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An R-Local Milnor–Moore Theorem

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Over a subring R of the rationals, we explore the properties of a new functor $\mathscr{K}$ from spaces to differential graded Lie algebras. We prove a Hurewicz theorem which identifies $H_*\mathscr{K}(X)$ with $\pi_*(\Omega X) \otimes R$ in a range of dimensions. Using it, we prove an R-local version of the Milnor–Moore theorem. © 1989 Academic Press, Inc.

INTRODUCTION

This paper continues the program, begun in [2], of exploring an R-local form of rational homotopy theory. Our overall goal, for a subring R of $\mathbb{Q}$, is to discover small algebraic models over R which encode as much information as possible about the R-local homotopy type of a space X. The paper [2] introduced a functor $\mathscr{E}\Omega$ which is weakly equivalent in the case $R = \mathbb{Q}$ with Quillen's differential graded Lie algebra functor $\mathscr{Q}$ [10]. This paper examines $\mathscr{E}\Omega$ when $R \neq \mathbb{Q}$.

The Milnor–Moore theorem [8] discusses the rationalized Hurewicz homomorphism

$$h'' \colon \pi_*(\Omega X) \otimes \mathbb{Q} \to H_*(\Omega X; \mathbb{Q}). \tag{1}$$

The homomorphism h'' is a bijection between $\pi_*(\Omega X) \otimes \mathbb{Q}$, known as the rational homotopy Lie algebra (with Samelson product) for X, and the Lie algebra $\mathscr{P}H_*(\Omega X; \mathbb{Q})$ of primitives in the rational Pontrjagin ring for ΩX. If $\mathbb{Q}$ is replaced in (1) by a subring R, h'' is in general neither one-to-one nor onto primitives.

In order to generalize the Milnor–Moore theorem to a coefficient ring $R \subsetneq \mathbb{Q}$, we first recall Quillen's differential graded Lie algebra functor $\mathscr{Q}$. Quillen [10] showed that $\mathscr{Q}$ enjoys the simultaneous properties that there are natural isomorphisms

$$H_*\mathscr{U}\mathscr{Q}X \xrightarrow{\approx} H_*(\Omega X; \mathbb{Q}) \tag{2a}$$

and

$$\pi_*(\Omega X) \otimes \mathbb{Q} \xrightarrow{\approx} H_*(\mathscr{Q}X), \tag{2b}$$

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where $\mathscr{U}$ denotes the enveloping algebra of a (differential) (graded) Lie algebra. Since $H_*\mathscr{U}L = \mathscr{U}H_*L$ when L is a differential graded Lie algebra over $\mathbb{Q}$ [10, Prop. B2.1], (2a) and (2b) tell us

$$\mathscr{P}H_*(\Omega X; \mathbb{Q}) \approx \mathscr{P}H_*\mathscr{U}\mathscr{Q}X = \mathscr{P}\mathscr{U}H_*\mathscr{Q}X = H_*\mathscr{Q}X \approx \pi_*(\Omega X) \otimes \mathbb{Q},$$

which is essentially the Milnor–Moore theorem. In order to generalize it we will show that certain spaces X have a differential graded Lie algebra model $\mathscr{L}X$ for which

$$H_*\mathscr{U}\mathscr{L}(X) \xrightarrow{\approx} H_*(\Omega X; R), \tag{3a}$$

while the relation

$$\pi_*(\Omega X) \otimes R \xrightarrow{\approx} H_*\mathscr{L}(X) \tag{3b}$$

is valid for a certain range of dimensions.

We will actually consider two Lie algebra models for X in this paper. The first one is derived from the functor $\mathscr{E}\Omega$ of [2]. Denoted $\mathscr{K}(\ )$, it is defined on r-connected spaces ($r \geqslant 1$) and is of interest up to dimension $r\rho - 1$ if $R \ni n^{-1}$ for $1 \leqslant n < \rho$. The functor $\mathscr{K}$ has homotopy-invariant homology, and (for $r \geqslant 2$) this homology coincides with $\pi_*(\Omega X) \otimes R$ through dimension $r + 2\rho - 4$. (This range represents an improvement over the usual mod ρ Hurewicz theorem [9] when r is small compared to ρ.) Because of this, the Lie algebra $H_*\mathscr{K}(X)$ may be thought of as an approximation to $\pi_*(\Omega X) \otimes R$.

Our second Lie algebra associated to X, denoted $\mathscr{L}(X)$, requires the further hypothesis that X be a CW complex of dimension $\leqslant r\rho$. Then $\mathscr{L}(X)$ and $\mathscr{K}(X)$ have the same homology below dimension $r\rho$, but $\mathscr{L}(X)$ also satisfies (3a).

The paper is organized as follows. We begin with some algebraic preliminaries. The most important of these is the observation that a homomorphism of differential graded Lie algebras induces an isomorphism in a range of dimensions if and only if the induced homomorphism on enveloping algebras does so. We introduce $\mathscr{K}$ and check that $H_*\mathscr{K}(\ )$ is preserved, on maps by homotopy and on spaces by weak homotopy equivalences. We prove the Hurewicz theorem for $H_*\mathscr{K}$, and discuss naturality with respect to products. We then recall the model $\mathscr{L}$ from [2] and verify the promosed R-local Milnor–Moore theorem. We conclude with an application to ρ-elliptic spaces, decomposing their loop spaces into well-known factors.

1. Algebraic Preliminaries

We collect here some definitions, notations, and lemmas which will be useful to us in the course of the paper. We work with (differential) graded Lie algebras and Hopf algebras over an arbitrary ring R. On the whole, our results are generalizations of results already known to hold over a field.

We fix once and for all a commutative ring with unity R. We reserve the symbol ρ for the least prime, if any, which is not a unit in R; if $R \supseteq \mathbb{Q}$, then $\rho = \infty$. We let $r \geqslant 1$ denote a connectivity parameter.

We begin with a brief review of graded Hopf algebras, for which the principal reference is the classic work [8]. For us, an *R-algebra* A is always non-negatively graded and connected (i.e., $A_0 = R$). It is *R-free* if it is free as an R-module, and *r-reduced* if $A_t = 0$ for $0 < t < r$. A Hopf algebra (A, ψ) has a coproduct $\psi: A \to A \otimes A$ satisfying $\psi(x) - x \otimes 1 - 1 \otimes x \in A_+ \otimes A_+$ for $x \in A_+$.

Given an algebra homomorphism $f: A \to B$, where A is a Hopf algebra with coproduct ψ, the *Hopf algebra kernel* of f, denoted $\mathrm{Hak}(f)$, is defined by

$$\mathrm{Hak}(f) = \{x \in A \mid (1 \otimes f)\, \psi(x) \in A \otimes 1\}.$$

The Hopf algebra kernel of f is a connected graded subalgebra of A, but it need not be a Hopf algebra.

A monomorphism of graded R-modules is *R-split* if it has a left inverse as R-modules. We will frequently compensate for relaxing the customary hypothesis that R is a field by supposing homomorphisms to be R-split.

LEMMA 1.1. *Let $f: (A, \psi) \to (B, \chi)$ be a surjective homomorphism between R-free coassociative Hopf algebras. Suppose the inclusion $K = \mathrm{Hak}(f) \to A$ is R-split. Choose any R-module splitting $\gamma: A \to K$ and define $g: A \to K \otimes B$ as the composite*

$$A \xrightarrow{\psi} A \otimes A \xrightarrow{1 \otimes f} A \otimes B \xrightarrow{\gamma \otimes 1} K \otimes B.$$

Filter A via $F_s A = ((1 \otimes f)\, \psi)^{-1}(A \otimes B_{\leqslant s})$ and $K \otimes B$ via $F_s(K \otimes B) = K \otimes B_{\leqslant s}$. Then

(a) *For each $s \geqslant 0$, $g(F_s A) \subseteq F_s(K \otimes B)$ and g induces an isomorphism $\bar{g}: F_s A / F_{s-1} A \to F_s(K \otimes B)/F_{s-1}(K \otimes B) \approx K \otimes B_s$;*

(b) *g is an isomorphism of right B-comodules;*

(c) $\mathrm{gr}(g)$: *$\oplus_s (F_s A / F_{s-1} A) \to K \otimes B$ is an isomorphism.*

Proof. This is the dual of [8, Prop. 1.7].

In this paper, a *Lie algebra* (over R) L is positively graded and satisfies the Jacobi identities with signs. It is *r-reduced* if $L_t = 0$ for $t < r$. We denote by $\mathscr{U}$ the universal enveloping algebra functor, from Lie algebras to Hopf algebras.

