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Resonant Bifurcations

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We consider dynamical systems depending on one or more real parameters, and assuming that, for some "critical" value of the parameters, the eigenvalues of the linear part are resonant, we discuss the existence—under suitable hypotheses—of a general class of bifurcating solutions in correspondence with this resonance. These bifurcating solutions include, as particular cases, the usual stationary and Hopf bifurcations. The main idea is to transform the given dynamical system into normal form (in the sense of Poincaré and Dulac) and to impose that the normalizing transformation is convergent, using the convergence conditions in the form given by A. Bruno. Some specifically interesting situations, including the cases of multiple-periodic solutions and of degenerate eigenvalues in the presence of symmetry, are also discussed in some detail.

1. INTRODUCTION

In this paper we consider dynamical systems of the form

$$\dot{u} = f(u, \lambda) \equiv A(\lambda)u + F(u, \lambda) \qquad u(t) \in \mathbf{R}^n,\ \lambda \in \mathbf{R}^p \tag{1}$$

depending on one or more real parameters λ, and, assuming that, for some "critical" value $\lambda = \lambda_0$ of the parameters, the matrix $A(\lambda_0)$ admits resonant eigenvalues, we want to discuss the existence—under suitable hypotheses—of a general class of "bifurcating solutions" $u = u_\lambda(t)$ in correspondence with this resonance. These bifurcating solutions include, as particular cases, the usual stationary bifurcation and the Hopf bifurcation (see, e.g., [1]). The main idea is to transform the given dynamical system into normal form (in the sense of Poincaré and Dulac [2–9]) and try to impose that the normalizing transformation be convergent. The imposition

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of convergence is essentially based on the application of the conditions given by Bruno [4, 5] and leads to some prescriptions which may be fulfilled thanks to the presence of the parameters λ; in this way, the appearance of these "resonant bifurcating" solutions can be automatically deduced.

Among these solutions, some attention is paid to discussing some situations of special interest (also from the physical point of view), including the cases of multiple-frequency periodic solutions and of degenerate eigenvalues in the presence of symmetry, giving examples for each situation.

2. BASIC ASSUMPTIONS AND PRELIMINARIES

We need some preliminary notions and results.

Let $u(t) \in \mathbf{R}^n$, let $\lambda \in \mathbf{R}^p$, and let $f = f(u, \lambda)$ be a vector-valued analytic function in a neighborhood of some u_0 and λ_0 (it is not restrictive to choose $u_0 = 0$ and $\lambda_0 = 0$), such that $f(\lambda, 0) = 0$ for each λ in a neighborhood of λ_0. Let us denote the linear part of f by $A(\lambda)u$, where

$$A(\lambda) = \nabla_u f(\lambda, 0), \tag{2}$$

and assume that, for some "critical" value of the parameters (we can assume that this value is just $\lambda_0 = 0$) the matrix

$$A_0 = A(0)$$

is semisimple, i.e., diagonalizable. Let us notice that, in the case A_0 is not semisimple, it could be uniquely decomposed into a sum of a semisimple and a nilpotent part: $A_0 = A^{(s)} + A^{(n)}$, and the foregoing discussion could be equally well applied to the semisimple part $A^{(s)}$, considering in particular normal forms with respect only to $A^{(s)}$. The introduction of normal forms with respect to a non-semisimple matrix requires a more difficult procedure and will not be considered here (cf. [10, 11]).

Up to a linear change of coordinates (possibly after complexification of the space), we will assume for convenience that the matrix A_0 is diagonal, with eigenvalues $\sigma_1, \ldots, \sigma_n$. The first important assumption is that, for the value $\lambda_0 = 0$, the eigenvalues exhibit a resonance; i.e., there are some nonnegative integers m_i such that

$$\sum_{i=1}^{n} m_i \sigma_i = \sigma_j, \qquad \sum_{i=1}^{n} m_i \geq 2 \tag{3}$$

for some index $j \in [1, \ldots, n]$.

Together with the given dynamical system (DS) (1), we need also to consider its normal form (NF) (in the sense of Poincaré and Dulac [2–9]), in a neighborhood of $u_0 = 0$, $\lambda_0 = 0$. As is well known, the idea is that of performing a near-identity coordinate transformation

$$u \to v = u + \varphi(u) \tag{4}$$

in such a way that in the new coordinates v the given DS takes its "simplest" form. To define this, consider the linear operator $\mathscr{A}$ (the "homological operator" [2–9]) defined on the space of vector-valued functions $h(v)$ by

$$\mathscr{A}(h) = A_0 v \cdot \nabla h - A_0 h. \tag{5}$$

Writing the NF in the form

$$\dot{v} = g(v, \lambda) = A_0 v + G(v, \lambda) \tag{6}$$

the nonlinear terms $G(v, \lambda)$ are then defined by the property [2–9, 11–14]

$$G(v, \lambda) \in \operatorname{Ker}(\mathscr{A}). \tag{7}$$

Actually, to be more precise, one should also consider, together with (1), the p equations

$$\dot{\lambda} = 0$$

(as in the usual suspension procedure), extend the homological operator $\mathscr{A}$ adding to the matrix A_0 the last p columns and p rows equal to zero, and similarly extend $G(v, \lambda)$ as an $(n+p)$-dimensional vector-valued function with the last p components equal to 0. It is important indeed to notice that it is essential here to consider the λ as independent variables; in this way, in particular, one has that the bilinear terms in v and λ are included in $G(v, \lambda)$; see also Remark 1 below.

Let us now briefly recall the following important results. The proof of these can be found, e.g., in [8, 11–14].

LEMMA 1. *Given the matrix A_0, the most general NF is given by* (6) *with*

$$G(v, \lambda) = \sum_i \beta_i(\lambda, \rho(v)) B_i v \qquad \text{where } B_i \in \mathscr{C}(A_0),\ \rho = \rho(v) \in \mathscr{I}_{A_0}, \tag{8}$$

where $\mathscr{C}(A_0)$ is the set of the matrices commuting with A_0, and $\mathscr{I}_{A_0}$ is the set of the constants of motion of the linear system

$$\dot{v} = A_0 v. \tag{9}$$

The sum in (8) *is extended to a set of independent matrices* B_i, *the constants of motion* $\rho(v)$ *can be chosen in the form of monomials* (*possibly fractional*), *and the functions* β_i *are series or rational functions of the* v_i (*see* [13], *for a detailed statement*, *and below*, *for the cases of interest for our discussion*).

It is now clear that the assumption (3) on the existence of some resonance among the eigenvalues of A_0 ensures that there are nontrivial constants of motion of the linear problem (9) and then nontrivial terms in the NF (8).

It is also well known that the coordinate transformations taking the given DS into NF is usually performed using recursive techniques and that in general the sequence of these transformations is purely formal: indeed, only very special conditions can assure the convergence of the normalizing transformation (NT).

Let us now recall the basic conditions, in the form given by Bruno and called respectively Condition ω and Condition A, which ensure this convergence. The first condition is (see [4, 5] for details).

Condition ω. Let $\omega_k = \min|(q, \sigma) - \sigma_j|$ for all $j = 1, \ldots, n$ and all n-tuples of nonnegative integers q_i such that $1 < \sum_{i=1}^n q_i < 2^k$ and $(q, \sigma) = \sum_i q_i \sigma_i \neq \sigma_j$. Then we require

$$\sum_{k=1}^{\infty} 2^{-k} \ln\left(\omega_k^{-1}\right) < \infty.$$

This is a actually very weak condition, devised to control the appearance of small divisors in the series of NT, and generalizes the Siegel-type condition,

$$\left|(q, \sigma) - \sigma_j\right| > \epsilon \left(\sum_{i=1}^{n} q_i \right)^{-\nu}$$

for some $\epsilon, \nu > 0$, or the much simpler condition $|(q, \sigma) - \sigma_j| > \epsilon > 0$, for all n-tuples q_i such that $(q, \sigma) \neq \sigma_j$ (see [2–5]). We explicitly assume from now on that this condition is always satisfied.

