Semiclassical treatment of the path-integral approach to special effects in superconductivity

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Received 27 July 1990

Abstract

Distinctive features of the flux mode regime in the Anderson–Josephson potential are analyzed using both semiclassical path-integral and instanton techniques. This potential governs the behavior of a superconducting ring interrupted by a weak Josephson junction. After establishing the relation between WKB and functional-integrals approximations, semiclassical expansions are derived from a discretized version of the Feynman propagator. The physical meaning of instantons is extended to the condensed matter problem of "energy spectrum" for weak-link ring states localized near the minimum energy regions. Under certain circumstances, an improved WKB approximation enables us to find expressions for the decay rates of metastable states, performing the reduction of the original Anderson–Josephson shape to a double-hump structure. Although we have not discussed here the temperature dependence of the Josephson plasma frequency, our calculations are consistent with recent theories for systems at finite temperatures with two metastable states separated by a potential energy barrier.

Keywords: Path-integral, superconductivity, decay rates of metastable states.

1. Introduction

Apart from their intrinsic interest, direct semiclassical (or WKB) quantization schemes yield new insights because there exists a well-known connection between the semiclassical approximation to the original Feynman path-integral and the formal equivalence of Feynman’s and Schrödinger’s approaches [9,16]. In the (nonlinear) real form of both approaches, Feynman’s and Schrödinger’s systems depend basically on $\hbar^2$ rather than $\hbar$ and are ripe for development in $\hbar^2$. In particular, the leading zeroth- and first-order in $\hbar$ terms in the WKB expansion are essentially classical as generated by the usual classical Hamilton–Jacobi equation.

In quantum field theory, the perturbation expansions present some analogy with semiclassical expansions generated by a path-integral, and the solutions of the complex classical field
equations (i.e., for complex field variables), namely instantons, are currently used in many branches of physics [2].

In another context, the interest in the formation of a natural bridge between WKB and functional-integral approximation has been increasing. This interest seems to be mainly motivated by Langer's fundamental work [20] based upon calculations for the quantum decay rate $\Gamma/\hbar$ (or for the quantum decay width $\Gamma$, in quantum field theory) of a metastable (virtual) level in one-dimensional potentials.

Although Langer's original treatment appeared to be particularly suited as we approach the top of the barrier, more recently, improved WKB calculations have been extended to near the bottom of the potential well. It would be then pertinent to obtain a closed-form expression for the decay rate of any energy level involving low-lying states, from which the more prominent characteristics of the motion in an unstable potential are reflected.

This work illustrates these ideas by studying some aspects concerning the behaviour of a thick superconducting ring containing a weak link constriction (i.e., a SQUID ring) subjected to an external applied flux $\Phi$. Assuming the included flux $\Phi$ in the ring to be a classical quantity, this behaviour is governed by the Anderson–Josephson potential [1,15]

$$U(\Phi, \Phi_s) = \frac{1}{2\Lambda} (\Phi - \Phi_s)^2 - \Lambda J_c \cos\left(2\pi \frac{\Phi}{\Phi_0}\right),$$  \hspace{1cm} (1)

where $\Lambda$ is the geometric inductance, $J_0 = \Phi_0/2\pi\Lambda$ the current associated to the flux quantum $\Phi_0 = \hbar/2e$ and $J_c$ the critical current amplitude.

Our motivation to adapt path integration to this condensed matter model, which has been successfully studied in field-theoretical contexts involving the dynamics of a simple Nambu–Goldstone field [12,29] and a Higgs-type Ginzburg–Landau model [26], has actually come from its suitability for reducing the original Lagrangian to simpler and more usual Lagrangians in which the WKB approximation works well [13,14]. To close this section, we would like to point out that in our approximate treatment the results on physical decay amplitudes are consistent with those which are theoretically and empirically suggested in recent literature [5,7,11,17].