LEMMA 1.2. *Let $f: L \to M$ be a surjection of R-free Lie algebras, and let K be its kernel. Then $\mathscr{U}(K) = \mathrm{Hak}(\mathscr{U}f)$ as subalgebras of $\mathscr{U}L$, and the inclusion of* Hak($\mathscr{U}f$) *into $\mathscr{U}L$ is R-split.*

Proof. This follows easily from the Poncaré–Birkhoff–Witt theorem, which enables us to write $\mathscr{U}L \approx \mathscr{U}K \otimes \mathscr{U}M$ as coalgebras and as left $\mathscr{U}K$-modules. In particular, $\mathscr{U}K \to \mathscr{H}L$ is R-split. A direct computation reveals that the composite

$$\mathscr{U}K \otimes \mathscr{U}M \xrightarrow{\approx} \mathscr{U}L \xrightarrow{\psi} \mathscr{U}L \otimes \mathscr{U}L \xrightarrow{1 \otimes \mathscr{U}f} \mathscr{U}L \otimes \mathscr{U}M$$

$$\xrightarrow{\approx} (\mathscr{U}K \otimes \mathscr{U}M) \otimes \mathscr{U}M \xrightarrow{1 \otimes \varepsilon \otimes 1} \mathscr{U}K \otimes \mathscr{U}M,$$

where $\varepsilon: \mathscr{U}M \to R$ is the augmentation, is the identity. It follows that $\mathrm{Hak}(\mathscr{U}f) = \mathscr{U}K \otimes 1 = \mathscr{U}K$ in $\mathscr{U}L$.

Now we introduce the differential. A *differential* on a (Lie, Hopf) R-algebra is a derivation of degree -1 whose square is zero; on a Hopf algebra, the coproduct must be a chain map. The term "differential graded (Lie, Hopf) algebra" is abbreviated dga (dgL, dgH). The kernel (resp. Hak) of a dgL (resp. dgH) homomorphism is a dgL (resp. dga). A dga (or dgL or dgH) is *n-acyclic* if its homology is n-reduced. A homomorphism is an *n-quism* if on homology it induces an isomorphism in dimensions below n and an epimorphism in dimension n. A *quism* is an n-quism for all n.

LEMMA 1.3. *Let $f: (L, d) \to (M, e)$ be a* dgL *surjection. Then f is an n-quism if and only if* ker(f) is n-acyclic.

Proof. Trivial, by the long exact sequence for homology.

LEMMA 1.4. *Let $f: (A, \psi) \to (B, \chi)$ and K be as in Lemma* 1.1, *and suppose further that there are differentials d on A and e on B which make f into a dgH homomorphism. Then $d(F_s A) \subseteq F_s A$, and the resulting first quadrant homology spectral sequence converging to $H_*(A)$ has $E_2^{st} = H_s(B; H_t(K))$.*

Proof. Straightforward.

LEMMA 1.5. *Let $f: (A, d, \psi) \to (B, e, \chi)$ be a surjection of R-free* *coassociative dgH's. Let* $(K, d) = \mathrm{Hak}(f)$, *and suppose* $K \to A$ *is R-split. Then f is an n-quism if and only if* (K, d) *is n-acyclic.*

Proof. Consider the spectral sequence associated to f in the previous lemma. The conclusion is a standard fact about spectral sequences whose E_2 term is given by $H_*(B; H_*(K))$.

Recall that ρ is defined so that $(\rho - 1)!$ is invertible in R. Fixing the integer $r \geq 1$, let $r' = 2[(r+2)/2]$ be the smallest even integer exceeding r, and put $N = N(r, \rho) = r'\rho - 2$.

LEMMA 1.6. *Let* $r \geq 1$ *and let* (L, d) *be an r-reduced dgL of the form* $L = \mathbb{L}\langle x_\alpha, y_\alpha \rangle$, *the free Lie R-algebra on the generating set* $\{x_\alpha, y_\alpha\}$, *where* $d(y_\alpha) = x_\alpha$ *and* $|x_\alpha| \geq r$. *Then* (L, d) *is N-acyclic. If* (M, e) *is any other R-free dgL, then the surjection*

$$(M, e) \amalg (L, d) \to (M, e) \tag{4}$$

sending L to zero is an N-quism.

Proof. The calculation of $H_*(\mathbb{L}\langle x_\alpha, y_\alpha \rangle, d)$, where $d(y_\alpha) = x_\alpha$, was essentially done in [4, Sect. 4].

Let (L', d') be the kernel of (4). One sees easily that L' is the free dgL on $\{\mathrm{ad}(w)(y_\alpha), \mathrm{ad}(w)(x_\alpha)\}$ as w runs through an R-basis for $\mathscr{U}M$. The differential d' satisfies

$$d'(\mathrm{ad}(w)(x_\alpha)) = \mathrm{ad}(e(w))(x_\alpha),$$

$$d'(\mathrm{ad}(w)(y_\alpha)) = \mathrm{ad}(e(w))(y_\alpha) + (-1)^{|w|}\,\mathrm{ad}(w)(x_\alpha).$$

Letting $\{z_\beta\}$ equal the set $\{\mathrm{ad}(w)(y_\alpha)\}$, we see easily that L' has the form $\mathbb{L}\langle z_\beta, d'(z_\beta)\rangle$, which makes L' N-acyclic. Apply Lemma 1.3.

LEMMA 1.7. *Let* (L, d) *be R-free and r-reduced. For* $n \leq N$, (L, d) *is n-acyclic if and only if* $\mathscr{U}(L, d)$ *is n-acyclic.*

Proof. First suppose L is free as a Lie R-algebra; then $\mathscr{U}L$ is free (i.e., it is a tensor algebra). If $\mathscr{U}L$ is n-acyclic, then the module $Q(\mathscr{U}L)$ of indecomposables is n-acyclic. Since $Q(L) = Q(\mathscr{U}L)$, in dimensions $\leq n$ it must be possible to pair off the generators of L, i.e., $L_{<n} = (\mathbb{L}\langle x_\alpha, y_\alpha \rangle)_{<n}$, where $d(y_\alpha) = x_\alpha$. We may now apply Lemma 1.6 to deduce that (L, d) is n-acyclic. Conversely, if L is n-acyclic, an induction on dimension using Lemma 1.6 shows that the generators of L may be paired off; i.e., L looks like some $\mathbb{L}\langle x_\alpha, y_\alpha \rangle$ with $d(y_\alpha) = x_\alpha$ in dimensions $<n$. Thus $\mathscr{U}(L, d)$ is also n-acyclic.

Now consider a general L. There exists a surjective quism $f\colon (M, e) \to$

(L, d), where M is a free r-reduced Lie algebra. Put $(K, e) = \ker(f)$. Because M is free and the inclusion $K \to L$ is R-split, K is also a free Lie algebra [4]. Because f is a quism, K is acyclic and consequently $\mathscr{U}K$ is N-acyclic. By Lemma 1.5, $\mathscr{U}f$ is an N-quism. Thus L is n-acyclic $\Leftrightarrow$ (because f is a quism) M is n-acyclic $\Leftrightarrow$ (because M is free) $\mathscr{U}M$ is n-acyclic $\Leftrightarrow$ ($\mathscr{U}f$ an N-quism) $\mathscr{U}L$ is n-acyclic.

LEMMA 1.8. *Let $f\colon (L, d) \to (M, e)$ be any homomorphism of r-reduced R-free dgL's.*

(a) *If f is surjective and $n \leqslant N$, then f is an n-quism if and only if $\mathscr{U}f$ is an n-quism.*

(b) *In general, for $n < N, f$ is an n-quism if and only if $\mathscr{U}f$ is an n-quism.*

Proof. First suppose f is surjective, and let $(K, d) = \ker(f)$. Then f is an n-quism $\Leftrightarrow$ (by 1.3) K is n-acyclic $\Leftrightarrow$ (by 1.7) $\mathscr{U}K$ is n-acyclic $\Leftrightarrow$ (by 1.2) $\mathrm{Hak}(\mathscr{U}f)$ is n-acyclic $\Leftrightarrow$ (by 1.5) $\mathscr{U}f$ is an n-quism.