The other condition, instead, is a quite strong restriction on the form of the NF. To state this condition in its simplest form, let us assume that there is a straight line through the origin in the complex plane which contains all the eigenvalues σ_i of A_0. Then the condition reads

Condition A. There is a coordinate transformation $u \to v$ changing $f = A_0 u + F$ to a NF $g = g(v)$ of the form

$$g(v) = A_0 v + \alpha(v) A_0 v$$

where $\alpha(v)$ is some scalar-valued power series (with $\alpha(0) = 0$).

In the case that there is no line in the complex plane which satisfies the above property, then Condition A should be modified [4, 5] (or even weakened: for instance, if there is a straight line through the origin such that all the σ_i lie on the same side in the complex plane with respect to this line, then the eigenvalues belong to a Poincaré domain [2–5] and the convergence is guaranteed without any other condition), but in all our applications below we shall assume for the sake of definiteness that the eigenvalues are either all real or purely imaginary; therefore, the above formulation of Condition A is enough to cover all the cases to be considered.

Remark 1. It can be useful to point out that it is essential in the present approach to use the suspension procedure for the parameters λ, i.e., to consider λ as additional variables. Indeed, let us consider, for instance, a simple standard Hopf-type two-dimensional bifurcation problem with one parameter ($p = 1$),

$$\dot{u} = A_0 u + \lambda I u + \text{higher order terms} \qquad \text{where } A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and I is the identity matrix. If λ is kept fixed $\neq 0$, the eigenvalues of the linear part $A_0 + \lambda I$ are $\sigma = \lambda \pm i$ and, as a consequence, there are no (analytic or fractional) constants of motion of the linearized problem, and the NF is trivially linear. Also, the NT would be convergent, as a consequence of Condition A (or, more simply, of the Poincaré criterion [2–5]). But no bifurcation can be found in this way. Instead, considering λ as an independent variable, the linear part of the problem is $\dot{u} = A_0 u$ and now there are nontrivial constants of motion in the NF, i.e., the functions of $r^2 = u_1^2 + u_2^2$ and λ.

Given the DS (1), it will be useful to rewrite its NF according to Lemma 1, observing that obviously $A_0 \in \mathscr{C}(A_0)$ and in view of Condition A, in the following "split" form (cf. (6), (8))

$$\dot{v} = g(v, \lambda) = A_0 v + \alpha(\lambda, \rho(v)) A_0 v + \sum_j{}^{*} \beta_j(\lambda, \rho(v)) B_j v$$

$$\rho(v) \in \mathscr{I}_{A_0}, \quad (10)$$

where (hereafter) $\sum_j^*$ is the sum extended to the matrices $B_j \neq A_0$. Following Bruno [4, 5], we can say that the convergence of the NT is granted if $\sum_j^* \beta_j(\lambda, \rho(v)) B_j v = 0$. Clearly, "convergence" stands for "convergence in some neighborhood of $u_0 = 0$, $\lambda_0 = 0$."

Let us remark, incidentally, that an algorithmic implementation of the procedure for obtaining the NF step by step is possible (cf., e.g., [15, 16]);

we stress, however, that actually, in this paper, we shall not need any explicit calculation of NF's.

3. THE TWO-DIMENSIONAL CASE

For the sake of simplicity we consider first of all the case of a two-dimensional DS, with some other simplifying assumptions; more general cases will be considered in subsequent sections.

THEOREM 1. *Given a two-dimensional DS*

$$\dot{u} = A(\lambda)u + F(u, \lambda) \qquad u \in \mathbf{R}^2,\ \lambda \in \mathbf{R} \tag{11}$$

($i.e.$, $n = 2$, $p = 1$), *assume that for* $\lambda_0 = 0$ *the two eigenvalues* σ_1, σ_2 *of* A_0 *are nonzero*, *are of opposite sign if real*, *and satisfy a resonance relation* (3), *which in this case can be more conveniently written in the form of commensurability of the two eigenvalues*,

$$\frac{\sigma_1}{s_1} = -\frac{\sigma_2}{s_2} = \theta_0, \tag{12}$$

where s_1, s_2 *are two positive*, *relatively prime integers*. *Assume also that*

$$s_2 \frac{d}{d\lambda} A_{11}(\lambda) + s_1 \frac{d}{d\lambda} A_{22}(\lambda)|_{\lambda=0} \neq 0. \tag{13}$$

Then there is a "resonant bifurcating" solution $\hat{u} = \hat{u}_\lambda(t)$ *of the form*

$$\begin{aligned} \hat{u}_1 &= c^{(1)} \exp(s_1 \theta(\lambda) t) + \sum_{\substack{n_1 = -\infty \\ n_1 \neq s_1}}^{+\infty} c_{n_1} \exp(n_1 \theta(\lambda) t) \\ \hat{u}_2 &= c^{(2)} \exp(-s_2 \theta(\lambda) t) + \sum_{\substack{n_2 = -\infty \\ n_2 \neq -s_2}}^{+\infty} c_{n_2} \exp(n_2 \theta(\lambda) t), \end{aligned} \tag{14}$$

where for $\lambda \to 0$

$$s_1 \theta(\lambda) \to \sigma_1, \qquad -s_2 \theta(\lambda) \to \sigma_2, \qquad c^{(1)}, c^{(2)} \to 0$$

and all terms in the two series are "higher order terms" ($h.o.t.$), $i.e.$, *terms vanishing more rapidly than the two leading terms, and the series are convergent in some time interval. There is also an analytic constant of motion along* *this solution, which, for small* λ, *has the form*

$$\rho(\hat{u}) = \left(c^{(1)}\right)^{s_2}\left(c^{(2)}\right)^{s_1} + h.o.t.$$

Proof. Consider the (possibly non-convergent) coordinate transformation $u \to v$ (4) which takes the given DS into NF (10). The assumption on the resonance of the eigenvalues σ_i and Lemma 1 imply that the DS in NF is expected to have the form

$$\dot{v} = g(\lambda, v) = A_0 v + \alpha(\lambda, \rho) A_0 v + \beta(\lambda, \rho) B v = (1 + \alpha) A_0 v + \beta B v, \tag{15}$$

where $\rho = \rho(v) = v_1^{s_2} v_2^{s_1}$ is a (monomial) constant of motion of the linear problem $\dot{v} = A_0 v$, B is any diagonal matrix independent of A_0 (i.e., such that $s_2 B_{11} + s_1 B_{22} \neq 0$), and λ can be viewed as a constant of motion of the enlarged system including $\dot{\lambda} = 0$. We now impose the Bruno Condition A (in the form given above, indeed σ_i are either real or purely imaginary) to ensure the convergence of the NT: this amounts to imposing

$$\beta(\lambda, \rho) = 0. \tag{16}$$

The expression of this function is clearly not known (unless the NF itself is known), but the manifold defined by $\beta = 0$ is analytic [4] and the relevant behavior of the first-order terms (in λ) of $\beta(\lambda, \rho)$ can be inferred from the original DS $\dot{u} = A(\lambda)u + \cdots$. Indeed the first-order terms (in λ) of $A(\lambda)u$, i.e., $A_0 u + \lambda(dA(\lambda)/d\lambda)|_{\lambda=0} u$, are not changed by the NT; therefore

$$\alpha(0,0) = \beta(0,0) = 0$$

and

$$s_2 \frac{d}{d\lambda} A_{11}(\lambda) + s_1 \frac{d}{d\lambda} A_{22}(\lambda)|_{\lambda=0} = (s_2 B_{11} + s_1 B_{22}) \left.\frac{\partial \beta}{\partial \lambda}\right|_{\lambda=0,\ \rho=0},$$