2. Short-time expansion

The retarded propagator $K(\Phi_Q, t_Q; \Phi_A, t_A)$ for $t_Q - t_A > 0$, which connects the flux state $\Phi_Q$ at the instant $t_Q$ and the flux state $\Phi_A$ at the instant $t_A$, is given in the Feynman's formulation of the quantum mechanics as

$$K(\Phi_Q, t_Q; \Phi_A, t_A) = \lim_{N \to \infty} \int d\Phi_1 \cdots d\Phi_N K_r(\Phi_Q, \Phi_{N-1}) \cdots K_r(\Phi_1, \Phi_A),$$  \hspace{1cm} (2)

where $K_r(\Phi, \Phi_{-1}) = K(\Phi, \epsilon; \Phi_{-1}, 0)$ is a short-time propagator ($\epsilon = (t_Q - t_A)/N$ is small).

Since the underlying system is one-dimensional, we may write

$$K_r(\Phi, \Phi_{-1}) = \left(C/2\pi i\hbar \epsilon \right)^{1/2} \exp\left(\frac{ic}{2\hbar \epsilon} (\Phi - \Phi_{-1})^2 - \frac{ie}{\hbar} U\left[\frac{1}{2}(\Phi + \Phi_{-1})\right]\right),$$  \hspace{1cm} (3)
where \( C \) is the effective geometric capacitance. This expression guarantees the formal equivalence between Feynman’s and Schrödinger’s approach, namely,

\[
\int d\Phi' K_s(\Phi, \Phi') \Psi(\Phi', t) = \exp \left( - \frac{i\epsilon}{\hbar} \left[ - \frac{\hbar^2}{2C} \frac{\partial^2}{\partial \Phi^2} + U(\Phi, \Phi_x) \right] \right) \Psi(\Phi, t), \tag{4}
\]

so that the wave-function in the zeroth-order (classical) approximation is of the form

\[
\Psi(\Phi, t) \equiv \Psi_{\text{WKB}}(\Phi, t) = D^{-1/2}(S_{\text{cl}}) \exp \left[ \frac{i}{\hbar} S_{\text{cl}}(\Phi, t) \right], \tag{5}
\]

where \( D(S_{\text{cl}}) \) is the conventional (\( U \)-dependent) WKB determinant which includes the universal normalizing factor, and \( S_{\text{cl}}(\Phi, t) \) is the classical action.

The dominant contribution to \( \Psi_{\text{WKB}}(\Phi, t) \) comes from the classical flux trajectory \( \Phi_{\text{cl}} \), which extremalizes the classical action function. As is known, starting from the propagator \( K(\Phi, t_0; \Phi_A, t_A) \), Feynman obtains the classical limit when \( S_{\text{cl}} > \frac{\pi}{\hbar} \).

The quantum probability \( P(Q, A) \) connecting two states of the ring, \( A \) and \( Q \), is given by

\[
P(Q, A) = \frac{2\pi\hbar(t_Q - t_A)}{C} |K(\Phi, t_0; \Phi_A, t_A)|^2. \tag{6}
\]

In order to find a formal expression for \( P(Q, A) \), we have introduced elsewhere [8] an equivalence relation which assigns the flux path \( \frac{1}{2}[\Phi(t) + \Phi(t)] \) to the pair \( [\Phi(t), \Phi(t)] \). Putting (2) and (3) into (6) and changing variables

\[
\varphi_j = \frac{\Phi_j + \Phi_j^*}{2}, \quad q_j = \frac{\varphi_j - \varphi_j^*}{\hbar}, \tag{7}
\]

where \( \varphi_j \) is the displacement charge as a function of the rate at which \( \Phi_j \) threads the ring (i.e., roughly \( \varphi_j \approx -C\Phi_j/\epsilon \)), the quantum probability reads

\[
P(Q, A) = (t_Q - t_A) \lim_{\epsilon \to 0} \frac{(2\pi)^\frac{1-N}{2}}{\epsilon} \int \cdots \int d\varphi_1 \cdots d\varphi_{N-1} \, dq_1 \cdots dq_{N-1} \]

\[
\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N-1} \left[ (\varphi_j - \varphi_j^*)(q_j - q_j^*) \right] + \frac{i\epsilon}{\hbar} U \left( \varphi_j - \frac{\hbar \epsilon}{2C} q_j \right) \right\}.
\tag{8}
\]