For general f, factor f as

$$(L, d) \xrightarrow{j} (L, d) \amalg (L', d') \xrightarrow{f'} (M, e).$$

Here f' is a surjective extension of f and (L', d') has the form $\mathbb{L}\langle x_\alpha, y_\alpha \rangle$ with $d'(y_\alpha) = x_\alpha$. If $n < r$ there is nothing to prove, and if $n \geqslant r$ then (L', d') may be chosen to be r-reduced. By Lemma 1.6, the surjection $(L, d) \amalg (L', d') \to (L, d)$ is an N-quism; since j is a right inverse for this surjection, it too induces an isomorphism of homology in dimensions below N. Thus j, and consequently $\mathscr{U}j$, are $(N-1)$-quisms. Since f' is surjective and $n \leqslant N - 1$, we now have f is an n-quism $\Leftrightarrow$ f' is an n-quism $\Leftrightarrow$ $\mathscr{U}f'$ is an n-quism $\Leftrightarrow$ $\mathscr{U}f = (\mathscr{U}f') \circ (\mathscr{U}j)$ is an n-quism.

LEMMA 1.9. *Let (L, d) be an r-reduced R-free dgL. The natural inclusion $j\colon (L, d) \to (\mathscr{U}L, d)$ of chain complexes induces*

$$j_*\colon H_n(L, d) \to H_n(\mathscr{U}L, d),$$

which is a monomorphism for $n < r\rho$.

Proof. This follows from the existence [2] of a differential-respecting direct sum decomposition

$$\mathscr{U}L = \bigoplus_{i=0}^{\rho} \mathscr{U}^i L,$$

valid in dimensions $\leqslant r\rho$.

LEMMA 1.10. *Let $f\colon (L, d) \to (M, e)$ be a homomorphism of dgL's, where L is r-reduced and R-free. For some $n < r\rho$, suppose $H_n(\mathscr{U}f)$ is monomorphic. Then $H_n(f)$ is monomorphic.*

Proof. In the diagram

$$\begin{array}{ccc} H_n(L, d) & \xrightarrow{f_*} & H_n(M, e) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle j_*} \\ H_n(\mathscr{U}L, d) & \xrightarrow{(\mathscr{U}f)_*} & H_n(\mathscr{U}M, e) \end{array}$$

the left and bottom arrows are injections, so the top arrow is also an injection.

PROPOSITION 1.11. *Let $f\colon (L, d) \to (M, e)$ be a homomorphism of r-reduced R-free dgL's. For some $n < r\rho$, suppose that $H_t(\mathscr{U}f)$ is an isomorphism for $t \leqslant n$. Then $H_t(f)$ is an isomorphism for $t \leqslant n$.*

Proof. Always, $r\rho \leqslant N$. By Lemma 1.8, $H_t(f)$ is isomorphic for $t < n$ and surjective for $t = n$. By Lemma 1.10, $H_n(f)$ is also injective.

For completeness, we also include here

LEMMA 1.12. *Suppose R is a field or a subring of $\mathbb{Q}$. Let (L, d) be r-reduced and R-free, and suppose that $H_*(\mathscr{U}L, d)$ is R-free in dimensions $< n$, where $n \leqslant N$. Then $H_*(L, d)$ is R-free in dimensions below n, and the natural homomorphism*

$$(\mathscr{U}j)_* \colon \mathscr{U}H_*(L, d) \to H_*(\mathscr{U}L, d)$$

is an isomorphism in dimensions $< n$ and an epimorphism in dimension n.

Proof. Incorporate the hypothesis that L is r-reduced into the proof of [1, Lemma 4.3].

2. THE FUNCTOR $\mathscr{K}$

Over an arbitrary ring R, we define a functor $\mathscr{K}$ from r-connected topological spaces to r-reduced dg$L$'s. We verify that the homology of $\mathscr{K}(\ )$ is a homotopy invariant. We also demonstrate that $H_*\mathscr{K}(X)$ is an abelian Lie algebra when X is an H-space. Since $\mathscr{K}$ agrees in low dimensions with the functor $\mathscr{E}\Omega$ of [2], and $\mathscr{E}\Omega$ is weakly equivalent to Quillen's functor $\mathscr{Q}$ when $R \supseteq \mathbb{Q}$, we may view $\mathscr{K}$ as a generalization of $\mathscr{Q}$.

Let TOP_r denote the category of r-connected pointed topological spaces and let MON_r be the category of $(r-1)$-connected topological monoids. Let (CU_*, ∂) denote the $(r-1)$st Eilenberg subcomplex of the cubical singular chain complex, with coefficients in R. In [2, Sect. 7] it is shown that the usual diagonal $\psi_Y: CU_*(Y) \to CU_*(Y) \otimes CU_*(Y)$ for $Y \in MON_r$ is naturally chain homotopic below dimension $r\rho$ to a new diagonal $\phi_Y: CU_{<r\rho}(Y) \to (CU_*(Y) \otimes CU_*(Y))_{<r\rho}$. This new $\phi_{(\ )}$ has the property that it is strictly cocommutative and coassociative below dimension $r\rho$. Putting $\mathscr{E}(Y) = \ker(\bar{\phi}_Y)$, $\bar{\phi}_Y$ being the reduced diagonal $\bar{\phi}_Y(u) = \phi_Y(u) - 1 \otimes u - u \otimes 1$, we have constructed a functor $\mathscr{E}$ from MON_r to dgL's for which $\mathscr{U}\mathscr{E}$ coincides in dimensions below $r\rho$ with $CU_*(\ )$.

When $R \supseteq \mathbb{Q}$, then $r\rho = \infty$ and $\mathscr{U}\mathscr{E} = CU_*(\ )$, so $\mathscr{E}$ provides a very nice approach to the rational homotopy theory of Y. However, when $\rho < \infty$, ϕ_Y is not defined above dimension $r\rho - 1$. We therefore feel free to make whatever definition is convenient in dimensions $\geqslant r\rho$, so as to simplify the properties of the resulting functor.

Definition. For $X \in TOP_r$, define $\mathscr{K}(X) = \{\mathscr{K}_n(X)\}_{n \geqslant 0} \subseteq CU_*(\Omega X)$ by

$$\mathscr{K}_n(X) = \begin{cases} \mathscr{E}_n \Omega(X) = (\ker \bar{\phi}_{\Omega X})_n & \text{if } n < r\rho; \\ (\ker \bar{\phi}_{\Omega X} \partial)_n & \text{if } n = r\rho; \\ CU_n(\Omega X) & \text{if } n > r\rho. \end{cases}$$

Clearly, $\mathscr{K}(\ )$ is a functor from TOP_r to r-reduced *dgL*'s.

Lemma 2.1.

(a) For $n \geqslant r\rho$, $H_n \mathscr{K}(X) = H_n(\Omega X; R)$.

(b) For $n < r\rho$, $(U\mathscr{K}(X)))_n = CU_n(\Omega X)$ and $H_n \mathscr{U}\mathscr{K}(X) = H_n(\Omega X; R)$.

Proof. The inclusion of chain complexes $\mathscr{K}(X) \to CU_*(\Omega X)$ is a surjection on cycles in dimension $r\rho$ and an isomorphism above $r\rho$, so it induces an isomorphism of homology in dimensions $\geqslant r\rho$.

We have already remarked that $(\mathscr{U}\mathscr{E}\Omega X)_{<r\rho} = CU_{r\rho}(\Omega X)$, which means that the induced dga homomorphism

$$g: \mathscr{U}\mathscr{K}(X) \to CU_*(\Omega X) \tag{5}$$

is an $(r\rho - 1)$-quism. It remains only to check that (5) injects on homology in dimension $r\rho - 1$.

Let $x \in \mathscr{U}\mathscr{K}(X)$ be any $(r\rho - 1)$-dimensional cycle for which $g(x)$ is a boundary, say $g(x) = \partial(y)$. We want to show that x is a boundary. In [2, Sect. 3] an R-homomorphism

$$\mu: (\mathscr{U}L \otimes \mathscr{U}L)_{\leqslant r\rho} \to (\mathscr{U}(L_{<r\rho}))_{\leqslant r\rho}$$

is defined for any R-free r-reduced Lie algebra L. This μ commutes with any differential that can be imposed on L. In dimensions below $r\rho$ we also have $\bar{\Delta}\mu\bar{\Delta}=\bar{\Delta}$, where $\bar{\Delta}$ denotes the reduced coproduct on $\mathscr{U}L$. We now drop the subscripts on $\psi_{\Omega X}$ and $\phi_{\Omega X}$.