having also used $s_2 \sigma_1 + s_1 \sigma_2 = 0$. Assumption (13) then shows that one can satisfy the condition $\beta(\lambda, \rho) = 0$, thanks to the implicit-function theorem, if λ and ρ are related by a function

$$\lambda = \lambda(\rho) \qquad \text{with } \lambda(0) = 0. \tag{17}$$

With $\beta = 0$, a solution of (15) is then

$$\hat{v}(t) = \exp((1 + \alpha) A_0 t) \hat{v}_0$$

or, putting $\theta(\lambda) = \theta_0(1+\alpha)$,

$$\begin{aligned} \hat{v}_1(t) &= \hat{v}_{10}\exp\big(s_1\theta(\lambda)t\big) \\ \hat{v}_2(t) &= \hat{v}_{20}\exp\big(-s_2\theta(\lambda)t\big) \end{aligned} \tag{18}$$

with the constraints

$$\big(\hat{v}_1(t)\big)^{s_2}\big(\hat{v}_2(t)\big)^{s_1} = \big(\hat{v}_{10}\big)^{s_2}\big(\hat{v}_{20}\big)^{s_1} = \rho \in \mathscr{I}_{A_0} \qquad \text{and} \qquad \lambda = \lambda(\rho) \tag{19}$$

and where $\alpha = \alpha(\lambda(\rho)) \to 0$, $\theta(\lambda) \to \theta_0$ for $\lambda, v, \rho \to 0$. On the other hand, in the analytic manifold defined by $\beta(\lambda, \rho) = 0$ the NT is convergent and this bifurcating solution corresponds to an analytic solution $\hat{u} = \hat{u}_\lambda(t)$ of the initial problem. The original coordinates u are in fact related to the new ones v by an analytic transformation (the inverse of (4))

$$u = v + \psi(v),$$

where ψ is a power series in the v_1, v_2, and the convergence is granted on some neighborhood of zero, say $|v_1| < R_1$, $|v_2| < R_2$. Notice now that the condition $\rho(v) = \text{const}$ does not ensure, if the eigenvalues are real (see the examples below), that the variables are bounded for all $t \in \mathbf{R}$; in this case, it will be sufficient to choose λ, together with $\hat{v}_{10}, \hat{v}_{20}$, small enough that the convergence is granted in some interval $T_1 < t < T_2$. ∎

EXAMPLE 1. Let the matrix elements $A_{ij}(\lambda)$ of $A(\lambda)$ in (11) satisfy

$$A_{11}(\lambda) \to 1, \quad A_{22}(\lambda) \to -2, \quad A_{12}(\lambda) \text{ and } A_{21}(\lambda) \to 0 \qquad \text{for } \lambda \to 0.$$

Then $\sigma_1 = 1$, $\sigma_2 = -2$, the constant of motion of the linear DS $\dot{v} = A_0 v$ is $\rho = v_1^2 v_2$, and, assuming $d(2A_{11} + A_{22})/d\lambda|_{\lambda=0} \neq 0$, there is a resonant bifurcating solution whose leading terms have the exponential behavior

$$\begin{aligned} \hat{u}_1 &= c^{(1)}\exp\big((1+\alpha)t\big) + \cdots \\ \hat{u}_2 &= c^{(2)}\exp\big(-2(1+\alpha)t\big) + \cdots \end{aligned}$$

with $\alpha, c^{(1)}, c^{(2)} \to 0$. To give a more explicit case, let us assume that the first terms of the DS (11) are

$$\dot{u} = \begin{pmatrix} 1+2\lambda & A_{12}(\lambda) \\ A_{21}(\lambda) & -2+\lambda \end{pmatrix} u + \begin{pmatrix} u_1^3 u_2 \\ u_1^2 u_2^2 \end{pmatrix} + \text{h.o.t.}$$

with A_{12}, A_{21} arbitrary vanishing functions of λ. In this case, the first nonlinear term is already in NF, with $B = I =$ identity (cf. (10)), and one easily gets $\theta(\lambda) = 1 + \lambda/3 + \cdots$, $\alpha = \lambda/3 + \cdots$; whereas Eq. (16) takes the form $\beta = \frac{5}{3}\lambda + \rho + \cdots = 0$, giving

$$\lambda = -\tfrac{3}{5}\hat{u}_1^2\hat{u}_2 + \cdots .$$

The next example, even if apparently similar, is actually a generalization of Theorem 1; indeed eigenvalues of the same sign will be involved. In this case, the convergence of the NF would be guaranteed with no other condition, thanks to the Poincaré criterion; however, we need even in this case to impose that Condition A is fulfilled, to obtain a bifurcating solution of the same form as discussed so far.

EXAMPLE 2. Let $A_0 = \operatorname{diag}(1,2)$ and $d(2A_{11}(\lambda) - A_{22}(\lambda))/d\lambda|_{\lambda=0} \neq 0$. Then $\rho = v_1^2/v_2$ and the bifurcating solution is expected along a manifold of the form $\lambda\hat{u}_2 \propto \hat{u}_1^2 + \cdots$, obtained as solution of the condition (16). The discussion is now identical to the previous cases.

COROLLARY 1 (Hopf Bifurcation). *If σ_1, σ_2 are imaginary, $\sigma_{1,2} = \pm i\omega_0$, then the condition* (13) *becomes the standard "transversality condition"*

$$\left.\frac{d\operatorname{Re}\sigma(\lambda)}{d\lambda}\right|_{\lambda=0} \neq 0$$

and the bifurcating solution is the usual Hopf bifurcation.

Proof. The proof of Theorem 1 holds also in the case of imaginary eigenvalues. In this case, $s_1 = s_2 = 1$, $\rho = v_1^2 + v_2^2 = r^2$, $\theta(\lambda) = i(\omega_0 + \alpha(\lambda(r^2)) = i\omega(\lambda)$, and a simple calculation shows indeed that

$$\left.\frac{d(A_{11} + A_{22})}{d\lambda}\right|_{\lambda=0} = 2\left.\frac{d\operatorname{Re}\sigma}{d\lambda}\right|_{\lambda=0}.$$

Notice that in this case the condition $\rho = r^2 =$ const ensures $|v_i| \leq r$ and then the convergence is true for all $t \in \mathbf{R}$. ∎

4. DS WITH DIMENSION $n > 2$: REDUCTION TO A LOWER DIMENSIONAL PROBLEM

In this section, we will consider a general situation in which, given an n-dimensional DS with $n > 2$, it is possible to reduce the problem to a lower dimensional case. A quite simple but useful result is the following.

LEMMA 2. *Consider an n-dimensional DS* ($n > 2$), *and assume that for $\lambda_0 = 0$ there are $r < n$ resonant eigenvalues, say $\sigma_1, \ldots, \sigma_r$, such that no* *resonance relation of the form*

$$\sum_{h=1}^{r} m_h \sigma_h = \sigma_k \qquad k = r+1, \dots, n \tag{20}$$

exists. Then the DS—once in NF—can be reduced to an r-dimensional problem, putting for the remaining $n - r$ variables

$$\hat{v}_k(t) \equiv 0 \qquad k = r+1, \dots, n.$$

If this r-dimensional problem admits a solution $\hat{v}_h(t)$, $h = 1, \dots, r$ (e.g., a bifurcating solution as in Theorem 1), *then the original DS admits a solution in which the $n - r$ components $\hat{u}_k(t)$, $k = r+1, \dots, n$, are "h.o.t." with respect to the first r components $\hat{u}_h(t)$.*

