\sinh\pi\zeta}{\cosh\pi\zeta}\hat{f}(\zeta), \tag{4.9}$$

which is meromorphic on the strip $\operatorname{Im} \zeta \in (h_0, h_1)$ in $\mathbb{C}$. Here $h_0 = 0$ corresponds to the a priori known regularity of the variational solution providing $\Psi \in \tilde{\boldsymbol{H}}^{-1/2}(\Gamma)$. The line $\operatorname{Im} \zeta = h_1$ corresponds to the regularity of the regular part Ψ_0 or Ψ_1 of the decompositions (a) or (b), where $h \leqslant s + \sigma - \frac{3}{2}$ because of the regularity of $\boldsymbol{g} \in \boldsymbol{H}^{s+\sigma}$. The poles of $\hat{\Psi}(\zeta - \mathrm{i})$ in the strip give the singular parts of the expansions (a) and (b) after application of the inverse Mellin transform and the Cauchy residue theorem. In order to find the singularity functions we can assume $\boldsymbol{f} = (f_1, f_2) \in \boldsymbol{C}_0^\infty[0, \infty)$. This implies that $\hat{\boldsymbol{f}}(\zeta)$ has meromorphic components $\hat{f}_j(\zeta)$, $j = 1, 2$, with simple poles at $\zeta = \mathrm{i}k$, $k \in \mathbb{N}_0$. Therefore, $\hat{\Psi}$ is also meromorphic and its poles can be determined from (4.9). Since the analysis given in [6,10] applies to our case almost word by word, it suffices to discuss the qualitative behavior of (4.9). Since the factor $\zeta \sinh \pi\zeta$ cancels the poles of $\hat{\boldsymbol{f}}(\zeta)$, there are only the simple poles at the zeros of

$$\hat{\mathring{V}}_0(\zeta) = -\frac{(\lambda + 3\mu)}{4\pi(\lambda + 2\mu)} \frac{\cosh \pi\zeta}{\zeta \sinh \pi\zeta}, \quad \text{i.e., } \zeta = \tfrac{1}{2}\mathrm{i}(2k - 1) \text{ with any integer } k. \tag{4.10}$$

The hypothesis $\Psi \in \tilde{\boldsymbol{H}}^{-1/2}(\Gamma)$ implies that because of (4.9), only the poles with $\operatorname{Im} \zeta > 0$ have to be considered, that is (4.10) with $k \in \mathbb{N}$. Finally, by [8, Lemma 4.3], the expansions (a) and (b) are obtained by taking the inverse Mellin transform of (4.9) in connection with the Cauchy residue theorem. Then the remainder terms Ψ_0 and Ψ_1 in (a) and (b) belong componentwise to the weighted Sobolev spaces $\mathring{W}_0^{1/2+\sigma}(\Gamma)$ and $\mathring{W}_0^{3/2+\sigma}(\Gamma)$, respectively, which near the crack tips are equivalent to $\tilde{\boldsymbol{H}}^{1/2+\sigma}(\Gamma)$ and $\tilde{\boldsymbol{H}}^{3/2+\sigma}(\Gamma)$, respectively [8, Lemma 2.6]. (For the weighted Sobolev spaces $\mathring{W}_0^{1/2+\sigma}(\Gamma)$ and $\mathring{W}_0^{3/2+\sigma}(\Gamma)$ and their equivalent norms, see [8,22].) Hence, by patching together the local results, the proof is complete. □

To incorporate the expansions of $[T(\boldsymbol{u})]|_\Gamma$ into the augmented Sobolev spaces for the purpose of boundary element approximation, we need mapping properties of $[T(\boldsymbol{u})]$ such as in Theorem 3.2. Let us begin with the following definition.