Put $L=\mathscr{K}(X)$, so that $(g\otimes g)\bar{\Delta}=\bar{\phi}g$ in dimensions below $r\rho$. Because $\psi\simeq\phi$ and $g(x)$ is a cycle, we may write $\bar{\phi}g(x)=\bar{\psi}g(x)+\partial(u)$. Because $g\otimes g$ is isomorphic on $((\mathscr{U}L)_+\otimes(\mathscr{U}L)_+)_{\leqslant r\rho}$, we may write $\bar{\psi}(y)+u=(g\otimes g)(v)$, and from $(g\otimes g)\,\partial(v)=\partial\bar{\psi}(y)+\partial(u)=\bar{\phi}g(x)=(g\otimes g)\,\bar{\Delta}(x)$ we may deduce $\partial(v)=\bar{\Delta}(x)$. Now consider $z=y-g\mu(v)$. Observe that

$$\bar{\phi}\partial(z)=\bar{\phi}g(x)-\bar{\phi}g\mu\partial(v)=(g\otimes g)\,\bar{\Delta}(x-\mu\bar{\Delta}(x))=(g\otimes g)(\bar{\Delta}-\Delta\mu\bar{\Delta})(x)=0.$$

This says that $z\in\ker(\bar{\phi}\partial)=\mathscr{K}_{r\rho}(X)$; in particular, $z\in\operatorname{im}(g)$. Thus $y\in\operatorname{im}(g)$, say $y=g(w)$. Then $g(\partial(w)-x)=\partial(y)-g(x)=0$, but g is one–to–one in dimension $r\rho-1$, so $\partial(w)=x$ as desired.

We now show that $H_*\mathscr{K}$ is a homotopy invariant. Let IX denote the reduced cylinder on X, i.e., $IX=(X\times I)/(x_0\times I)$, x_0 being the base point of X. Let $j_0,j_1\colon X\to IX$ denote the inclusions of X into the two ends of the cylinder, and let $q\colon IX\to X$ be the projection back onto X. Thus $qj_0=id_X=qj_1$. Two maps $f_0,f_1\colon X\to Y$ are homotopic in TOP_r if and only if $f_0\vee f_1$ extends over IX.

LEMMA 2.2. *Let $f\colon X\to Y$ be a weak R-equivalence in TOP_r; i.e., f induces an isomorphism on $\pi_*(\ )\otimes\mathbb{Z}[(\rho-1)!]^{-1}$. Then $H_*\mathscr{K}(f)$ is an isomorphism. In particular, $H_*\mathscr{K}(q)$ is an isomorphism.*

Proof. In dimensions $n\geqslant r\rho$, $H_*\mathscr{K}(f)$ is simply the isomorphism $H_*(\Omega f; R)$. In dimensions below $r\rho$, apply Proposition 1.11 and Lemma 2.1.

LEMMA 2.3. *Let $f_0,f_1\colon X\to Y$ be homotopic in TOP_r. Then $H_*\mathscr{K}(f_0)=H_*\mathscr{K}(f_1)$.*

Proof. It suffices to show that the two inclusions $j_0,j_1\colon X\to IX$ induce the same homomorphism on $H_*\mathscr{K}(\ )$. They do, because each of $H_*\mathscr{K}(j_0)$ and $H_*\mathscr{K}(j_1)$ is a right inverse for the isomorphism $H_*\mathscr{K}(q)$.

LEMMA 2.4. *Let $j\colon\mathscr{K}(X)\to\mathscr{U}\mathscr{K}(X)$ be the natural inclusion. The induced homomorphism on homology $g_*j_*\colon H_*\mathscr{K}(X)\to H_*(\Omega X; R)$ is one-to-one.*

Proof. In dimensions $\geqslant r\rho$, g_*j_* is clearly the identity. In dimensions $<r\rho$, combine Lemmas 1.9 and 2.1.

LEMMA 2.5. *Let $X\in TOP_r$ be an H-space or, more generally, suppose* *that the Pontrjagin ring $H_*(\Omega X; R)$ is commutative. Then $H_*\mathscr{K}(X)$ is an abelian Lie algebra.*

Proof. For $x, y \in H_*\mathscr{K}(X)$, $g_*j_*[x, y] = [g_*j_*(x), g_*j_*(y)] = 0$, the second bracket denoting the Pontrjagin commutator in $H_*(\Omega X; R)$. Since g_*j_* is one-to-one, $[x, y] = 0$.

3. The Hurewicz Theorem

We prove here a strong form of the Hurewicz theorem for $\mathscr{K}$. Specifically, for $X \in TOP_r$ and $R \subseteq \mathbb{Q}$, $H_n\mathscr{K}(X)$ agrees with $\pi_n(\Omega X) \otimes R$ for $n < \min(r + 2\rho - 3, r\rho - 1)$. When $R = \mathbb{Q}$, we obtain a new proof of the Milnor–Moore theorem.

Throughout this section we will suppose that R is a subring of $\mathbb{Q}$. We fix an $r \geqslant 1$ and define $\mathscr{K}$: $TOP_r \to$ dg$L$'s as in Section 2, for this r. We begin by computing $H_*\mathscr{K}(S^{m+1})$.

LEMMA 3.1. *For $m \geqslant r$ and $r \leqslant q < r\rho$,*

$$H_q\mathscr{K}(S^{m+1}) = \begin{cases} R & \text{if } q = m; \\ R & \text{if } q = 2m \text{ and } m \text{ is odd}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is a straightforward application of Lemmas 1.12 and 2.1.

Recall that the Hurewicz homomorphism h, a natural transformation between homotopy and homology, is defined by fixing a generator $z_m \in H_m(S^m)$. Let z'_m denote the image of z_m under the inclusion i: $S^m \to \Omega S^{m+1}$ of the m-sphere into its James construction. For $X \in TOP_1$, suppose $[\alpha] \in \pi_m(\Omega X)$ is a homotopy class and let $\tilde{\alpha}$: $S^{m+1} \to X$ be the adjoint of a representative α for this class. Then $(\Omega\tilde{\alpha})\, i = \alpha$, so

$$(\Omega\tilde{\alpha})_*(z'_m) = \alpha_*(z_m) = h[\alpha].$$

Let $z''_m \in H_m(\Omega S^{m+1}; R)$ be the R-reduction of the class z'_m. Define a natural transformation h'': $\pi_m(\Omega X) \to H_m(\Omega X; R)$ by $h''[\alpha] = (\Omega\tilde{\alpha})_*(z''_m)$; clearly $h''[\alpha]$ is the R-reduction of $h[\alpha]$. When $r \leqslant m < r\rho$, we may choose a generator ξ_m of $H_m\mathscr{K}(S^{m+1})$ such that $g_*j_*(\xi_m) = z''_m$, where j: $\mathscr{K}(\ ) \to \mathscr{U}\mathscr{K}(\ )$ is the natural inclusion and g is given by (5).

DEFINITION 3.2. Given $X \in TOP_r$ and $[\alpha] \in \pi_m(\Omega X)$, $r \leqslant m < r\rho$, let

$$\mathfrak{h}[\alpha] = H_m\mathscr{K}(\tilde{\alpha})(\xi_m) \in H_m\mathscr{K}(X).$$

LEMMA 3.3. *$\mathfrak{h}$ is a well-defined natural transformation between $\pi_{<r\rho}(\Omega X)$ and $H_{<r\rho}\mathscr{K}(\ )$.*

Proof. Clear, using Lemma 2.3.

LEMMA 3.4. *For $X \in TOP_r$ and $r \leqslant m < r\rho$, this diagram commutes*

$$\begin{array}{ccc} \pi_m(\Omega X) & \xrightarrow{\ \mathfrak{h}\ } & H_m\mathscr{K}(X) \\ {\scriptstyle h''}\downarrow & & \downarrow{\scriptstyle j_*} \\ H_m(\Omega X; R) & \underset{g_*}{\overset{=}{\longleftarrow}} & H_m\mathscr{U}\mathscr{K}(X) \end{array} \tag{6}$$

where g is given by (5). Furthermore, $\mathfrak{h}$ is a homomorphism of abelian groups and

$$\mathfrak{h}[x, y] = [\mathfrak{h}(x), \mathfrak{h}(y)], \tag{7}$$

the bracket on the left denoting a Samelson product and the bracket on the right denoting a (Pontrjagin) commutator.