Proof. Let us consider the NF variables v_i, $i = 1, \dots, n$, and let us introduce the shorthand notation $v' \equiv (v_1, \dots, v_r)$ and $v'' \equiv (v_{r+1}, \dots, v_n)$ —and, correspondingly,

$$A_0 = \begin{pmatrix} A'_0 & \\ & A''_0 \end{pmatrix} \qquad \text{with } A'_0 = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$$

and similarly for A''_0. According to Lemma 1, the matrices B_i commuting with A_0 necessarily split in block form:

$$B_i = \begin{pmatrix} B'_i & \\ & B''_i \end{pmatrix}.$$

Therefore the DS in NF will be

$$\begin{aligned} \dot{v}' &= A'_0 v' + \sum \beta'_i B'_i v' = (1+\alpha) A'_0 v' + \sum_j{}^{*} \beta'_j B'_j v' \\ \dot{v}'' &= A''_0 v'' + \sum \beta''_i B''_i v'' = (1+\alpha) A''_0 v'' + \sum_j{}^{*} \beta''_j B''_j v'', \end{aligned} \tag{21}$$

where β'_i and β''_i are functions (of λ and) of all the constants of motion $\rho(v', v'')$. Considering in particular the terms β'', the assumption that there are no resonances of the form (20) excludes the occurrence of fractional constants of motion in $\mathcal{I}_{A_0}$ of the form $\beta'' = \rho'(v')/v_k$ for some $k = r + 1, \dots, n$; then $v'' = 0$ solves the second set of equations of the system in NF. Transformation into the initial coordinates u shows the final statement. ∎

Remark 2. A quite common situation, which is actually a special case of Lemma 2, is that the n eigenvalues σ_i are such that the only analytic or fractional constants of motion ρ which can be constructed depend only on r variables v', with the above notation (e.g., if $\sigma_i = 2, -2, \sqrt{2}, \sqrt{3}$). In this case, no resonance of the form (20) is clearly admitted and Lemma 2 holds true; notice in particular that in this case the NF (21) becomes automatically "triangular" (cf. [13]). It is also clear that, given n eigenvalues σ_i, it can happen that several different reductions are possible: e.g., if $\sigma_i = 1, -2, \sqrt{2}, -\sqrt{2}, \sqrt{3}$, Lemma 2 can allow a reduction of the NF either into a four-dimensional problem or equally well into two independent two-dimensional problems.

Remark 3. We can clearly combine Theorem 1 and Lemma 2 to obtain a resonant bifurcating solution with $n \geq 2$, $r = 2$, $p = 1$. It can be noted that the leading terms of the resonant bifurcating solution obtained in this way can be characterized as the kernel of the linear operator T defined by

$$T = \frac{d}{dt} - A_0$$

acting on the linear space of the vectors $w \equiv (w_1, \dots, w_n)$, where the w_i are formal series in powers of $\exp(\theta_0 t)$,

$$w_i = \sum_{m_i = -\infty}^{+\infty} c_{m_i} \exp(m_i \theta_0 t) \qquad i = 1, \dots, n, \tag{22}$$

where, using previously introduced notation (12),

$$\theta_0 = \frac{\sigma_1}{s_1} = -\frac{\sigma_2}{s_2}.$$

It is easily seen, indeed, that the hypothesis on σ_i is equivalent to the property that T has precisely a two-dimensional kernel, generated by $w_1 = \exp(s_2 \theta_0 t) = \exp(\sigma_1 t)$ and $w_2 = \exp(-s_1 \theta_0 t) = \exp(\sigma_2 t)$; see (14).

An especially interesting case occurs clearly when one of the eigenvalues is zero. This can be considered as a particular case of Lemma 2; we have indeed:

COROLLARY 2 (Stationary Bifurcation). *Assume that for $\lambda_0 = 0$ the matrix A_0 of the given DS admits just one eigenvalue* (*say σ_1*) *equal to zero. Then, with the standard condition*

$$\left. \frac{dA_{11}(\lambda)}{d\lambda} \right|_{\lambda = 0} \neq 0, \tag{23}$$

there is a stationary bifurcating solution of the form $\hat{u} = \hat{u}_1 + \cdots$.

Proof. With the notation introduced in the proof of Lemma 2 we put here $v' = v_1$, $v'' = (v_2, \ldots, v_n)$, and now $v_1 \in \mathscr{I}_{A_0}$ as $\sigma_1 = 0$. Writing the problem in the form (21), there cannot be terms of the form $\beta'' = \rho'(v_1)/v''$ (indeed, $\sigma_2, \ldots, \sigma_n \neq 0$), and therefore $v'' = 0$ solves the second set of equations for v''. The remaining equation is then a one-dimensional equation

$$\dot{v}_1 = \sum_j{}^* \beta_j' B_j' v_1 \equiv \beta^*(\lambda, v_1) v_1$$

and the assumption (23) is easily seen to be equivalent to $\partial\beta^*/\partial\lambda|_{\lambda=0} \neq 0$, which ensures the existence of a stationary bifurcation, solving $\dot{v}_1 = 0$ with $\lambda = \lambda(v_1)$, as in the usual situations. As in the previous cases, the proof is completed by coming back to the original coordinates u. ∎

5. DS WITH DIMENSION $n > 2$: A GENERAL RESULT

We now consider an n-dimensional DS: according to the above section, we can assume, for concreteness, that there is a resonance involving all the n eigenvalues (see also Remark 4 below).

Before giving the main result of this section, the following property may be useful (the proof is straightforward).

LEMMA 3. *Given a DS in NF $\dot{v} = g(v) = A_0 v + G(v)$, the constants of motion $\rho \in \mathscr{I}_{A_0}$ of the linear part are in general not constants of motion of the full DS, but their time dependence can be expressed as a function only of the ρ themselves: $(d\rho/dt)_g = \Phi(\rho)$, where $(d/dt)_g$ is the Lie derivative along the DS. If the DS satisfies Condition A, then these constants of motion ρ are also constants of motion of the full DS in NF $\dot{v} = g(v)$.*

THEOREM 2. *Consider the DS* (1) *and assume that for the value $\lambda_0 = 0$ the eigenvalues σ_i of A_0 are distinct, are real or purely imaginary, and satisfy a resonance relation* (3). *Assume also that $p = n - 1$, i.e., that there are $n - 1$ real parameters $\lambda \equiv (\lambda_1, \ldots, \lambda_{n-1})$ and finally that putting*

$$a_k^{(i)} = \left.\frac{\partial A_{ii}(\lambda)}{\partial \lambda_k}\right|_{\lambda=0} \qquad i = 1, \ldots, n,\ k = 1, \ldots, n-1 \tag{24}$$

the $n \times n$ matrix D constructed according to the following definition (notice *that only the diagonal terms $A_{ii}(\lambda)$ of $A(\lambda)$ are involved) is not singular:*

$$\det D \equiv \det\begin{pmatrix} \sigma_1 & a_1^{(1)} & a_2^{(1)} & \cdots & a_{n-1}^{(1)} \\ \sigma_2 & a_1^{(2)} & \cdots & & \\ \cdots & & & & \\ \sigma_n & a_1^{(n)} & \cdots & & a_{n-1}^{(n)} \end{pmatrix} \neq 0. \tag{25}$$

Then there is, in a neighborhood of $u_0 = 0$, $\lambda_0 = 0$, $t = 0$, a bifurcating solution of the form

$$\hat{u}_i(t) = \big(\exp(\hat{\alpha}(\lambda)A_0 t)\big)\hat{u}_{0i}(\lambda) + h.o.t. \qquad i = 1, \ldots, n, \tag{26}$$

where $\hat{\alpha}(\lambda)$ is some function of the λ's such that $\hat{\alpha}(\lambda) \to 1$ for $\lambda \to 0$.