Definition 4.2. (a) Let $s > 1$. We define

$$\boldsymbol{Z}^s(\Gamma) := \left\{ \Psi = \sum_{i=1}^{2} \vec{\alpha}_i \rho_i^{-1/2} \chi_i + \Psi_0 \,\middle|\, \vec{\alpha}_i \in \mathbb{R}^2,\ \Psi_0 \in \tilde{\boldsymbol{H}}^s(\Gamma) \right\}$$

equipped with

$$\|\Psi\|_{\boldsymbol{Z}^s(\Gamma)} := \begin{cases} \sum_{i=1}^{2} |\vec{\alpha}_i| + \|\Psi_0\|_{\tilde{\boldsymbol{H}}^s(\Gamma)} & \text{for } 0 \leqslant s < 1, \\ \|\Psi\|_{\tilde{\boldsymbol{H}}^s(\Gamma)} & \text{for } s < 0. \end{cases}$$

(b) For $1 \leqslant s < 2$, we define

$$\boldsymbol{Z}^s(\Gamma) := \left\{ \Psi = \sum_{i=1}^{2} \left(\vec{\alpha}_i \rho_i^{-1/2} + \vec{\beta}_i \rho_i^{1/2} \right) \chi_i + \Psi_1 \,\middle|\, \vec{\alpha}_i, \vec{\beta}_i \in \mathbb{R}^2,\ \Psi_1 \in \tilde{\boldsymbol{H}}^s(\Gamma) \right\}$$

equipped with

$$\|\Psi\|_{\boldsymbol{Z}^s(\Gamma)} := \sum_{i=1}^{2} |\vec{\alpha}_i| + |\vec{\beta}_i| + \|\Psi_1\|_{\tilde{\boldsymbol{H}}^s(\Gamma)}.$$

These augmented spaces allow us to extend the mapping properties in Theorem 3.2 to higher order spaces. The following results are similar to those in [10,32,36].

Theorem 4.3. *For fixed* σ, $|\sigma| < \frac{1}{2}$, *the operator* A_Γ *defined by* (3.6) *possesses the following mapping properties*:

$$A_\Gamma: \boldsymbol{Z}^{1/2+\sigma}(\Gamma) \times \mathbb{R}^3 \to \boldsymbol{H}^{3/2+\sigma}(\Gamma) \times \mathbb{R}^3$$

and

$$A_\Gamma: \boldsymbol{Z}^{3/2+\sigma}(\Gamma) \times \mathbb{R}^3 \to \boldsymbol{H}^{5/2+\sigma}(\Gamma) \times \mathbb{R}^3$$

with

$$\{\vec{\alpha}_1, \vec{\alpha}_2, \Psi_0, \vec{\omega}\} \mapsto \begin{pmatrix} V_\Gamma\left(\sum_{i=1}^{2} \vec{\alpha}_i \rho_i^{-1/2}\chi_i + \Psi_0\right) + M\vec{\omega} \\ \Lambda_\Gamma\left(\sum_{i=1}^{2} \vec{\alpha}_i \rho_i^{-1/2}\chi_i + \Psi_0\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{b} \end{pmatrix}, \tag{4.11}$$

and

$$\{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\beta}_1, \vec{\beta}_2, \Psi_1, \vec{\omega}\} \mapsto \begin{pmatrix} V_\Gamma\left(\sum_{i=1}^{2} \left(\vec{\alpha}_i \rho_i^{-1/2} + \vec{\beta}_i \rho_i^{1/2}\right)\chi_i + \Psi_1\right) + M\vec{\omega} \\ \Lambda_\Gamma\left(\sum_{i=1}^{2} \left(\vec{\alpha}_i \rho_i^{-1/2} + \vec{\beta}_i \rho_i^{1/2}\right)\chi_i + \Psi_1\right) \end{pmatrix} = \begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{b} \end{pmatrix},$$

respectively, are continuous and bijective. Furthermore, there hold the corresponding a priori estimates

$$\|\Psi\|_{\boldsymbol{Z}^{1/2+\sigma}(\Gamma)} + |\vec{\omega}| \leqslant c\{\|\boldsymbol{g}\|_{\boldsymbol{H}^{3/2+\sigma}(\Gamma)} + |\boldsymbol{b}|\},$$

and

$$\|\Psi\|_{\boldsymbol{Z}^{3/2+\sigma}(\Gamma)} + |\vec{\omega}| \leqslant c\{\|\boldsymbol{g}\|_{\boldsymbol{H}^{5/2+\sigma}(\Gamma)} + |\boldsymbol{b}|\}. \tag{4.12}$$