Then, we are able to construct an expansion in powers of \( \hbar^2 \),

\[
P(Q, A) = P_0(Q, A) + P_1(Q, A) \hbar^2 + \ldots, \tag{9}
\]

whose terms may be obtained from the expression

\[
P_j(Q, A) = j! \lim_{\hbar \to 0} \frac{d^j P(Q, A)}{d(\hbar^2)^j}. \tag{10}
\]

Our semiclassical approximation, which will be valid as long as it is sufficient to take a few terms in (9) to calculate \( P(Q, A) \) with a negligible error, provides an interesting tool for extending the range of applications of Feynman’s approach to quantum mechanics.
3. The instanton approach

Now we shall consider equilibrium total flux in the ring, so that an isothermal representation of the steady-state ($\partial U/\partial \Phi = 0$) obeys [4]

$$\Phi - \Phi_0 = \Lambda J(\Phi),$$

where $J(\Phi)$ denotes the critical current in the ring, i.e.,

$$J(\Phi) = J_0 \sin \left( \frac{2\pi \Phi}{\Phi_0} \right),$$

and verifies the flux periodicity condition $J(\Phi + \Phi_0) = J(\Phi)$.

Let us concentrate on well-defined discrete flux states in the ring which correspond to an included flux given by $\Phi = n\Phi_0$, $n = 0, 1, 2, \ldots$. In this case, inserting (11) and (12) into (1) and making use of the series expansions of $\sin(2\pi \Phi/\Phi_0)$ and $\cos(2\pi \Phi/\Phi_0)$ up to fourth order in $2\pi \Phi/\Phi_0$, the inverted double-hump potential may be written as

$$V(\Phi) = -U(\Phi) = \frac{2C\Omega_0^2}{3\Lambda^2 J_0^2} \left( \Phi^2 - \frac{3}{8} \Lambda^2 J_0^2 \right)^2,$$

where the energy origin has been conveniently shifted and

$$\Omega_0 = (\Lambda C)^{-1/2} \frac{J_0}{J_c} \left( 1 + \frac{J_0}{J_c} \right)^{1/2}$$

is the Josephson plasma frequency.

We now turn our attention to the Feynman path-integral representation of the propagator

$$K(\Phi_Q, t_Q : \Phi_A, t_A) = \int(\Phi_0, t_0) \mathcal{D}[\Phi(t)] \exp\left( \frac{i}{\hbar} S_{cl}[\Phi(t)] \right),$$

where

$$\mathcal{D}[\Phi(t)] = \lim_{N \to \infty} \left( \frac{C}{2\pi i \hbar \epsilon} \right)^{N/2} \prod_{j=1}^{N} d\Phi(t_j)$$

is the Feynman path differential measure in flux space and

$$S_{cl}[\Phi(t)] = \int_{t_a}^{t_Q} dt \left[ \frac{1}{2} C \left( \frac{d\Phi}{dt} \right)^2 - V(\Phi) \right]$$

is the classical action. In order to see how the instanton comes about, we make the substitution (Wick transformation)

$$t \to \tau = -it.$$

The propagator (15) becomes

$$K(\Phi_Q, \tau_Q : \Phi_A, \tau_A) = \int(\Phi_0, \tau_0) \mathcal{D}[\Phi(\tau)] \exp\left( -\frac{1}{\hbar} S_{cl}[\Phi(\tau)] \right),$$

and now, the action for the particle of "effective mass" $C$ moving in the potential $U(\Phi)$ is given by

$$S_{cl}[\Phi(\tau)] = \int_{\tau_a}^{\tau_Q} d\tau \left[ \frac{1}{2} C \left( \frac{d\Phi}{d\tau} \right)^2 + V(\Phi) \right].$$
As mentioned above, with regard to the WKB approximation to the wave-function, the main contribution to (19) comes from the minima of (20), i.e., for flux paths $\Phi(\tau)$ such that

