Tight bounds for the generalized Marcum $Q$-function

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1. Introduction and preliminary results

For $\nu$ unrestricted real number let $I_\nu$ be the modified Bessel function [29, p. 77] of the first kind of order $\nu$, and let $b \mapsto Q_\nu(a,b)$ be the generalized Marcum $Q$-function, defined by

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu e^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at) \, dt,$$

where $b \geq 0$ and $a, \nu > 0$. When $\nu = 1$, the function

$$b \mapsto Q_1(a,b) = \int_b^\infty t e^{-\frac{t^2+a^2}{2}} I_0(at) \, dt$$

is known in literature as Marcum $Q$-function. The Marcum $Q$-function and the generalized Marcum $Q$-function, defined above, are widely used in radar communications and particularly in the study of target detection by pulsed radars with single or multiple observations [15,16] and have important applications in error performance analysis of multichannel dealing with partially coherent, differentially coherent, and non-coherent detections over fading channels [15,19,22]. Since, the precise computation of the Marcum $Q$-function, generalized Marcum $Q$-function, respectively is quite difficult, in the last few decades many engineers, statisticians and mathematicians established approximation formulas and bounds for the function $b \mapsto Q_\nu(a,b)$. For more details on approximations, lower and upper bounds we refer to the most recent papers...
[2,5,7,9,12,13,20,21,27,28] and to the references therein. In this field \( v \) is the number of independent samples of the output of a square-law detector, and hence in most of the papers the authors deduce lower and upper bounds for the generalized Marcum \( Q \)-function with order \( v \) integer. However, as in [5], in our analysis of this paper \( v \) is not necessarily an integer number.

An important contribution to the subject, concerning lower and upper bounds for the Marcum \( Q \)-function \( b \mapsto Q_1(a,b) \), is the publication of Corazza and Ferrari [9], which was the starting point of our paper [5]. In [5] we have shown that all results of Corazza and Ferrari from [9] can be extended to the generalized Marcum \( Q \)-function with \( v \) real order. Recently, Wang [28] has improved the results from [9], and motivated by the results of Wang, in this paper we extend all the results from [28]. Moreover, in both cases \( b \geq a \) and \( b < a \) we improve Wang's upper bounds and we give the best possible upper bound for \( Q_v(a,b) \). This paper, which is the direct continuation of [5], is organized as follows: in this section we present some preliminary results which will be useful to deduce new lower and upper bounds for the generalized Marcum \( Q \)-function. Our main results of this section are some monotonicity properties of some functions which involve the modified Bessel functions of the first kind and the key tools are some classical results of Gronwall [11], Simpson and Spector [23], which have been used in wave mechanics and finite elasticity. In Section 2, as we mentioned above, we show that all results of Wang [28] can be extended to the generalized Marcum \( Q \)-function with \( v \) real order. In the case of \( v = n \) positive integer we deduce closed forms of the lower and upper bounds deduced in the general case, and these bounds can be applied without any difficulty to approximate the generalized Marcum \( Q \)-function of integer order. These results complement and improve the results established in [5] (see Remarks 3 and 5 for further details). Our notation is standard and the basic ideas are taken from Wang's paper [28]. However, we found that there are some incompleteness on the above mentioned paper [28], and in the next section we also clarify these things in order to have complete rigorous proofs. More precisely, Eqs. (5) and (16) in [28] are stated without proof, even if some computer generated pictures appears, which suggest the validity of Eqs. (5) and (16). But, computer generated pictures can be misleading and rigorous mathematical proof is required. In part b of Lemma 1 below we prove a somewhat stronger version of Eqs. (5) and (16) (for further comments see also Remark 4). Another shortcoming in [28] is that the proofs of Eqs. (10) and (22) are missing too. In part c of Lemma 1 we present the preliminary result from which the general version of the above equations will be deduced (see also Remark 4 for further comments).