Proof. First note that for $0 \leqslant \tau < 1$, we have $\rho_i^{-1/2} \notin \tilde{H}^\tau(\Gamma)$, and for $1 \leqslant \tau < 2$ we have $\rho_i^{-1/2} \notin \tilde{H}^\tau(\Gamma)$ and $\rho_i^{1/2} \notin \tilde{H}^\tau(\Gamma)$. Therefore, the injectivity of A_Γ for $-1 < \tau < 0$, as shown in Theorem 3.2, implies injectivity of A_Γ for $\tau = \frac{1}{2} + \sigma$ and for $\tau = \frac{3}{2} + \sigma$, respectively.

For proving surjectivity with $\boldsymbol{g} \in \boldsymbol{H}^{s+\sigma}(\Gamma) \subset \boldsymbol{H}^{1/2}(\Gamma)$, first solve (3.1) for $[T(\boldsymbol{u})] = \boldsymbol{\Phi} \in \tilde{\boldsymbol{H}}^{-1/2}(\Gamma)$ and then apply Theorem 4.1 to obtain $\vec{\alpha}_i$ and Ψ_0 or $\vec{\alpha}_i$, $\vec{\beta}_i$ and Ψ_1 for $s = \frac{3}{2}$ or $\frac{5}{2}$, respectively. Note that according to the proof of Theorem 4.1, these decompositions of Ψ are uniquely determined for given $\boldsymbol{g}$.

Hence, A_Γ in (4.11) is a bijective linear mapping from $\boldsymbol{Z}^{s+\sigma}(\Gamma) \times \mathbb{R}^3$ onto $\boldsymbol{H}^{s+\sigma}(\Gamma) \times \mathbb{R}^3$. The mapping is also continuous since $V_\Gamma: \tilde{\boldsymbol{H}}^{s-1+\sigma}(\Gamma) \to \boldsymbol{H}^{s+\sigma}(\Gamma)$ is continuous, Λ_Γ is linear and the spaces spanned by $\rho_i^{-1/2}$ for $s = \frac{3}{2}$ or $\rho_i^{-1/2}$ and $\rho_i^{1/2}$ for $s = \frac{5}{2}$ are finite dimensional. The closed-graph theorem then implies continuity of the inverse mapping which is equivalent to the a priori estimates (4.12). □

We remark that, in particular, one may take $\boldsymbol{b}=\mathbf{0}$ in Theorem 4.3. Then (4.11) coincides with the integral equation (3.1) with

$$[T(\boldsymbol{u})]\,|_{\Gamma}=\sum_{i=1}^{2}\vec{\alpha}_i\rho_i^{-1/2}\chi_i+\Psi_0 \quad\text{or}\quad [T(\boldsymbol{u})]\,|_{\Gamma}=\sum_{i=1}^{2}\left(\vec{\alpha}_i\rho_i^{-1/2}+\vec{\beta}_i\rho_i^{1/2}\right)\chi_i+\Psi_1,$$

depending on the given data $\boldsymbol{g}\in\boldsymbol{H}^{3/2+\sigma}(\Gamma)$ or $\boldsymbol{g}\in\boldsymbol{H}^{5/2+\sigma}(\Gamma)$. It is this system (4.11) that we will solve in the next section for $\vec{\alpha}_i$, Ψ_0, $\vec{\omega}$ and $\vec{\alpha}_i$, $\vec{\beta}_i$, Ψ_1, $\vec{\omega}$ by an augmented boundary element method originally developed for a closed smooth curve [24,34,36]. In our augmented method we use besides regular splines for Ψ_0, Ψ_1, the special singular elements $\rho_i^{-1/2}\chi_i$, $\rho_i^{1/2}\chi_i$ as in Theorems 4.1 and 4.3. Our procedure has the advantage that we are able not only to obtain higher rates of convergence, but also to compute the stress intensity factors simultaneously together with the approximate desired boundary charges $[T(\boldsymbol{u})]\,|_{\Gamma}$.