$$\frac{d^2 \Phi}{d\tau^2} = -\frac{dV(\Phi)}{d\Phi}. \quad (21)$$

It may be verified [28] that for (13),

$$\Phi(\tau) = \frac{1}{4}\sqrt{6} A J_0 \tanh \left[ \frac{1}{\sqrt{2}} \Omega_0 (\tau - \tau_0) \right] \quad (22)$$
satisfies (21) being an instanton solution, where the parameter $\tau_0$ indicates its centre and the classical minima correspond to

$$\Phi = \pm \frac{1}{4}\sqrt{6} A J_0. \quad (23)$$

The charge ("momentum") associated with (22) is

$$\mathcal{Q} = -C \Phi = -\frac{1}{4}\sqrt{3} \Omega_0 A J_0 \text{sech}^2 \left[ \frac{1}{2} \sqrt{2} \Omega_0 (\tau - \tau_0) \right], \quad (24)$$

which is localized about the centre $\tau_0$. For $\mathcal{Q} \to \infty$ and $\tau_\pm \to -\infty$, from (13) the action (20) becomes [10]

$$S \approx \frac{1}{4} C \Omega_0 A^2 J_0^2, \quad (25)$$

and the splitting of the ground state into two nearby levels is obtained as

$$E = \frac{1}{4} \hbar \Omega_0 \left[ 1 \pm \Lambda J_0 \left( \frac{6 C \Omega_0}{\pi \hbar} \right)^{1/2} \exp \left( - \frac{C \Omega_0 A^2 J_0^2}{2 \hbar} \right) \right]. \quad (26)$$

In order to check this result, we will now try to apply the orthodox WKB method. Taking into account the double-hump structure of the potential, the usual WKB penetration integral reads

$$W_2 = 4 \int_0^b d\Phi \left\{ 2 C \left[ V(\Phi) - \frac{1}{2} \hbar \Omega_0 \right] \right\}^{1/2}, \quad (27)$$

where

$$b = \left( \frac{1}{8} \Lambda^2 J_0^2 - \frac{1}{2} \Lambda J_0 \left( \frac{3 \hbar}{C \Omega_0} \right)^{1/2} \right)^{1/2} \quad (28)$$
is the classical turning point. By assuming that the ground state energy $\frac{1}{2} \hbar \Omega_0$ is much smaller than the height of the barrier, (27) can be put in the form [22]

$$W_2 = C \Omega_0 A^2 J_0^2 - \hbar \left[ \ln \left( \frac{6 C \Omega_0 A^2 J_0^2}{\hbar} \right) + 1 \right] + \cdots \quad (29)$$

and the split is [19, p.176]

$$\Delta E_{\text{WKB}} = \frac{\hbar \Omega_0}{\pi} \exp \left[ - \frac{1}{2 \hbar} W_2 \right] = \frac{\hbar \Omega_0 \Lambda J_0}{\pi} \left( \frac{6 e C \Omega_0}{\hbar} \right)^{1/2} \exp \left( - \frac{C \Omega_0 A^2 J_0^2}{2 \hbar} \right) = \sqrt{\frac{e}{\pi}} \Delta E_{\text{inst}}. \quad (30)$$

The factor $(e/\pi)^{1/2}$ corresponds to the zeroth-order correction to the barrier penetration factor, $\exp[-W_2/(2\hbar)]$, near the bottom of the potential well, i.e., where the harmonic approximation works well. In fact, if one takes within this approximation the square of the ratio of the
approximate WKB wave-function for the ground state to the exact harmonic-oscillator wave-function evaluated deep inside the classically inaccessible region, one gets precisely the value of (30), as pointed out by various authors (see, e.g., [3]).

4. Decay rates

In this section, we present WKB calculations of the decay rates for the Anderson–Josephson potential (1) confining our attention to well-defined flux values in the ring. Clearly, the complete solvability (i.e., the determination of all eigenfunctions) is not important since the distinctive features of the dynamic decay process are reflected in its behaviour near the top and bottom of the well.