Proof. As in the particular cases examined in the previous sections, let us consider the given problem transformed into NF: $\dot{v} = A_0 v + G(v, \lambda)$ in the new coordinates v. Let us write $G(v, \lambda)$ in the more convenient form

$$G(v, \lambda) = \sum_{i=1}^{n} \kappa_i(\lambda, \rho(v))K_i v,$$

where $K_i \equiv \operatorname{diag}(0, \ldots, 1, \ldots, 0)$ is the diagonal matrix with 1 at the ith position, or also

$$\dot{v}_i = \sigma_i v_i + \kappa_i(\lambda, \rho(v))v_i \qquad (\text{no sum over } i = 1, \ldots, n),\ \rho(v) \in \mathscr{I}_{A_0}. \tag{27}$$

Now recall that the functions $\rho(v)$ and $\kappa(\lambda, \rho)$ can be fractional in the components v_i, but in such a way that each term $\kappa_i K_i v$ is a polynomial, so that the only admitted fractional terms necessarily have the form, e.g., $(v_2^{s_2} \cdot \cdots \cdot v_n^{s_n})/v_1$ and so on; the assumption that the σ_i are distinct ensures that the functions κ cannot be of zero degree in the v_i (i.e., of the form v_2/v_1, e.g.); then when $v \to 0$ all terms $\kappa_i v_i$ vanish more rapidly than v and one finds

$$\frac{\partial}{\partial v_i}\big(\kappa_i(\lambda, \rho(v))v_i\big)\big|_{v=0} = \kappa_i(\lambda, 0) \qquad (\text{no sum over } i).$$

Then Eq. (27) can be written, at the lowest order,

$$\dot{v}_i = \sigma_i v_i + \kappa_i(\lambda, 0)v_i + \cdots = \sigma_i v_i + \sum_{k=1}^{n-1} q_{ik}\lambda_k v_i + \cdots + \cdots, \tag{28}$$

where q_{ik} are the elements of a constant matrix with n rows and $(n-1)$ columns. On the other hand, considering the original DS

$$\dot{u} = A(\lambda)u + \cdots$$

its diagonal bilinear terms (in u and λ) $a_k^{(i)}\lambda_k u_i$ (using the definition (24)) are just NF terms and are not changed by the normalizing procedure; therefore, $a_k^{(i)} = q_{ik}$. Now according to Condition A, the NF is convergent (or, better, is obtained by a convergent NT) if one can rewrite (27) in the split form as in (10),

$$\dot{v} = A_0 v + \alpha(\lambda, \rho(v)) A_0 v + \sum_j{}^* \beta_j(\lambda, \rho(v)) B_j v, \tag{29}$$

where α, β_j are suitable combinations of the κ_i and can satisfy the $(n-1)$ conditions

$$\beta_j(\lambda, \rho) = 0. \tag{30}$$

In fact, the hypothesis (25) ensures precisely that one is able to do this and also to satisfy $\beta_j(\lambda, \rho) = 0$, by means of the implicit-function theorem, giving some $(n-1)$ relations (here the ρ are considered as independent variables)

$$\lambda_j = \lambda_j(\rho). \tag{31}$$

Once these $n-1$ conditions are satisfied, i.e., on the manifold defined just by (31) (which is an analytic manifold—see [4]), the convergence of the NT taking the initial DS into

$$\dot{v} = A_0 v + \alpha(\lambda) A_0 v = \hat{\alpha}(\lambda) A_0 v \tag{32}$$

is granted. This DS can be easily solved, giving

$$\hat{v}(t) = \exp((\hat{\alpha}(\lambda) A_0 t))\hat{v}_0 \tag{33}$$

with the $n-1$ relations

$$\lambda_j = \lambda_j(\rho(\hat{v})) \qquad \rho = \rho(\hat{v}(t)) = \rho(\hat{v}_0) \in \mathscr{I}_{A_0}. \tag{34}$$

The desired result is then obtained, with remarks similar to those in the proof of Theorem 1, coming back to the initial coordinates by means of the inverse (convergent) transformation $v \to u = v + \psi(v)$, where $\psi(v)$ are series of monomials of the v_i $(i = 1, \ldots, n)$. ∎

It is immediately seen that, in particular, condition (13) of Theorem 1 is nothing but a special case of (25). As a generalization of Corollary 1, the case of purely imaginary eigenvalues is particularly interesting, because it corresponds to the case of coupled oscillators with multiple frequencies and gives, with the above hypotheses, the existence of multiple-periodic bifurcating solutions. We have indeed [17]:

COROLLARY 3. *With the same notation as before, let* $n = r = 4$ *and* $\sigma_1 = -\sigma_2 = i$, $\sigma_3 = -\sigma_4 = mi$ *(with* $m = 2, 3, \dots$*): then, with* $\lambda \equiv (\lambda_1, \lambda_2, \lambda_3) \in \mathbf{R}^3$ *and* $\det D \neq 0$, *there is a double-periodic bifurcating solution preserving the frequency resonance*

$$\omega_1 : \omega_2 = 1 : m.$$

EXAMPLE 3. Consider a four-dimensional DS, with $u \in \mathbf{R}^4$, $\lambda \in \mathbf{R}^3$, describing two coupled oscillators with unperturbed frequencies $\omega_1 = 1$, $\omega_2 = 2$,

$$\dot{u} = A(\lambda)u + F(u, \lambda)$$

$$\text{where } A(\lambda) = \begin{pmatrix} \lambda_1 + \lambda_3 & -(1 + \lambda_2) & \lambda_1 & 0 \\ 1 + \lambda_2 & \lambda_1 - \lambda_3 & 0 & \lambda_1 \\ \lambda_2 & 0 & \lambda_3 & -2 \\ 0 & -\lambda_2 & 2 & \lambda_3 \end{pmatrix}.$$

For $\lambda = 0$, the eigenvalues of A_0 are $\pm i, \pm 2i$, and it is easily seen that condition (25) is satisfied (the above procedure and notation can be extended without difficulty to the complex space). The NF will have the form, denoting by $w_a = v_1 + iv_2$, $w_b = v_3 + iv_4$ the new coordinates, in complex form

$$\dot{w}_a = iw_a + \left(\lambda_1 + \beta_1^{(1)}(\lambda, \rho) + i\left(\lambda_2 + \beta_2^{(1)}(\lambda, \rho)\right)\right)w_a$$
$$\dot{w}_b = 2iw_b + \left(\lambda_3 + \beta_3^{(1)}(\lambda, \rho) + i\beta_4^{(1)}(\lambda, \rho)\right)w_b,$$

where we have put

$$\beta_i(\lambda, \rho) = \beta_i(\lambda, 0) + \beta_i^{(1)}(\lambda, \rho)$$

and the β_i are real functions of the three (functionally independent) constants of motion

$$\rho_1 = |w_a|^2 = v_1^2 + v_2^2 \equiv r_a^2, \qquad \rho_2 = |w_b|^2 = v_3^2 + v_4^2 \equiv r_b^2,$$
$$\rho_3 = \left(w_a^2\overline{w_b} + \text{c.c.}\right) \equiv 2r_a^2 r_b \cos 2\varphi,$$

where φ is the time phase-shift between the two components w_a and w_b. Notice that fractional constants of motion $\rho(w)$ may appear in this problem, e.g., w_a^2/w_b or $\overline{w_a}w_b/w_a$ (which are functionally—but not polynomially—dependent on the three above), but this would not alter the result, as shown in the proof of the theorem. The above NF can be transformed into the split form (29)

$$\dot{w}_a = iw_a + i\alpha(\lambda, \rho(w))w_a + \beta_a(\lambda, \rho(w))w_a$$
$$\dot{w}_b = 2iw_b + 2i\alpha(\lambda, \rho(w))w_b + \beta_b(\lambda, \rho(w))w_b$$

with

$$\alpha = -\lambda_2 - \beta_2^{(1)} + \beta_4^{(1)}$$
$$\beta_a = \lambda_1 + \beta_1^{(1)} + i\big(2\lambda_2 + 2\beta_2^{(1)} - \beta_4^{(1)}\big)$$
$$\beta_b = \lambda_3 + \beta_3^{(1)} + i\big(2\lambda_2 + 2\beta_2^{(1)} - \beta_4^{(1)}\big).$$