It is also worth mentioning that the generalized Marcum \( Q \)-function has an important interpretation in probability theory, namely that (as a function of \( b \)) is the complement (with respect to unity) of the cumulative distribution function of the non-central chi distribution with \( 2v \) degrees of freedom. We recall that in probability theory and in economic theory the complement (with respect to unity) of a cumulative distribution function is called a survival (or a reliability) function. More precisely, as we pointed out in [26], the generalized Marcum \( Q \)-function, i.e. \( b \mapsto Q_v(a,b) \) is exactly the reliability function of the non-central chi distribution with \( 2v \) degrees of freedom (where \( v \) is not necessarily an integer and non-centrality parameter \( a \) (see also [5] for further details). Thus, for all \( b \geq 0 \) and \( a,v > 0 \) the function \( b \mapsto Q_v(a,b) \) can be rewritten as

\[
Q_v(a,b) = 1 - \frac{1}{a^{v+1}} \int_0^b t^v e^{-\frac{t^2 + a^2}{2}} I_{v-1}(at) \, dt.
\]

The following technical lemma, which may be of independent interest, is one of the crucial facts in the proof of our main results of Section 2. We note that part b of Lemma 1 below improves part b of Lemma 1 [5].

**Lemma 1.** The following assertions are true:

a. The function \( f_v : (0, \infty) \to \mathbb{R} \), defined by

\[
f_v(x) = e^x \left[ \frac{I_v(x)}{I_{v+1}(x)} - 1 \right],
\]

is decreasing on \((0, \rho_v] \) and increasing on \([ \rho_v, \infty) \) for all \( v \geq 0 \), where \( \rho_v \) is the unique simple positive root of the equation

\[
(x + 2v + 1)I_{v+1}(x) = xI_v(x).
\]

b. The function \( g_v : (0, \infty) \to \mathbb{R} \), defined by

\[
g_v(x) = x^{-v} I_v(x),
\]

is decreasing for all \( v \geq 0 \), where \( \lambda_v = f_v(\rho_v) \) is the largest positive constant (depending on \( v \)) for which the function \( g_v \) is decreasing on \((0, \infty) \).

c. The function \( h_v : (0, \infty) \to \mathbb{R} \), defined by

\[
h_v(x) = \frac{x^{v+1} I_v(x)}{e^x - e^{-x}},
\]

is strictly increasing for all \( v \geq 0 \).
**Proof.** a. Using the recurrence relations [29, p. 79]

\[ x[I_v(x)]' = xI_{v+1}(x) + vI_v(x) \]

and

\[ x[I_{v+1}(x)]' = xI_v(x) - (v + 1)I_{v+1}(x), \]

we have

\[ f'_v(x) = e^xI_v(x) \frac{x}{xI_{v+1}(x)} \left[ x + 2v + 1 - \frac{xI_v(x)}{I_{v+1}(x)} \right]. \]

Consider the function \( v : (0, \infty) \to \mathbb{R} \), defined by

\[ v(x) = xI_v(x)/I_{v+1}(x), \]

which is of special interest in finite elasticity [23,24]. Using the asymptotic formula [1, p. 377]

\[ I_v(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left[ 1 - \frac{4v^2 - 1}{1!(8x)} + \frac{(4v^2 - 1)(4v^2 - 9)}{2!(8x)^2} - \cdots \right], \tag{3} \]

which holds for large values of \( x \) and for fixed \( v \geq 0 \), one has

\[ \lim_{x \to \infty} v(x) = \infty, \quad \lim_{x \to \infty} \frac{v(x)}{x} = 1 \quad \text{and} \quad \lim_{x \to \infty} \left[ v(x) - x \right] = v + 1/2. \]

On the other hand we have

\[ \lim_{x \to 0} v(x) = 2(v + 1). \]

Thus we have that the graph of the function \( v \) has the skew asymptote \( y = x + v + 1/2 \), which is parallel with the straight line \( y = x + 2v + 1 \). Due to Simpson and Spector [23] it is known that \( v \) is strictly increasing and strictly convex. Since \( 2v + 1 > v + 1/2 \), geometrically this means that the equation \( f'_v(x) = 0 \), i.e. \( x + 2v + 1 = v(x) \) has exactly one positive solution. If we would accept the situation when there are two roots, i.e. two intersection points, then this would contradicts the fact that the graph of \( v \), has the skew asymptote \( y = x + v + 1/2 \). More precisely, suppose that the equation \( x + 2v + 1 = v(x) \) has at least two solutions. Without loss of generality, let us denote the first two roots with \( x_1 \) and \( x_2 \), i.e. for which we have \( v(x_1) = x_1 + 2v + 1 \) and \( v(x_2) = x_2 + 2v + 1 \). Using Taylor’s formula, for all \( x_1 < x_2 \) there exists \( x_3 \in (x_1, x_2) \) such that

\[ v(x_1) = v(x_2) + (x_1 - x_2)v'(x_2) + \frac{1}{2!}(x_1 - x_2)^2v''(x_3) > v(x_2) + (x_1 - x_2)v'(x_2) = x_2 + 2v + 1 + (x_1 - x_2)v'(x_2). \]