Proof. For $[\alpha] \in \pi_m(\Omega X)$, $g_* j_* \mathfrak{h}[\alpha] = g_* j_* H_m\mathscr{K}(\tilde{\alpha})(\xi_m) = g_* H_m\mathscr{U}\mathscr{K}(\tilde{\alpha})(j_*\xi_m) = H_m(\Omega\tilde{\alpha}; R)(g_* j_* \xi_m) = (\Omega\tilde{\alpha})_*(z''_m) = h''[\alpha]$. As to (7), $g_* j_*$ is one-to-one by Lemma 2.4; since h'' is linear and bracket-preserving, so is $\mathfrak{h}$.

We will show that $\mathfrak{h}$ is an R-local isomorphism in a certain range of dimensions. Fix the notation $D = D(r, \rho) = \min(r + 2\rho - 3, r\rho - 1)$.

LEMMA 3.5. *Let $Z \in TOP_1$. Suppose $H_m(Z; R) = M_1 \oplus M_2$ for some R-modules M_1 and M_2. Identify $H_m(K(M_1, m); R)$ with M_1 via the Hurewicz isomorphism h''. There is a map $\varepsilon: Z \to K(M_1, m)$ whose effect on $H_m(\ ; R)$ is precisely the projection of $M_1 \oplus M_2$ onto M_1.*

Proof. This is a standard fact, based upon the natural identification of $H^m(Z; M_1)$ with the group of homotopy classes $[Z; K(M_1, m)]$.

PROPOSITION 3.6. *Let $X \in TOP_r$, and let Y be the generalized Eilenberg–MacLane space $Y = \prod_{n=r}^{D} K(H_n\mathscr{K}(X), n)$. There is a map $\varepsilon: \Omega X \to Y$ inducing an R-local D-equivalence. Furthermore, this diagram commutes for $t \leqslant D$*

$$\begin{array}{ccccc} \pi_t(Y) & \underset{\eta_t}{\overset{=}{\longrightarrow}} & H_t\mathscr{K}(X) & \xrightarrow{\ j_*\ } & H_t\mathscr{U}\mathscr{K}(X) \\ {\scriptstyle h''}\downarrow & & & & \downarrow{\scriptstyle g_*}\ {\scriptstyle =} \\ H_t(Y; R) & & \xleftarrow{\qquad \varepsilon_* \qquad} & & H_t(\Omega X; R) \end{array} \tag{8}$$

where η_t is the obvious identification of $\pi_t(Y)$ with $H_t\mathscr{K}(X)$.

Proof. By Lemma 1.9, we have for $m < r\rho$ a direct sum decomposition

$$H_m(\Omega X; R) = M_1 \oplus M_2,$$

where $M_1 = H_m\mathscr{K}(X)$. Apply Lemma 3.5 to obtain maps $\varepsilon_m\colon \Omega X \to K(H_m\mathscr{K}(X), m)$ for $m \leqslant D$, such that $(\varepsilon_m)_* g_* j_*$ is the Hurewicz isomorphism h''. Multiply the $\{\varepsilon_m\}$ together to obtain $\varepsilon\colon \Omega X \to Y$. Denote by ν_m the inclusion of $K(H_m\mathscr{K}(X), m)$ into Y.

For some particular $t \leqslant D$ consider the diagram

$$\begin{array}{ccccc}
\pi_*(Y) & \xleftarrow{(\nu_t)_\#} & \pi_* K(H_t\mathscr{K}(X), t) & \xrightarrow{=} & H_t\mathscr{K}(X) \\
\downarrow{\scriptstyle h''} & & \approx\downarrow{\scriptstyle h''} & & \downarrow{\scriptstyle j_*} \\
H_*(Y; R) & \xleftarrow{(\nu_t)_*} & H_*(K(H_t\mathscr{K}(X), t); R) & & H_t\mathscr{U}\mathscr{K}(X) \\
& {\scriptstyle \varepsilon_*}\nwarrow & \uparrow{\scriptstyle (\varepsilon_t)_*} & \swarrow{\scriptstyle g_*}\approx & \\
& & H_*(\Omega X; R) & &
\end{array} \tag{9}$$

In (9), the two right-most groups are to be viewed as graded R-modules which are zero except in dimension t. The left square commutes, and by our construction of ε_t so does the right pentagon. The triangle does not commute, but we claim it does so when restricted to $\operatorname{im}(g_* j_*)$. This amounts to the assertion that $(\varepsilon_m)_*(u) = 0$ when $m \neq t$ for $u \in \operatorname{im}(g_* j_*)$. When $m > t$, $(\varepsilon_m)_*(u) = 0$ because $H_t(K(H_m\mathscr{K}(X), m); R) = 0$. When $m < t$, note that any $u \in \operatorname{im}(g_* j_*)$ is primitive (i.e., $(\Delta_*)(u) = \omega(u \otimes 1 + 1 \otimes u)$, where Δ is the diagonal map and ω is the homology external product). But $H_t(K(H_m\mathscr{K}(X), m);\ R)$ contains no non-zero primitives (because $m < t \leqslant D < m + 2\rho - 2$), so $(\varepsilon_m)_*(u) = 0$. It now follows quickly that (8) also commutes, since η_t may be viewed as $(\nu_t)_\#^{-1}$.

Finally, let us verify that $H_t(\varepsilon; R)$ is an isomorphism for $t < D$ and an epimorphism for $t = D$. Since we will primarily utilize well-known information about the homology of Eilenberg–MacLane spaces with field coefficients, we omit most details. Let F denote any field which is an R-module. Lemma 1.12 and the universal coefficient theorem show that

$$\bigoplus_{m=r}^{D} \left(H_m\mathscr{K}(X) \otimes F \oplus \operatorname{Tor}(H_{m-1}\mathscr{K}(X), F)\right) \tag{10}$$

accounts for the primitives of $H_{\leqslant D}(\Omega X; F)$; and (10) also describes all the primitives of $H_{\leqslant D}(Y; F)$. One checks easily that the homomorphism of coalgebras

$$H_*(\varepsilon; F)\colon H_*(\Omega X; F) \to H_*(Y; F) \tag{11}$$

induces a bijection on primitives in dimensions $\leqslant D$. Because both the source and the target of (11) are r-reduced Hopf F-algebras, this is enough to make (11) into an isomorphism in dimensions $\leqslant D$.

Furthermore, the composite

$$\begin{aligned} &H_{D+1}(K(H_r\mathscr{K}(X), r); F) \\ &\xrightarrow{(v_r)_*} H_{D+1}(Y; F) \to \operatorname{cok}(H_{D+1}(\varepsilon; F)) = H_{D+1}(Y, \Omega X; F) \end{aligned}$$

is surjective. Since the top arrow of the diagram

$$\begin{array}{ccc} H_{D+1}(K(H_r\mathscr{K}(X), r)) \otimes F & \longrightarrow & H_{D+1}(K(H_r\mathscr{K}(X), r); F) \\ \downarrow & & \downarrow \\ H_{D+1}(Y, \Omega X) \otimes F & \longrightarrow & H_{D+1}(Y, \Omega X; F) \end{array}$$

is also surjective, we see that the bottom arrow must be surjective too. This fact, together with the already-established vanishing of $H_{\leqslant D}(Y, \Omega X; F)$ for any R-module field F, shows that $H_{\leqslant D}(Y, \Omega X) \otimes R = 0$. This is the desired conclusion.

THEOREM 3.7. *Let* $R \subseteq \mathbb{Q}$, $r \geqslant 1$, $D = \min(r + 2p - 3, rp - 1)$. *The Hurewicz homomorphism*

$$\mathfrak{h}: \pi_*(\Omega X) \to H_*\mathscr{K}(X),$$

defined in dimensions $< rp$, *is a natural bracket-preserving homomorphism. It is an R-local isomorphism in dimensions below D and an R-local epimorphism in dimension D.*

Proof. In view of Lemmas 3.3 and 3.4 and Proposition 3.6, it suffices to verify that $\mathfrak{h}$ coincides with $\eta_t \varepsilon_\#$ in each dimension $t \leqslant D$. Consider the diagram

$$\begin{array}{ccccc} \pi_t(\Omega X) & & \xrightarrow{\varepsilon_\#} & & \pi_t(Y) \\ & \searrow^{\mathfrak{h}} & & \swarrow^{=}_{\eta_t} & \\ \downarrow h'' & & H_t\mathscr{K}(X) & & \downarrow h'' \\ & & \swarrow_{g_* j_*} & & \\ H_t(\Omega X; R) & & \xrightarrow[\varepsilon_*]{} & & H_t(Y; R) \end{array}$$

The quadrilateral commutes by Proposition 3.6, and the outer square expresses a familiar naturality relation. The left-hand triangle commutes by Lemma 3.4. Deduce that $h''(\eta_t)^{-1}\mathfrak{h} = h''\varepsilon_\#$. But h'' is monomorphic when applied to an R-local generalized Eilenberg–MacLane space, so $(\eta_t)^{-1}\mathfrak{h} = \varepsilon_{\#}$, as desired.