One can impose the convergence of the NT solving for $\lambda_i = \lambda_i(\rho)$ the conditions $\beta_a = \beta_b = 0$, which actually give three real conditions, and obtain a bifurcating double-periodic solution, with frequencies

$$\omega_1 = 1 + \alpha(\lambda) \qquad \text{and} \qquad \omega_2 = 2(1 + \alpha(\lambda)).$$

Just to give a concrete example, let us imagine that the NF is such that

$$\beta_1^{(1)} = -\rho_1, \qquad \beta_2^{(1)} = 0, \qquad \beta_3^{(1)} = -\rho_2, \qquad \beta_4^{(1)} = \rho_3.$$

Then the leading terms of the solution, in the original real variables u_i, are

$$\hat{u}_1 = r_a \cos \omega t, \qquad \hat{u}_2 = r_a \sin \omega t, \qquad \hat{u}_3 = r_b \cos 2\omega(t + \varphi),$$
$$\hat{u}_4 = r_b \sin 2\omega(t + \varphi)$$

with the constraints

$$\lambda_1 = r_a^2, \qquad \lambda_3 = r_b^2, \qquad \lambda_2 = r_a^2 r_b \cos 2\varphi, \qquad \omega = 1 + \lambda_2$$

producing (see especially the role of λ_2) a sort of amplitude−phase−frequency locking in the solution.

Remark 4. As already remarked, it can happen that, among the n resonant eigenvalues σ_i, as considered in Theorem 2, one can find some $r < n$ eigenvalues σ_h in such a way that the assumption of Lemma 2 is satisfied; therefore the problem can be reduced—as explained in the above section—to an r-dimensional problem. In this case, if one can also find $r - 1$ parameters λ_h in such a way that the corresponding $r \times r$ matrix D is not singular, then the existence of another bifurcating solution is ensured by the same Theorem 2 (see the end of the final Example 6 for a (quite simple) case).

6. DEGENERATE EIGENVALUES AND THE PRESENCE OF SYMMETRIES

We now consider the case of multiple eigenvalues of the matrix A_0. This situation is a little bit more involved: indeed, the presence in this case of constants of motion $\rho \in \mathscr{I}_{A_0}$ of the form $\rho = v_i / v_j$ prevents the direct application of the argument used in the previous theorems. Another difficulty is related to the greater number of matrices $B_i \in \mathscr{C}(A_0)$ (if an eigenvalue σ_i has multiplicity d, then any $d \times d$ matrix acting on the subspace of the corresponding eigenvectors clearly commutes with A_0): this would require the presence of a greater number of parameters λ, to satisfy the Condition A.

However, the presence of degenerate eigenvalues is usually connected to the existence of some symmetry property of the problem, and we will restrict to consider this simpler—and probably more realistic and physically interesting—case.

We refer here to the case of "geometric" or Lie point-symmetries [18, 19]: a vector function $s(u) = Lu + S(u)$ is said to be (the infinitesimal generator of) a symmetry for the given DS if $s(u)$ is not proportional to $f(u)$ and the vector fields $X_f = f \cdot \nabla$ and $X_s = s \cdot \nabla$ commute,

$$[X_f, X_s] = 0 \tag{35}$$

or, introducing the Lie–Poisson bracket $\{\cdot, \cdot\}$ between two vector functions $h^{(1)}(u), h^{(2)}(u)$ defined by

$$\{h^{(1)}, h^{(2)}\}_i = (h^{(1)} \cdot \nabla) h_i^{(2)} - (h^{(2)} \cdot \nabla) h_i^{(1)} \tag{36}$$

if, equivalently,

$$\{f, s\} = 0. \tag{35'}$$

The symmetry is linear if $S = 0$. Notice that linear and nonlinear symmetries are changed into the other under (nonlinear) coordinate transformations $u \to v$. It can also be remarked that a DS in NF always admits a linear symmetry [8, 11, 12]:

LEMMA 4. *Any NF admits the linear symmetry $s_{A_0} = A_0 v$:*

$$\{A_0 v, g(v)\} = 0.$$

This is in fact a restatement of (7) and (5) using the above definition (36), indeed $\mathscr{A}(g) = \{A_0 v, g\}$.

We need the following important properties of Lie–point symmetries of a DS.

LEMMA 5. *Assume that the given DS admits a symmetry $s(u) = Lu + S(u)$, where L is semisimple and not zero. Then one has in particular*

$$[A_0, L] = 0$$

and there is a NF of the DS which admits the linear symmetry $s_L = Lv$.

The proof is well known and can be found, e.g., in [8, 11, 12].

An (unpleasant) consequence of the degeneracy of the eigenvalues σ_i of A_0 is a larger arbitrariness in the choice of the matrices $B_i \in \mathscr{C}(A_0)$ to express the NF (8): consider, e.g., the case in $\mathbf{R}^3$

$$A_0 = \mathrm{diag}(1, 1, -2),$$

then the same resonant term can be written in two apparently different forms (the notation is obvious),

$$\begin{pmatrix} x^2yz \\ 0 \\ 0 \end{pmatrix} = x^2z \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \rho_1 B_1 v$$

$$= xyz \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \rho_2 B_2 v \qquad \rho_1, \rho_2 \in \mathscr{I}_{A_0}, B_1, B_2 \in \mathscr{C}(A_0).$$

Assume now that the symmetry "removes the degeneracy"; that is, recalling that A_0 and L commute, assume that there is a simultaneous basis of eigenvectors for A_0 and L such that any two eigenvectors with the same eigenvalue under A_0 are distinguished by a different eigenvalue under L. Then there are precisely n independent matrices, which we now denote by $\tilde{B}_i$, commuting with both A_0 and L:

$$\tilde{B}_i \in \mathscr{C}(A_0) \cap \mathscr{C}(L). \tag{37}$$

These matrices $\tilde{B}_i$ also commute with each other (they in fact can be taken to be diagonal: this is true upon complexification of the space, in the case the eigenvalues of A_0 or of L are not real; see Example 4 below). We can then express the NF by using precisely these n matrices $\tilde{B}_i$,

$$\begin{aligned} G(v, \lambda) &= \sum_{i=1}^{n} \beta_i(\lambda, \rho(v)) \tilde{B}_i v \qquad \left(\tilde{B}_i \in \mathscr{C}(A_0) \cap \mathscr{C}(L)\right) \\ &= \alpha(\lambda, \rho(v)) A_0 v + \sum_j{}^{*} \beta_j(\lambda, \rho(v)) \tilde{B}_j v \end{aligned} \tag{38}$$

using previously introduced notation. The convenience of this choice is immediately evident. Indeed, recalling Lemma 5, the NF admits the symmetry $s_L = Lv$, i.e.,

$$\{Lv, G(v)\} = 0, \tag{39}$$

which implies

$$\sum_{i=1}^{n} Lv \cdot \nabla \beta_i \tilde{B}_i v = 0$$

or, due to the independence of the $\tilde{B}_i v$,

$$\alpha(\lambda, \rho(v)), \beta_j(\lambda, \rho(v)) \in \mathscr{I}_L \qquad (j = 1, \dots, n-1), \tag{40}$$

where $\mathscr{I}_L$ is the set of the constants of motion of the linear problem $\dot{v} = Lv$, and this means that the functions β_i in (38) (and the ρ as well) can be chosen as simultaneous constants of motion of the two linear problems $\dot{v} = A_0 v$ and $\dot{v} = Lv$:

$$\rho(v), \beta_i(\lambda, \rho) \in \mathscr{I}_{A_0} \cap \mathscr{I}_L. \tag{38$'$}$$