5. Boundary element method

Here we derive asymptotic error estimates for the Galerkin approximation of the solution of the system (3.1). For conformity of the method we assume that $\boldsymbol{S}_h=(S_h^{t,k})^2$ is a family of regular finite-element subspaces

$$\boldsymbol{S}_h\subset\boldsymbol{Z}^{-1/2}(\Gamma)\subset\tilde{\boldsymbol{H}}^{-1/2}(\Gamma). \tag{5.1}$$

Recall that the parameters in $S_h^{t,k}$ have the following meaning (roughly speaking): h with $0<h\leqslant h_0$ is the maximum meshsize of the partition of Γ; $t-1$ is the degree of piecewise polynomials; k describes the conformity, that is $S_h^{t,k}\subset H^k(\Gamma)$. The *Galerkin approximation* of the solution $(\Phi,\vec{\omega})\in\tilde{\boldsymbol{H}}^{-1/2}(\Gamma)\times\mathbb{R}^3$ of the system (3.1) for given $\boldsymbol{g}\in\boldsymbol{H}^{1/2}(\Gamma)$ and $\boldsymbol{b}\in\mathbb{R}^3$ is the pair $(\Phi^h,\vec{\omega}^h)\in\boldsymbol{S}_h\times\mathbb{R}^3$ satisfying the Galerkin equations

$$\left\langle A_\Gamma\begin{pmatrix}\Phi^h\\ \vec{\omega}^h\end{pmatrix},\begin{pmatrix}\Psi\\ \vec{k}\end{pmatrix}\right\rangle_{\boldsymbol{L}^2(\Gamma)\times\mathbb{R}^3}=\left\langle\begin{pmatrix}\boldsymbol{g}\\ \boldsymbol{b}\end{pmatrix},\begin{pmatrix}\Psi\\ \vec{k}\end{pmatrix}\right\rangle_{\boldsymbol{L}^2(\Gamma)\times\mathbb{R}^3} \tag{5.2}$$

for all $\Psi\in\boldsymbol{S}_h$, $\vec{k}\in\mathbb{R}^3$; that is

$$\begin{aligned}\left(V_\Gamma\Phi^h,\Psi\right)_{\boldsymbol{L}^2(\Gamma)}+\vec{\omega}^h\left(\tilde{\Lambda}_\Gamma 1,\Psi\right)_{\boldsymbol{L}^2(\Gamma)}&=\left(\boldsymbol{g},\Psi\right)_{\boldsymbol{L}^2(\Gamma)}\\ &=\left(V_\Gamma\Phi,\Psi\right)_{\boldsymbol{L}^2(\Gamma)}+\vec{\omega}\left(\tilde{\Lambda}_\Gamma 1,\Psi\right)_{\boldsymbol{L}^2(\Gamma)}\end{aligned} \tag{5.3}$$

for all $\Psi\in\boldsymbol{S}_h$ and the side condition

$$\Lambda_\Gamma\Phi^h=\boldsymbol{b}=\Lambda_\Gamma\Phi. \tag{5.4}$$

The solvability of the Galerkin equations (5.2) and the quasi-optimality of the Galerkin error in the energy norm follow immediately from Gårding's inequality (3.10) together with uniqueness of the system (3.1). The arguments here are now standard [19,30]. We summarize the results in the following theorem.