We may recover at once the classic Milnor–Moore theorem.

COROLLARY 3.8. *Suppose $R = \mathbb{Q}$ and $r = 1$. For $X \in TOP_1$, there is a natural bracket-preserving rational isomorphism*

$$\mathfrak{h}: \pi_*(\Omega X) \to H_*\mathscr{K}(X),$$

as well as a natural isomorphism

$$\mathscr{U}H_*\mathscr{K}(X) \approx H_*(\mathscr{U}\mathscr{K}(X)) \xrightarrow[g_*]{=} H_*(\Omega X; \mathbb{Q}).$$

Furthermore, the composite

$$\pi_*(\Omega X) \xrightarrow{\mathfrak{h}} H_*\mathscr{K}(X) \xrightarrow{j_*} H_*\mathscr{U}\mathscr{K}(X) \xrightarrow[g_*]{=} H_*(\Omega X; \mathbb{Q})$$

coincides with the Hurewicz homomorphism h''.

4. PRODUCTS

In low dimensions, there is a natural chain equivalence of $\mathscr{K}(X \times Y)$ with $\mathscr{K}(X) \oplus \mathscr{K}(Y)$. In this section we will prove this fact.

We assume R and r fixed, as usual. For X, $Y \in TOP_r$ let $X \xrightarrow{i_1} X \times Y \xrightarrow{p_1} X$ and $Y \xrightarrow{i_2} X \times Y \xrightarrow{p_2} Y$ be the inclusions and projections. Define a natural transformation of dgL's

$$\lambda = \lambda_{XY}: \mathscr{K}(X \times Y) \to \mathscr{K}(X) \oplus \mathscr{K}(Y)$$

by $\lambda = (\mathscr{K}(p_1), \mathscr{K}(p_2))$.

LEMMA 4.1. *If $R \subseteq \mathbb{Q}$, let $\mathfrak{h}$ be as in Definition 3.2. The diagram*

$$\begin{array}{ccc} \pi_*\Omega(X \times Y) & \xrightarrow[((p_1)_*, (p_2)_*)]{\approx} & \pi_*(\Omega X) \oplus \pi_*(\Omega Y) \\ \downarrow \mathfrak{h} & & \downarrow \mathfrak{h} \oplus \mathfrak{h} \\ H_*\mathscr{K}(X \times Y) & \xrightarrow{\lambda} & H_*\mathscr{K}(X) \oplus H_*\mathscr{K}(Y) \end{array}$$

commutes in dimensions $< r\rho$.

Proof. The naturality of $\mathfrak{h}$.

Let us relate λ to the Eilenberg–Zilber equivalence

$$\eta\colon CU_*\Omega(X\times Y)\to CU_*(\Omega X)\otimes CU_*(\Omega Y).$$

Recall that $\mathscr{K}$ is defined using the diagonal approximation ϕ. By [2, Sect. 7] there is a natural chain equivalence $\zeta\colon CU_*\Omega(X\times Y)\to CU_*(\Omega X)\otimes CU_*(\Omega Y)$, homotopic to η, for which $\phi=\zeta\circ CU_*(\Omega\Delta)$. The equivalence $\zeta=\zeta_{XY}$ is strictly coassociative and cocommutative below dimension $r\rho$. That is, the diagram

$$\begin{array}{ccc} CU_*(\Omega(X\times Y\times Z)) & \xrightarrow{\zeta_{X\times Y,Z}} & CU_*(\Omega(X\times Y))\otimes CU_*(\Omega Z) \\ \downarrow{\scriptstyle \zeta_{X,Y\times Z}} & & \downarrow{\scriptstyle \zeta_{XY}\otimes 1} \\ CU_*(\Omega X)\otimes CU_*(\Omega(Y\times Z)) & \xrightarrow{1\otimes\zeta_{YZ}} & CU_*(\Omega X)\otimes CU_*(\Omega Y)\otimes CU_*(\Omega Z) \end{array} \tag{12}$$

commutes in dimensions $<r\rho$; when $T(x,y)=(y,x)$ and $\tau(a\otimes b)=(-1)^{|a||b|}b\otimes a$, so does

$$\begin{array}{ccc} CU_*(\Omega(X\times Y)) & \xrightarrow{CU_*(\Omega T)} & CU_*(\Omega(Y\times X)) \\ \downarrow{\scriptstyle \zeta_{XY}} & & \downarrow{\scriptstyle \zeta_{YX}} \\ CU_*(\Omega X)\otimes CU_*(\Omega Y) & \xrightarrow{\tau} & CU_*(\Omega Y)\otimes CU_*(\Omega X) \end{array} \tag{13}$$

Furthermore, if $\varepsilon\colon A\to R$ denotes the augmentation for a connected (differential) graded algebra,

$$(1\otimes\varepsilon)\,\zeta=CU_*(\Omega p_1)\colon CU_*(\Omega(X\times Y))\to CU_*(\Omega X)$$

and

$$(\varepsilon\otimes 1)\,\zeta=CU_*(\Omega p_2)\colon CU_*(\Omega(X\times Y))\to CU_*(\Omega Y).$$

Since ζ is (naturally) homotopic to η, they induce the same isomorphism on homology.

LEMMA 4.2. *For $n<r\rho$, $\zeta(\mathscr{K}_n(X\times Y))\subseteq\mathscr{K}_n(X)\otimes 1+1\otimes\mathscr{K}_n(Y)$.*

Proof. By (12) and (13), we deduce the equation

$$(1\otimes\tau\otimes 1)\,(\zeta_{XY}\otimes\zeta_{XY})\,\phi_{\Omega(X\times Y)}=(\phi_{\Omega X}\otimes\phi_{\Omega Y})\,\zeta_{XY}\colon CU_*(\Omega(X\times Y))$$
$$\to CU_*(\Omega X)\otimes CU_*(\Omega X)\otimes CU_*(\Omega Y)\otimes CU_*(\Omega Y). \tag{14}$$

So let $u \in \mathscr{K}_n(X \times Y)$, i.e., $u \in CU_n\Omega(X \times Y)$ and $\phi(u) = u \otimes 1 + 1 \otimes u$. Write $\zeta(u) = \sum a_i \otimes b_i$, $\{a_i \otimes b_i\}$ being linearly independent, and note that

$$(1 \otimes \tau \otimes 1)(\zeta \otimes \zeta)\,\phi(u) = \sum a_i \otimes 1 \otimes b_i \otimes 1 + \sum 1 \otimes a_i \otimes 1 \otimes b_i. \tag{15}$$

However, if $\zeta(u)$ contains any term $a_i \otimes b_i$ with $|a_i| > 0$ and $|b_i| > 0$, then $(\phi \otimes \phi)\,\zeta(u)$ contains $a_i \otimes 1 \otimes 1 \otimes b_i$, in contrast with (15). Thus $\zeta(u) = a \otimes 1 + 1 \otimes b$ for some a and b. If either a or b is non-primitive, then $(\phi \otimes \phi)\,\zeta(u)$ cannot look like the right-hand side of (15). Thus $a \in \mathscr{K}_n(X)$ and $b \in \mathscr{K}_n(Y)$, as desired.

LEMMA 4.3. *View $\mathscr{K}(X)$ as $\mathscr{K}(X) \otimes 1 \subseteq CU_*(\Omega X) \otimes 1$ and $\mathscr{K}(Y)$ as $1 \otimes \mathscr{K}(Y) \subseteq 1 \otimes CU_*(\Omega Y)$. Then $\zeta_{XY}(u) = \lambda(u)$ for $u \in \mathscr{K}_{<r\rho}(X \times Y)$.*

Proof. Suppose $u \in \mathscr{K}_n(X \times Y)$, $n < r\rho$. We already know that $\zeta(u) \in \mathscr{K}_n(X) \oplus \mathscr{K}_n(Y)$. But we also know that the component of $\zeta(u)$ lying in $CU_*(\Omega X) \otimes 1$ equals $(1 \otimes \varepsilon)\,\zeta(u) = CU_n(\Omega p_1)(u) = \mathscr{K}(p_1)(u)$, and likewise the component of $\zeta(u)$ lying in $1 \otimes CU_n(\Omega Y)$ equals $\mathscr{K}(p_2)(u)$.