The assumption that L has removed the degeneracy has the other consequence that no constants of motion of the form v_i/v_j are admitted in $\mathscr{I}_{A_0} \cap \mathscr{I}_L$ (and then in the NF (38)); therefore, the functions β_j appearing in the Σ_j^* in (38) can be written in the form

$$\beta_j(\lambda, \rho) = \beta_j(\lambda, 0) + \beta_j^{(1)}(\lambda, \rho) \tag{41}$$

with $\beta_j^{(1)}(\lambda, 0) = 0$. On the other hand, considering the first-order terms (in u) of the DS in its initial form, and writing

$$\begin{aligned} \dot{u} &= A_0 u + A^{(1)}(\lambda) u + \text{higher order terms} \\ &= A_0 u + \sum_i b_i(\lambda) \tilde{B}_i u + \sum_\ell c_\ell(\lambda) C_\ell u + \text{h.o.t.} \\ &= A_0 u + a(\lambda) A_0 u + \sum_j{}^{*} b_j(\lambda) \tilde{B}_j u + \sum_\ell c_\ell(\lambda) C_\ell u + \text{h.o.t.}, \end{aligned} \tag{42}$$

where $A^{(1)}(0) = 0$, the sum Σ_ℓ includes all linear terms which are not resonant (i.e., $[A_0, C_\ell] \neq 0$) and therefore disappear after the NT. The remaining terms are instead not changed by the NT and therefore one gets

$$a(\lambda) = \alpha(\lambda, 0), \quad b_j(\lambda) = \beta_j(\lambda, 0) \qquad (j = 1, \dots, n-1). \tag{43}$$

Then, assuming that there are $n-1$ parameters λ_k and that the $(n-1) \times (n-1)$ matrix $\tilde{D}$, introduced by means of the following definition, is not singular, i.e., that

$$\det \tilde{D} \equiv \det \left. \frac{\partial b_j}{\partial \lambda_k} \right|_{\lambda = 0} \neq 0 \qquad b_j = b_j(\lambda) = \beta_j(\lambda, 0), \tag{44}$$

one can proceed exactly as in Theorem 1 and conclude with the existence of a convergent NT and of a bifurcating solution on some manifold $\lambda_j = \lambda_j(\rho)$.

We can then state:

THEOREM 3. *Let* $\dot{u} = A(\lambda)u + F(\lambda, u)$ *be a DS with* $n - 1$ *real parameters* λ_j; *assume that* A_0 *has degenerate resonant eigenvalues and that the DS admits a Lie point-symmetry generated by* $s(u) = Lu + S(u)$, *where* $L \neq 0$ *is semisimple, such that* L *removes the degeneracy of the eigenvalues of* A_0. *Let* $\tilde{B}_i$ *be* n *independent matrices in* $\mathscr{C}(A_0) \cap \mathscr{C}(L)$, *and write the linear part* (*in* u) *of the DS in the form* (42). *Then, if the matrix* $\tilde{D}$ *defined in* (44) *is not singular, there is a resonant bifurcating solution of the same form* (26) *as in Theorem* 2.

This result can be well illustrated by two examples (notice that in the second one we will consider also the possibility of extending the above procedure in the presence of a *discrete* symmetry (i.e., not a continuous Lie point-symmetry)).

EXAMPLE 4. Introducing the 2×2 matrices

$$J_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad I_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \tag{45}$$

consider the four-dimensional DS, describing two coupled oscillators with the same unperturbed frequency $\omega = 1$.

$$\dot{u} = f(u, \lambda) = A_0 u + \sum \varphi_i\left(r_a^2, r_b^2, r_a \times r_b, \lambda\right) M_i u \qquad \text{with } A_0 = \begin{pmatrix} J_2 & \\ & J_2 \end{pmatrix}, \tag{46}$$

where the eigenvalues of A_0 are $\sigma_i = \pm i$ (doubly degenerate), $r_a^2 = u_1^2 + u_2^2$, $r_b = u_3^2 + u_4^2$, $r_a \times r_b = u_1 u_4 - u_2 u_3$, and the M_i are matrices of the form

$$M_i = \begin{pmatrix} N' J_2 & N'' J_2 \\ J_2 N'' & J_2 N' \end{pmatrix} \qquad \text{with } N', N'' \text{ arbitrary } 2 \times 2 \text{ matrices}.$$

This DS admits the linear symmetry generated by $Lu \cdot \nabla$, where

$$L = \begin{pmatrix} & J_2 \\ J_2 & \end{pmatrix}.$$

There are four matrices $\tilde{B}_i \in \mathscr{C}(A_0) \cap \mathscr{C}(L)$, namely

$$A_0, \qquad L, \qquad I = \begin{pmatrix} I_2 & \\ & I_2 \end{pmatrix}, \qquad H = \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix}.$$

Let us then write down explicitly the linear part (in u) of the DS (46) as in (42), i.e.,

$$\begin{aligned} \nabla f|_{u=0} &= A(\lambda) = A_0 + A^{(1)}(\lambda) \\ &= A_0 + a(\lambda)A_0 + \big(b_1(\lambda)I + b_2(\lambda)L + b_3(\lambda)H\big) + \sum_{\ell=1}^{4} c_\ell(\lambda)C_\ell, \end{aligned}$$

where a, b_j, c_ℓ are combinations of the $\varphi_i(0,0,0,\lambda)$. When in NF the last sum disappears, whereas the other terms remain unchanged, and the NF will have the form

$$\dot{v} = A_0 v + \alpha A_0 v + \big(\beta_1(\lambda,\rho)I + \beta_2(\lambda,\rho)L + \beta_3(\lambda,\rho)H\big)v, \quad (47)$$

where $\rho(v) = v_1^2 + v_2^2 + v_3^2 + v_4^2,\ v_1v_3 + v_2v_4 \in \mathscr{I}_{A_0} \cap \mathscr{I}_L$, and

$$\beta_j(\lambda, 0) = b_j(\lambda) \qquad j = 1,2,3.$$

The assumption (44) $\det(\partial b_j/\partial\lambda_k)|_{\lambda=0} \neq 0$ then ensures, as already discussed, the existence of a resonant bifurcating solution, which in this case is a periodic solution with frequency $\omega = 1 + \cdots$ and preserving the strict frequency resonance $1:1$.

EXAMPLE 5. As an even simpler example, let us consider the case of a three-dimensional DS with real eigenvalues: let

$$\dot{u} = A_0 u + \varphi_1(r^2, z, \lambda)Lu + \varphi_2(r^2, z, \lambda)Iu, \quad (48)$$

where $u \in \mathbf{R}^3$, $\lambda \in \mathbf{R}^2$, $r^2 = u_1^2 + u_2^2$, $z = u_3$, and, using the 2×2 matrices introduced in (45),

$$A_0 = \begin{pmatrix} I_2 & \\ & -2 \end{pmatrix} \qquad L = \begin{pmatrix} J_2 & \\ & 0 \end{pmatrix}.$$

This DS admits the linear symmetry SO_2 generated by $Lu \cdot \nabla$. There are three matrices $\tilde{B}_i$ in $\mathscr{C}(A_0) \cap \mathscr{C}(L)$, namely A_0, L, I (notice that, actually, in this case, $\mathscr{C}(A_0) \subset \mathscr{C}(L)$, but the DS (48) is not in NF, because r^2 and z are not in $\mathscr{I}_{A_0}$). Proceeding as before, the NF will have the form

$$\dot{v} = A_0 v + \alpha A_0 v + \beta_1 Lv + \beta_2 Iv, \quad (49)$$

where now β_j (and α, of course) are functions of $\rho = (v_1^2 + v_2^2)v_3$ and λ only. Then we need two real parameters λ_k, and—with the assumption (44)—a bifurcating solution is obtained

$$\hat{u}_1 = \hat{u}_{10} \exp(1 + \alpha)t + \text{h.o.t.}$$
$$\hat{u}_2 = \hat{u}_{20} \exp(1 + \alpha)t + \text{h.o.t.}$$
$$\hat{u}_3 = \hat{u}_{30} \exp(-2(1 + \alpha))t + \text{h.o.t.}$$

along with some conditions

$$\lambda_1 = \lambda_1(r^2 z), \qquad \lambda_2 = \lambda_2(r^2 z).$$

This example can be useful to show that also the presence of *discrete* symmetries (e.g., exchange or reflection symmetries) can be of some help in this approach: a DS admits a discrete symmetry R, where R is a nonsingular matrix, if

$$f(Ru) = Rf(u) \tag{50}$$

and it is known that the NF also admits the same symmetry [5]. The possible role of this fact in our argument can be illustrated by the following modification of the above example.