Theorem 5.1. *Let* $(\Phi,\vec{\omega})\in\tilde{\boldsymbol{H}}^{-1/2}(\Gamma)\times\mathbb{R}^3$ *denote the exact solution of* (3.1) *and* $(\Phi^h,\vec{\omega}^h)\in\boldsymbol{S}_h\times\mathbb{R}^3$ *be the Galerkin solution of* (5.2). *Then there hold*:

(i) *there exists an* $h_0>0$ *such that for any* $0<h\leqslant h_0$ *the Galerkin operator* $\mathcal{G}_h$ *from* $\tilde{\boldsymbol{H}}^{-1/2}(\Gamma)\times\mathbb{R}^3$ *into* $\boldsymbol{S}_h\times\mathbb{R}^3$ *defined by* (5.2) *as* $\mathcal{G}_h(\Phi,\vec{\omega})=(\Phi^h,\vec{\omega}^h)$ *is uniformly bounded independently of* h;

(ii) *for decreasing meshsizes $h \to 0$ there exist constants C, $\tilde{C} > 0$ independently of h, $\boldsymbol{g}$ and $\boldsymbol{b}$ such that for any $\epsilon > 0$*

$$\begin{aligned} \| \Phi - \Phi^h \|_{\tilde{H}^{-1/2}(\Gamma)} + | \vec{\omega} - \vec{\omega}^h | &\leqslant C \inf_{\Psi \in S_h} \{ \| \Phi - \Psi \|_{\tilde{H}^{-1/2}(\Gamma)} \} \\ &\leqslant C h^{1/2-\epsilon} \| \Phi \|_{\tilde{H}^{-\epsilon}(\Gamma)} \\ &\leqslant \tilde{C} h^{1/2-\epsilon} \{ \| \boldsymbol{g} \|_{H^{1-\epsilon}(\Gamma)} + | \boldsymbol{b} | \}. \end{aligned} \tag{5.5}$$

Remark 5.2. The asymptotic rate of convergence (5.5) is due to the approximation property of S_h and the restricted regularity of the exact solution Φ caused by the singular terms $\rho_i^{-1/2}\chi_i$ belonging to $H^{-\epsilon}(\Gamma)$ for any $\epsilon > 0$.

For higher convergence rates we augment the space of test and trial functions with the special singular elements $\rho_i^{-1/2}\chi_i$, $\rho_i^{1/2}\chi_i$, $i = 1, 2$, constituting the augmented finite-dimensional finite-element spaces $Z_h^{1/2}(\Gamma)$, $Z_h^{3/2}(\Gamma)$ on Γ:

$$\begin{aligned} Z_h^{1/2}(\Gamma) &:= \left\{ \Psi = \Psi_0 + \sum_{i=1}^{2} \vec{\alpha}_i \rho_i^{-1/2} \chi_i \mid \Psi_0 \in \mathring{H}_h^1(\Gamma) \text{ and } \vec{\alpha}_i \in \mathbb{R}^3 \right\}, \\ Z_h^{3/2}(\Gamma) &:= \left\{ \Psi = \Psi_1 + \sum_{i=1}^{2} \left(\vec{\alpha}_i \rho_i^{-1/2} + \vec{\beta} \rho_i^{1/2} \right) \chi_i \mid \Psi_1 \in \mathring{H}_h^2(\Gamma) \text{ and } \vec{\alpha}_i, \vec{\beta}_i \in \mathbb{R}^3 \right\}, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \mathring{H}_h^1(\Gamma) &:= \left\{ \Psi_0 \in S_h : \Psi_0(Z_i) = 0,\ i = 1, 2 \text{ and } S_h = \left(S_h^{2,1}\right)^2 \right\}, \\ \mathring{H}_h^2(\Gamma) &:= \left\{ \Psi_1 \in S_h : \Psi_1(Z_i) = \Psi_1'(Z_i) = 0,\ i = 1, 2 \text{ and } S_h = \left(S_h^{3,2}\right)^2 \right\}. \end{aligned} \tag{5.7}$$