LEMMA 4.4. *Let g be as in (5), and let $j\colon \mathscr{K}(\ ) \to \mathscr{U}\mathscr{K}(\ )$ be the natural inclusion. This diagram commutes in dimensions $< r\rho$:*

$$\begin{array}{ccc}
\mathscr{K}(X \times Y) & \xrightarrow{\lambda} & \mathscr{K}(X) \oplus \mathscr{K}(Y) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} \\
\mathscr{U}\mathscr{K}(X \times Y) & \xrightarrow{\mathscr{U}\lambda} & \mathscr{U}\mathscr{K}(X) \otimes \mathscr{U}\mathscr{K}(Y) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g \otimes g} \\
CU_*(\Omega(X \times Y)) & \xrightarrow{\zeta} & CU_*(\Omega X) \otimes CU_*(\Omega Y)
\end{array}$$

Proof. Lemma 4.3 tells us that $(g \otimes g)\,j\lambda = \zeta g j$. Since $\mathscr{K}(X \times Y)$ generates $\mathscr{U}\mathscr{K}(X \times Y)$ as an R-algebra, the lower square also commutes.

PROPOSITION 4.5. *λ is an $r\rho$-quism.*

Proof. To begin with, λ has a right inverse (namely, $\mathscr{K}(i_1) + \mathscr{K}(i_2)$), so λ is surjective on homology in all dimensions. Also, ζ is a quism, and in dimensions below $r\rho$, g and $g \otimes g$ are isomorphisms. By Proposition 1.11, λ induces an homology isomorphism below dimension $r\rho$.

5. The Model $\mathscr{L}$

We consider next the dgL model $\mathscr{L}(\ )$, introduced in [2, Sect. 8], for r-connected CW complexes of dimension $\leqslant r\rho$. We demonstrate that $H_n\mathscr{L}$ and $H_n\mathscr{K}$ coincide for $n < r\rho$. Combining this with Theorem 3.7, we obtain the promised R-local version of the Milnor–Moore theorem.

Let CW_r^m be the full subcategory of TOP_r consisting of m-dimensional CW complexes having a trivial r-skeleton. In [2, Sect. 8] the Adams–Hilton model for a space $X \in CW_r^{r\rho}$ is used in order to define a Lie R-algebra model $\mathscr{L}(X)$ for X. We will not repeat the properties of $\mathscr{L}$ here, except to mention the existence of a quism $\theta = \theta_X : \mathscr{U}\mathscr{L}(X) \to CU_*(\Omega X)$ which is natural up to homotopy.

LEMMA 5.1. *Let $X \in CW_r^{r\rho}$. There is a dgL homomorphism $\sigma = \sigma_X$: $\mathscr{L}(X) \to \mathscr{K}(X)$ which induces an isomorphism on homology in dimensions below $r\rho$. Also, $g_*(\mathscr{U}\sigma_X)_* = (\theta_X)_* : H_*\mathscr{U}\mathscr{L}(X) \to H_*(\Omega X; R)$, where g is given by (5). For any map $f: X \to Y$, this diagram commutes up to dgL homotopy:*

$$\begin{array}{ccc} \mathscr{L}(X) & \xrightarrow{\mathscr{L}(f)} & \mathscr{L}(Y) \\ \downarrow{\scriptstyle \sigma_X} & & \downarrow{\scriptstyle \sigma_Y} \\ \mathscr{K}(X) & \xrightarrow{\mathscr{K}(f)} & \mathscr{K}(Y) \end{array} \tag{16}$$

Proof. Because $\mathscr{L}(X)$ is generated in dimensions below $r\rho$, the dga homomorphism θ may be factored as $\theta = g\theta'$, where $\theta' : \mathscr{U}\mathscr{L}(X) \to \mathscr{U}\mathscr{K}(X)$. It is easy to see that θ and θ' are homomorphisms of Hopf algebras up to homotopy [2]. Since $\mathscr{L}(X)$ is also free, we may apply [2, Theorem 6.3] to obtain a dgL homomorphism $\sigma: \mathscr{L}(X) \to \mathscr{K}(X)$ for which $\mathscr{U}\sigma \simeq \theta'$. By Proposition 1.11, σ induces an isomorphism on $H_{<r\rho}(\ )$ because θ' does. We also have $g(\mathscr{U}\sigma) \simeq g\theta' = \theta$, so $g_*(\mathscr{U}\sigma)_* = \theta_*$. As to (16), note that

$$g\mathscr{U}(\sigma_Y \mathscr{L}(f)) \simeq \theta_Y \mathscr{U}\mathscr{L}(f) \simeq CU_*(\Omega f)\,\theta_X \simeq g\mathscr{U}(\mathscr{K}(f)\,\sigma_X).$$

Since g is an isomorphism below $r\rho$ and a surjection at $r\rho$, and $\mathscr{L}(X)$ is generated in dimensions below $r\rho$, we have $\mathscr{U}(\sigma_Y \mathscr{L}(f)) \simeq \mathscr{U}(\mathscr{K}(f)\,\sigma_X)$. Now apply the Lemaire–Aubry theorem ([3] or [2, Prop. 3.3]).

LEMMA 5.2. *Suppose $R \subseteq \mathbb{Q}$. For $r \leqslant m < r\rho$, let ξ'_m denote the generator of $H_m\mathscr{L}(S^{m+1})$ which is sent via σ_* to $\xi_m \in H_m\mathscr{K}(S^{m+1})$. For $X \in CW_r^{r\rho}$, define $\mathfrak{h}' : \pi_m(\Omega X) \to H_m\mathscr{L}(X)$ by $\mathfrak{h}'[\alpha] = \mathscr{L}(\tilde{\alpha})(\xi'_m)$. Then $\sigma_*\mathfrak{h}' = \mathfrak{h}$. In* *particular,* $\mathfrak{h}'$ *is natural, bracket-preserving, R-locally isomorphic below dimension* $D = \min(r + 2p - 3, rp - 1)$, *and R-locally epimorphic in dimension D.*

Proof. Straightforward.

We now have all the necessary pieces for

THEOREM 5.3 (R-local Milnor–Moore theorem). *Suppose* $R \subseteq \mathbb{Q}$, *and let* $X \in CW_r^{rp}$. *The dgL* $\mathscr{L}(X)$ *enjoys the following properties:*

(a) *There is a natural bracket-preserving homomorphism*

$$\mathfrak{h}': \pi_*(\Omega X) \to H_*\mathscr{L}(X),$$

defined in dimensions $< rp$, *which is an R-local isomorphism in dimensions below* $D = \min(r + 2p - 3, rp - 1)$ *and an R-local epimorphism in dimension D.*

(b) *There is a natural isomorphism*

$$(\theta_X)_*: H_*\mathscr{U}\mathscr{L}(X) \xrightarrow{\approx} H_*(\Omega X; R),$$

valid in all dimensions.

(c) *The composite*

$$\pi_*(\Omega X) \xrightarrow{\mathfrak{h}'} H_*\mathscr{L}(X) \xrightarrow{j_*} H_*\mathscr{U}\mathscr{L}(X) \xrightarrow{\theta_*} H_*(\Omega X; R)$$

coincides with the usual Hurewicz homomorphism h″, in dimensions below rp.

6. ρ-ELLIPTIC SPACES

We conclude the paper with an interesting application of the R-local Milnor–Moore theorem. We show that the loop space on certain finite complexes decomposes into a finite product of well-known factors.

By [2, Theorem 9.1], $X \in CW_r^{rp}$ implies that $H_*(\Omega X; \mathbb{Z}_p)$ is a primitively generated Hopf algebra. By analogy with the situation for rational coefficients, we call X *ρ-elliptic* if $H_*(\Omega X; \mathbb{Z}_p)$ is, as a Hopf algebra, the enveloping algebra of a finite dimensional Lie $\mathbb{Z}_p$-algebra.