From a purely formal point of view, the decay rate in a metastable domain of a quantum mechanical system may be defined as [18]

$$\Gamma = 2 \lim_{\tau \to +\infty} \left\{ \frac{1}{\tau} \text{Re} \left[ i \log \langle \Psi_{\text{WKB}} | \exp \left( \frac{H\tau}{\hbar} \right) | \Psi_{\text{WKB}} \rangle \right] \right\},$$

where $H$ is the Hamiltonian of the system.

In the special case of a symmetric double-hump potential $U(\Phi)$ as typified by (13), the quantum decay rate $\Gamma(E)/\hbar$ (of the lowest metastable state) near the top of the well is obtained from the expression [25]

$$\Gamma(E) = \frac{\hbar \Omega_0}{\ln[64U(0)/\hbar \Omega_0]} \ln \left\{ 1 + \exp \left\{ -\frac{2\pi}{\hbar \Omega_0} [U(0) - E] \right\} \right\}.$$ 

Combining (32) and (33), in Fig. 1 we represent $-\ln[\Gamma(E)/(\hbar \Omega_0)]$ as a function of the weak link capacitance. We have considered $J_c = 0.81 \mu A$, $A = 5 \cdot 10^{-10}$ H and $C$-values within the

![Fig. 1. Rate diagram near the top of the well as obtained from (32) and (33). The circuit parameters are $J_c = 0.81 \mu A$ and $A = 5 \cdot 10^{-10}$ H.](image)
Fig. 2. Rates diagram near the bottom of the well as obtained from (34)-(36). We consider here the first three levels \((k = 0, 1, 2)\) and values of \(J_c\) and \(\Lambda\) as for Fig. 1.

range of experimental validity [23]. Quantum flux escape out of both walls of the potential \(U(\Phi)\), as a model for the decay of a metastable state of a physical system [27], begins to play a role if \(C\) ("effective mass") decreases, i.e., when the de Broglie wave-length is large enough in comparison with the width of the barrier.

Following [25], in order to find the decay rate \(\Gamma(E_k)/\hbar\) of the \(k\)th level in \(U(\Phi)\) near the bottom of the well, we may write

\[
\Gamma(E_k) = \hbar \Omega_0 (\pi \alpha_B)^{-1} \exp \left( -\frac{1}{2\hbar} W_2 \right), \quad k = 0, 1, 2, \ldots,
\]

where for the first three levels,

\[
\alpha_B^{-1}(k = 0) = \left( \frac{\pi}{e} \right)^{1/2} \approx 1.08, \quad \alpha_B^{-1}(k = 1) \approx 1.03, \quad \alpha_B^{-1}(k = 2) \approx 1.02,
\]

and the barrier penetration factor becomes

\[
\exp \left( -\frac{1}{2\hbar} W_2 \right) = \left[ \frac{3C \Omega_0 \Lambda^2 J_0^2}{\hbar (k + \frac{1}{2})} \right]^{1/2} \exp \left( -\frac{C \Omega_0 \Lambda^2 J_0^2}{2\hbar} \right).
\]

Obviously, for \(k = 0\), (34)–(36) connect the semiclassical approximation to the path-integral formalism with instanton techniques (see (30)).

The rates diagram implied in (34)–(36) is represented in Fig. 2 for the three lower-lying states using the circuit parameters above mentioned. As long as the weak link capacitance increases, the quantum decay rate of the lowest metastable state seems to be independent of \(C\) until below \(10^{-15}\) F. This value goes into the accessible region for quantum fluxoid transitions in the framework of the harmonic oscillator approximation [8]. It is apparent that quantum flux escape from the \(k = 1\) and 2 levels is favoured by very small values of the capacitance.

Several authors [6,21,24] have pointed out the sharp transitions between the two different regimes that are present in the rate diagram of a system at finite temperatures and zero damping [11]. In their works, the attention is generally focused on an energy barrier with the shape of a quadratic-plus-cubic potential. A perhaps remarkable feature with regard to this widely studied
system is that its ground state (or nearly ground state) exhibits a similar behaviour to that one would expect here for large $k$. Damping mechanisms in unstable SQUID rings will be discussed separately.

References