Here we have incorporated transition conditions at the crack tips Z_i, $i = 1, 2$, so that

$$\mathring{H}_h^1(\Gamma) \subset \tilde{H}^1(\Gamma) \quad \text{and} \quad \mathring{H}_h^2(\Gamma) \subset \tilde{H}^2(\Gamma)$$

hold when $S_h^{2,1} \subset H^1(\Gamma)$, and $S_h^{3,2} \subset H^2(\Gamma)$, respectively. Consequently, we have

$$Z_h^{1/2}(\Gamma) \subset Z^{1/2+\sigma}(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma), \qquad Z_h^{3/2}(\Gamma) \subset Z^{3/2+\sigma}(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma). \tag{5.8}$$

We note that these augmented finite-element spaces satisfy the following two approximation properties:

$$\inf_{\Psi^h \in Z_h^l} \| \Psi - \Psi^h \|_{Z^s} \leqslant C h^{r-s} \| \Psi_0 \|_{\tilde{H}^r(\Gamma)} \tag{5.9}$$

for $-\frac{1}{2} \leqslant s \leqslant r < l + 1$ with $l = \frac{1}{2}$ and $l = \frac{3}{2}$, respectively, and

$$\inf_{\Psi^h \in Z_h^l} \| \Psi - \Psi^h \|_{\tilde{H}^{-1/2}(\Gamma)} \leqslant C h^{r+1/2-\delta} \| \Psi_0 \|_{\tilde{H}^r(\Gamma)} \tag{5.10}$$

for $l + 1 \leqslant r \leqslant l + \frac{3}{2}$ and for any $\delta > 0$.

These properties can be derived from the well-known approximation properties of the polynomial spline spaces $\mathring{H}_h^{l+1/2}(\Gamma)$ as follows: for (5.9) we have

$$\inf_{\Psi^h \in Z_h^{l+1/2}} \| \Psi - \Psi^h \|_{Z^s} = \inf_{\Psi_0^h \in \mathring{H}_h^{l+1/2}(\Gamma);\, \vec{\alpha}_i^h, \vec{\beta}_i^h \in \mathbb{R}^2} \Big\| \left(\Psi_0 - \Psi_0^h\right) + \sum_{i=1}^{2} \left(\vec{\alpha}_i - \vec{\alpha}_i^h\right) \rho_i^{-1/2} \chi_i$$

$$+ \delta_{l-1/2,1} \sum_{i=1}^{2} \left(\vec{\beta}_i - \vec{\beta}_i^h\right) \rho_i^{1/2} \chi_i \Big\|_{Z^s}$$

$$\leqslant \inf_{\Psi_0^h \in \mathring{H}_h^{l+1/2}(\Gamma)} \| \Psi_0 - \Psi_0^h \|_{\tilde{H}^s(\Gamma)} \leqslant C h^{r-s} \| \Psi_0 \|_{\tilde{H}^r(\Gamma)},$$

since $\mathring{H}^{l+1/2}(\Gamma) \subset \tilde{H}^r(\Gamma)$ for $r < \frac{3}{2}$ in case $l = \frac{1}{2}$ and for $r < \frac{5}{2}$ if $l = \frac{3}{2}$.

The second estimate (5.10) will be needed in order to apply the Aubin–Nitsche trick where $s = -\frac{1}{2}$ and $l + 1 \leqslant r \leqslant l + \frac{3}{2}$. Here,

$$\inf_{\Psi^h \in Z_h^l} \| \Psi - \Psi^h \|_{\tilde{H}^{-1/2}(\Gamma)} \leqslant \inf_{\Psi_0^h \in \mathring{H}_h^{l+1/2}(\Gamma)} \| \Psi_0 - \Psi_0^h \|_{\tilde{H}^{-1/2}(\Gamma)}$$