THEOREM 6.1. *Let* $R = \mathbb{Z}_{(p)}$, *the integers localized at a prime* $p < \infty$. *Let* $X \in CW_r^{rp}$ *be ρ-elliptic, and write* $H_*(\Omega X; \mathbb{Z}_p) = \mathscr{U}L$ *as Hopf algebras. Write* $L = \mathrm{Span}_{\mathbb{Z}_p}(x_1, ..., x_n)$, *and suppose that each* $|x_i| \leqslant D = \min(r + 2p - 3, rp - 1)$. *Suppose further that in the homology Bockstein spectral sequence for*

ΩX, no non-zero differentials originate on any of the even-dimensional x_i's. Then ΩX has the R-local homotopy type of a finite product. Each factor in the product is either an odd sphere, the loop space on an odd sphere, or the fiber of the degree ρ^k self-map (for some k) on an odd sphere.

Proof. Using Lemma 1.12, $\mathscr{U}L = H_*(\Omega X; \mathbb{Z}_\rho) = H_*(\mathscr{U}\mathscr{L}(X) \otimes \mathbb{Z}_\rho)$, so L coincides with $H_*(\mathscr{L}(X) \otimes \mathbb{Z}_\rho)$ in dimensions below $r\rho$. In view of Lemma 1.9, $(\mathscr{L}(X) \otimes \mathbb{Z}_\rho)_{\leqslant r\rho}$ is a direct summand of $(\mathscr{U}\mathscr{L}(X) \otimes \mathbb{Z}_\rho)_{\leqslant r\rho}$ as differential $\mathbb{Z}_\rho$-modules, hence the Bockstein spectral sequence (henceforth BSS) for ΩX has as a direct summand the BSS for $\mathscr{L}(X) \otimes \mathbb{Z}_\rho$. The E^1 term of the latter BSS is precisely L. This means that L has a $\mathbb{Z}_\rho$-basis consisting of some elements which survive the BSS and others which come in pairs connected by a Bockstein differential. Without loss of generality we may assume that $\{x_1, ..., x_n\}$ satisfies

$$x_1 = \beta^{k_1}(x_2), \qquad x_3 = \beta^{k_2}(x_4), ..., \qquad x_{2s-1} = \beta^{k_s}(x_{2s}),$$

where s is the number of such pairs, β^k denotes the kth Bockstein differential, and $x_{2s+1}, ..., x_n$ survive the BSS.

We will now realize each x_i as an Hurewicz image. Consider separately the cases $2s < i$ with $|x_i|$ odd, $2s < i$ with $|x_i|$ even, and $i \leqslant 2s$. When $2s < i$, x_i survives the BSS and represents the generator of an infinite cyclic summand of $H_{\leqslant D}\mathscr{L}(X)$. By Theorem 5.3, x_i is the mod ρ reduction of the Hurewicz image of some homotopy class $\gamma_i\colon S^t \to \Omega X$, where $t = |x_i|$. If t is odd, let $\hat{\gamma}_i = \gamma_i$, let $\hat{S}_i$ denote S^t, and denote the generator of $H_t(\hat{S}_i; \mathbb{Z}_\rho)$ by z_i. Of course $(\hat{\gamma}_i)_*(z_i) = x_i$. If t is even, put $\hat{S}_i = \Omega S^{t+1}$(= James construction on S^t), and let $\hat{\gamma}_i\colon \hat{S}_i \to \Omega X$ denote the canonical extension of γ_i to an H-map. Letting z_i denote the generator of $H_t(\hat{S}_i; \mathbb{Z}_\rho)$, we know that $H_*(\hat{S}_i; \mathbb{Z}_\rho)$ is spanned by $\{z_i^m\}_{m \geqslant 0}$ and that $(\hat{\gamma}_i)_*(z_i^m) = x_i^m$.

For $1 \leqslant j \leqslant s$, x_{2j-1} and x_{2j} survive only until the (k_j)th stage of the BSS. So x_{2j-1} is the mod ρ reduction of the generator of a cyclic summand of $H_{<D}\mathscr{L}(X)$ of order ρ^{k_j}. By Theorem 5.3 there is a homotopy class of order ρ^{k_j} whose Hurewicz image is this generator. The homotopy class extends over the mod ρ^{k_j} Moore space,

$$\gamma_j\colon P^t(\rho^{k_j}) \to \Omega X,$$

where $t = |x_{2j}|$. Note that $|x_{2j-1}|$ and $|x_{2j}|$ span the homology image (coefficients $\mathbb{Z}_\rho$) of γ_j.

According to our hypotheses, t is odd. By [4], γ_j may be extended to a map

$$\hat{\gamma}_j\colon \hat{S}_i \to \Omega X,$$

where $\hat{S}_j = S^t\{\rho^{k_j}\}$ is the homotopy-theoretic fiber of the Brouwer degree ρ^{k_j} self-map on S^t. Observe that

$$H_*(\hat{S}_j; \mathbb{Z}_\rho) = \text{Span } \mathbb{Z}_\rho\{z_{2j-1}^m, z_{2j-1}^m z_{2j}\}_{m \geqslant 0},$$

where $|z_{2j}| = t$ and $|z_{2j-1}| = t-1$ and $\beta^{k_j}(z_{2j}) = z_{2j-1}$. The map $\hat{\gamma}_j$ is not typically an H-map, but it may be selected so as to send z_{2j-1}^m to x_{2j-1}^m and $z_{2j-1}^m z_{2j}$ to $x_{2j-1}^m x_{2j}$.

Consider the space

$$A = \hat{S}_1 \times \cdots \times \hat{S}_s \times \hat{S}_{2s+1} \times \cdots \times \hat{S}_n. \tag{17}$$

Its mod ρ homology may be described as the $\mathbb{Z}_\rho$-span of the elements

$$\{z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}\}$$

as $e = (e_1, \ldots e_n)$ runs through the set T consisting of all length n sequences of nonnegative integers satisfying $e_i \leqslant 1$ when $|z_i|$ is odd. By applying the Poincaré–Birkhoff–Witt theorem to $H_*(\Omega X; \mathbb{Z}_\rho) = \mathscr{U}L$, we may write this graded algebra as the $\mathbb{Z}_\rho$-span of the products (in this order)

$$\{x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}\}$$

as $\{e_1, \ldots e_n\}$ runs through T.

Define $\gamma: A \to \Omega X$ to be the composite

$$A \xrightarrow{\hat{\gamma}_1 \times \hat{\gamma}_2 \times \cdots \times \hat{\gamma}_s \times \hat{\gamma}_{2s+1} \times \cdots \times \hat{\gamma}_n} \Omega X \times \cdots \times \Omega X \xrightarrow{\nu} \Omega X,$$

where ν denotes the H-space multiplication. Clearly, $(z_1^{e_1} \cdots z_n^{e_n})$ is sent via γ_* to $x_1^{e_1} \cdots x_n^{e_n}$. Thus γ induces an isomorphism on mod ρ homology. Since A and ΩX are both H-spaces with locally finite homology, γ is an R-local equivalence [9]. Thus (17) is our desired product decomposition for ΩX.

EXAMPLE. Fix $t \geqslant 2$, $m \geqslant 2$, $k \geqslant 1$. Let V denote the $(2mt-1)$-skeleton of a minimal $\mathbb{Z}_{(\rho)}$-local CW decomposition for $\Omega S^{2t+1}\{\rho^k\}$. Thus

$$V = (pt) \cup e^{2t-1} \cup e^{2t} \cup e^{4t-1} \cup e^{4t} \cup \cdots \cup e^{2mt-1} \in CW_{2t-2}^{2mt-1}$$

when $m < \rho$. For $m < \rho$ we have

$$H^*(V; \mathbb{Z}_\rho) = \Lambda(a_{2t-1}) \otimes \mathbb{Z}_\rho[b_{2t}]/(b_{2t}^m),$$

subscripts signifying dimension. The Eilenberg–Moore spectal sequence yields

$$H_*(\Omega V; \mathbb{Z}_\rho) = \mathbb{Z}_\rho[x_{2t-2}] \otimes \Lambda(y_{2t-1}) \otimes \mathbb{Z}_\rho[z_{2mt-2}]$$

with $\beta^k(y_{2t-1}) = x_{2t-2}$. Thus $H_*(\Omega V; \mathbb{Z}_p) = \mathscr{U}L$, where L is the abelian Lie algebra on $\{x_{2t-2}, y_{2t-1}, z_{2mt-2}\}$. As long as

$$p \geqslant t(m-1)+2$$

we have $2mt-2 < (2t-2)+2p-3 = D$, so Theorem 6.1 applies. Deduce that, localized at p,

$$\Omega V \approx S^{2t-1}\{p^k\} \times \Omega S^{2mt-1}.$$

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