EXAMPLE 5′. The situation is same as Example 5, but now assume the DS admits the discrete symmetry

$$R = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}.$$

Then, in both the initial DS (48) and its NF (49), the term containing the matrix L disappears; therefore, Condition A requires the vanishing of only one term in the NF and a bifurcating solution can be obtained with the presence of just *one* real parameter λ.

7. FINAL REMARK ON THE ROLE OF CONDITION A

To stress and illustrate the relevance of the role played in our argument by Condition A, let us consider this final example, which is essentially an adjustment (in view of the present discussion) of an example given, with quite different purposes, in [20].

EXAMPLE 6. With $u \in \mathbf{R}^3$ and $\lambda \in \mathbf{R}^2$, let

$$\dot{u} = A(\lambda)u + \begin{pmatrix} -(r^2 + 3z^2)u_1 \\ -(r^2 + 3z^2)u_2 \\ (3r^2 + z^2)z \end{pmatrix},$$

$$\text{where } A(\lambda) = \begin{pmatrix} \lambda_1 + \lambda_2 & 1 & 0 \\ -1 & \lambda_1 & 0 \\ 0 & 0 & -\lambda_1 \end{pmatrix} \tag{51}$$

with $r^2 = u_1^2 + u_2^2$, $z = u_3$. It is not difficult to see (cf. [20]) that, if $\lambda_2 = 0$, this system admits a family of heteroclinic orbits connecting the origin to the circle $r^2 = \lambda_1$ and another family of heteroclinic orbits, living on the manifold $r^2 + z^2 = \lambda_1$, connecting the circle $r^2 = \lambda_1$ to the point $P \equiv (0, 0, \lambda_1)$. When $\lambda_2 \neq 0$, this heteroclinic structure breaks down, and an application of the Melnikov theory [20–23] shows the occurrence of transversal intersections of stable and unstable manifolds, with the consequent appearance of the chaotic behavior described by the classical Birkhoff–Smale horseshoe-like structure (actually, we need here a simple three-dimensional version of the standard Melnikov theorem; see, e.g., [20]).

On the other hand, the NF of the above DS (51) exhibits a perfectly regular (i.e., nonchaotic) behavior: indeed, the NF, according to Lemma 4, must possess the linear symmetry generated by A_0, which implies that the NF is symmetric under rotations around the z-axis; then the NF is essentially a two-dimensional problem and therefore no chaos is admitted. This clearly implies that the NT cannot be convergent.

If one now imposes Condition A on the NF of the DS (51), it can be easily seen that this condition is satisfied only along the circle defined by

$$\lambda_1 = 3\left(v_1^2 + v_2^2\right) + v_3^2 \qquad \lambda_2 = 4\left(v_3 - v_1^2 - v_2^2\right),$$

where in fact a (Hopf-type) periodic solution occurs. Then, in conclusion, we are here in the presence of a regular solution (where Condition A is indeed satisfied), which is completely surrounded by chaotic solutions: this clearly confirms the crucial role played by Condition A in the argument.

Just for completeness, and in agreement with Lemma 2 and Corollary 2, let us remark that this example admits (quite trivially) a reduction according to Remark 4: indeed, in correspondence with the eigenvalue $\sigma_3 = 0$, we also get the stationary bifurcating solution $\lambda_1 = z^2$ with $u_1 = u_2 = 0$.

REFERENCES

1. S.-N. Chow and J. K. Hale, ''Methods of Bifurcation Theory,'' Springer-Verlag, New York, 1982.
2. V. I. Arnold, ''Geometrical Methods in the Theory of Differential Equations,'' Springer-Verlag, Berlin, 1982.
3. V. I. Arnold and Yu. S. Il'yashenko, Ordinary differential equations, *in* ''Dynamical Systems I'' (D. V. Anosov and V. I. Arnold, Eds.), Encyclopaedia of Mathematical Sciences, Vol. 1, pp. 1–148, Springer-Verlag, Berlin, 1988.
4. A. D. Bruno, Analytical form of differential equations, *Trans. Moscow Math. Soc.* **25** (1971), 131–198; **26** (1972), 199–239.
5. A. D. Bruno, ''Local Methods in Nonlinear Differential Equations,'' Springer-Verlag, Berlin, 1989.
6. J.-C. van der Meer, ''The Hamiltonian Hopf Bifurcation,'' Springer-Verlag, Berlin, 1985.
7. H. W. Broer and F. Takens, Formally symmetric normal forms and genericity, *in* Dynamics Reported: Expositions in Dynamical Systems (New Series), Vol. 2, pp. 39–59, Springer-Verlag, Berlin/New York, 1989.
8. G. Iooss and M. Adelmeyer, ''Topics in Bifurcation Theory and Applications,'' World Scientific, Singapore, 1992.
9. S.-N. Chow, C. Li, and D. Wang, ''Normal Forms and Bifurcations of Planar Vector Fields,'' Cambridge Univ. Press, Cambridge, UK, 1994.
10. D. Arnal, M. Ben Ammar, and G. Pinczon, The Poincaré–Dulac theorem for nonlinear representations of nilpotent Lie algebras, *Lett. Math. Phys.* **8** (1984), 467–476.
11. G. Cicogna and G. Gaeta, ''Symmetry and Perturbation Theory in Nonlinear Dynamics,'' Springer–Verlag, Berlin, 1999.
12. C. Elphick, E. Tirapegui, M. E. Brachet, P. Coullet, and G. Iooss, A simple global characterization for normal forms of singular vector fields, *Physica D* **29** (1987), 95–127.
13. S. Walcher, On differential equations in normal form, *Math. Ann.* **291** (1991), 293–314.
14. G. Cicogna and G. Gaeta, Poincaré normal forms and Lie point symmetries, *J. Phys. A* **27** (1994), 461–476, 7115–7124.
15. R. H. Rand and D. Armbruster, ''Perturbation Methods, Bifurcation Theory and Computer Algebra,'' Springer-Verlag, Berlin, 1987.
16. C. Cicogna and G. Gaeta, Lie point symmetries and nonlinear dynamical systems, *Math. Comput. Modelling*, **25** (1997), 101–113.
17. C. Cicogna, Multiple-periodic bifurcation and resonance in dynamical systems, *Nuovo Cimento B* **113** (1998), 1425–1430.
18. P. J. Olver, ''Applications of Lie Groups to Differential Equations,'' Springer-Verlag, Berlin, 1986.
19. L. V. Ovsjannikov, ''Group Properties of Differential Equations,'' Acad. Sci. USSR, Novosibirsk, 1962 (English transl. by G. W. Bluman, 1967); ''Group Analysis of Differential Equations,'' Academic Press, New York, 1982.
20. P. Holmes, Unfolding a degenerate nonlinear oscillator: A codimension two bifurcation, *Ann. New York Acad. Sci.* **357** (1980), 473–488.
21. V. K. Melnikov, Stability of the center to time periodic perturbation, *Trans. Moscow Math. Soc.* **12** (1963), 3–57.
22. S.-N. Chow, J. K. Hale, and J. Mallet-Paret, An example of bifurcation to homoclinic orbits, *J. Differential Equations* **37** (1980), 351–373.
23. J. Guckenheimer and P. J. Holmes, ''Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields,'' Springer-Verlag, Berlin, 1983.