$$\leqslant \inf \| \Psi_0 - P_h \Psi_0 \|_{\tilde{H}^{-1/2}(\Gamma)}$$

$$= \inf \Big\{ \sup_{\| \Phi \|_{H^{1/2}(\Gamma)} \leqslant 1} \langle \Psi_0 - P_h \Psi_0,\, \Phi - \Phi_0^h \rangle_{L_2(\Gamma)} \Big\},$$

where P_h is the orthogonal projection in $L_2(\Gamma)$ onto $\mathring{H}_h^{l+1/2}(\Gamma)$. The latter can further be estimated by

$$\inf \| \Psi_0 - P_h \Psi_0 \|_{\tilde{H}^{-1/2}(\Gamma)} \leqslant \inf \| \Psi_0 - \Psi_0^h \|_{L_2(\Gamma)} \cdot \sup_{\| \Phi \|_{H^{1/2}(\Gamma)} \leqslant 1} C h^{1/2-\delta} \| \Phi \|_{\tilde{H}^{1/2-\delta}(\Gamma)}$$

$$\leqslant C h^{1/2-\delta} \| \Psi_0 - I_h \Psi_0 \|_{L^2(\Gamma)}$$

with any $\delta > 0$; and I_h the interpolation of Ψ_0 by splines in $H^{l+\sigma}(\Gamma)$ at the break- or midpoints, respectively. The interpolation, as is well known, provides the approximation property

$$\| \Psi_0 - I_h \Psi_0 \|_{L^2(\Gamma)} \leqslant C h^r \| \Psi_0 \|_{H^r(\Gamma)}$$

for $\frac{1}{2} < r < l + 1$. Hence,

$$\inf_{\Psi^h \in Z_h^l} \| \Psi - \Psi^h \|_{\tilde{H}^{-1/2}(\Gamma)} \leqslant C h^{r+1/2-\delta} \| \Psi_0 \|_{H^r(\Gamma)} \tag{5.11}$$

for $l + 1 \leqslant r \leqslant l + \frac{3}{2}$ and for any $\delta > 0$.

Moreover, we shall also need the inverse property for the augmented spaces (also called "inverse assumption" in finite-element analysis). The inverse property has been established in [8,36] and reads as follows.

Inverse property. *Let the family of partitions of Γ be quasi-uniform. Then the augmented spaces satisfy*

$$\| \Psi^h \|_{Z^r} \leqslant M h^{s-r-\epsilon} \| \Psi^h \|_{Z^s}, \quad \text{for } \Psi^h \in Z_h^l \tag{5.12}$$

for $-\frac{1}{2} \leqslant s < r < l + 1$, $l = \frac{1}{2}$ or $\frac{3}{2}$, respectively, where $\epsilon = 0$ for $0 \notin [s, r]$ and for $1 \notin [s, r]$. Otherwise, (5.12) holds with any $\epsilon > 0$.

The proof of (5.12) can be found in [8, Lemma 5.7]. In view of the decomposition (Theorem 4.1) we can improve the error estimates (5.5) by solving Galerkin's equations (5.2) in the augmented finite-element spaces. With the conformity (5.8), similar to [9, Section 3], we have the next theorem.

Theorem 5.3. (i) *There exists a maximum mesh width* $h_0 > 0$ *such that the Galerkin equations* (5.2) *are uniquely solvable in* $\boldsymbol{Z}_h^l(\Gamma)$, $l = \frac{1}{2}$ *or* $\frac{3}{2}$, *for any* h, $0 < h \leqslant h_0$.

(ii) *Let* $-\frac{3}{2} - l < s \leqslant r < l + \frac{1}{2}$. *Then for* $h \to 0$ *there holds*

$$\begin{aligned} \| \Phi - \Phi^h \|_{\boldsymbol{Z}^s(\Gamma)} + |\vec{\omega} - \vec{\omega}^h| &\leqslant C h^{r-s-\epsilon} \| \Phi \|_{\boldsymbol{Z}^r(\Gamma)} \\ &\leqslant C h^{r-s-\epsilon} \{ \| \boldsymbol{g} \|_{\boldsymbol{H}^{r+1}(\Gamma)} + |\boldsymbol{b}| \} \end{aligned} \tag{5.13}$$

provided $\boldsymbol{g} \in \boldsymbol{H}^{r+1}(\Gamma)$, *where the constant* $C > 0$ *is independent of the exact solution* $(\Phi, \vec{\omega})$ *of* (3.1), *of the Galerkin solution* $(\Phi^h, \vec{\omega}^h) \in \boldsymbol{Z}_h^l(\Gamma) \times \mathbb{R}^3$, *the given data* $(\boldsymbol{g}, \boldsymbol{b})$ *as well as the mesh width* h, *where* $\epsilon > 0$ *if* $s \geqslant 0$ *and* $\epsilon = 0$ *if* $s < 0$.

Proof. First let us consider the case $s = -\frac{1}{2}$. Here, we can use Cea's lemma, which follows here from (3.13) and uniqueness in exactly the same manner as for a closed smooth curve Γ in [20]. We obtain here the estimate

$$\| \Phi - \Phi^h \|_{\tilde{\boldsymbol{H}}^{-1/2}(\Gamma)} + |\vec{\omega} - \vec{\omega}^h| \leqslant c \inf_{\Psi^h \in \boldsymbol{Z}_h^l(\Gamma)} \| \Phi - \Psi^h \|_{\tilde{\boldsymbol{H}}^{-1/2}(\Gamma)}.$$

Then the approximation property (5.11) implies (5.13). In case $s < -\frac{1}{2}$, the estimate (5.13) can be obtained from the Aubin–Nitsche lemma as given in [9, Lemma 2.1] which avoids the adjoint operator in $\tilde{\boldsymbol{H}}^s(\Gamma)$. In particular, the assumption in [9, (2.4)] is here satisfied due to (5.9), namely

$$\begin{aligned} \inf_{\substack{\Psi^h \in \boldsymbol{Z}_h^l \\ \vec{\lambda} \in \mathbb{R}^3}} \left\| A_\Gamma^{-1} \begin{pmatrix} \boldsymbol{g} \\ \boldsymbol{b} \end{pmatrix} - \begin{pmatrix} \Psi^h \\ \vec{\lambda} \end{pmatrix} \right\|_{\tilde{\boldsymbol{H}}^{-1/2}(\Gamma) \times \mathbb{R}^3} &= \inf_{\Psi^h \in \boldsymbol{Z}_h^l} \| \Phi - \Psi^h \|_{\tilde{\boldsymbol{H}}^{-1/2}(\Gamma)} \\ &\leqslant C h^{-s-1/2-\delta} \| \Phi_0 \|_{\tilde{\boldsymbol{H}}^{-s-1}(\Gamma)} \\ &\leqslant C h^{-s-1/2-\delta} \{ \| \boldsymbol{g} \|_{\boldsymbol{H}^{-s}(\Gamma)} + |\boldsymbol{b}| \}. \end{aligned}$$

The remaining case follows from the inverse property (5.12) in the standard manner by making use of (5.13) with $s = -\frac{1}{2}$. □

Remark 5.4. For $s \geqslant l - \frac{1}{2}$, $l = \frac{1}{2}$ or $\frac{3}{2}$ and for any $\epsilon > 0$, the estimate (5.13) provides an explicit error estimate of order $h^{r-l+1/2-\epsilon}$ for the stress intensity factors with $l = \frac{1}{2}$ for $\vec{\alpha}_i$ and $l = \frac{3}{2}$ for $\vec{\alpha}_i$ and $\vec{\beta}_i$ if $\boldsymbol{g}$ is sufficiently smooth, since for such s the norm in $\boldsymbol{Z}^s(\Gamma)$ includes the stress intensity factors explicitly.

Although numerical experiments have not been presented, it should be mentioned that in the present paper our scheme should be at least as efficient as in the case of the Laplacian. In the latter, numerical experiments are available in [24] and they are in agreement with our asymptotic error estimates.

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