From this we have \( v'(x_2) > 1 \). Now, using again the Taylor formula for arbitrary \( x > x_2 \), we obtain that there exists \( x_4 \in (x_2, x) \) such that

\[ v(x) = v(x_2) + (x - x_2)v'(x_2) + \frac{1}{2!}(x - x_2)^2v''(x_4) > v(x_2) + (x - x_2) = x + 2v + 1. \]

But this contradicts the fact that the straight line \( y = x + v + 1/2 \) is the skew asymptote of \( v \). Hence, indeed the equation \( x + 2v + 1 = v(x) \) has exactly one positive root, which we denote with \( \rho_v \).

Now we prove that the function \( f_v \) is decreasing on \((0, \rho_v)\). For this first let us rewrite the derivative of \( f_v \) as follows

\[ f'_v(x) = \frac{e^xI_v(x)}{x[I_{v+1}(x)]^2} \left[ (x + 2v + 1)I_{v+1}(x) - xI_v(x) \right] = \frac{e^xI_v(x)}{x[I_{v+1}(x)]^2} \left[ \sum_{n \geq 0} \frac{(x + 2v + 1)(x/2)^{2n+v+1}}{n!(n + v + 2)} - \sum_{n \geq 0} \frac{x(x/2)^{2n+v}}{n!(n + v + 1)} \right] = \frac{e^xI_v(x)}{x[I_{v+1}(x)]^2} \sum_{n \geq 0} \frac{x^{2n+v+1}}{2^{2n+v}n!(n + v + 1)} \left[ \frac{x + 2v + 1}{2(n + v + 1)} - 1 \right]. \]

Using this series representation we obtain that \( f'_v(x) < 0 \) for all \( x \in (0, 1) \), i.e. the function \( f_v \) is strictly decreasing on \((0, 1)\). Since the equation \( f'_v(x) = 0 \) has exactly one solution it is clear that the function \( f_v \) has exactly one global minimum in \( \rho_v \).
and \( \rho_v > 1 \). Now suppose that there exists an interval \([x_5, x_6] \subset (0, \rho_v)\) on which the function \( f_v \) is increasing. Since \( f_v \) is continuous it follows that there exists an \( x_7 \in (x_5, \rho_v) \) such that \( f_v(x_5) = f_v(x_7) \). But, from Rolle's theorem this implies that there exists an \( x_8 \in (x_5, x_7) \) such that \( f'_v(x_8) = 0 \), which contradicts the fact that the equation \( f'_v(x) = 0 \) has exactly one solution. Hence there is no subinterval of \((0, \rho_v)\) on which the function \( f_v \) is increasing, and consequently indeed \( f_v \) is decreasing on the whole interval \([0, \rho_v]\).

Since
\[
\lim_{x \to +\infty} \left[ v_v(x) - x \right] = \nu + \frac{1}{2} \quad \text{and} \quad \lim_{x \to +\infty} xe^{-x} = 0,
\]
we have
\[
\lim_{x \to +\infty} f_v(x) = \lim_{x \to +\infty} \frac{v_v(x) - x}{xe^{-x}} = \infty.
\]

On the other hand recall that the function \( f_v \) has exactly one global minimum in \( \rho_v \) and it is decreasing on \((0, \rho_v)\). From this, in view of the continuity of \( f_v \), a similar argument yields that the function \( f_v \) is increasing on the whole interval \([\rho_v, +\infty)\), and with this the proof of this part is complete.

b. In view of the recurrence relation [29, p. 79]
\[
\left[ x^{-\nu} I_{\nu}(x) \right]' = x^{-\nu} I_{\nu+1}(x),
\]
to prove that the function \( g_v \) is decreasing on \((0, +\infty)\) we need to show that
\[
\frac{d}{dx} \left[ x^{-\nu} I_{\nu}(x) \right] = \frac{x^{-\nu} I_{\nu+1}(x)}{e^{x+\lambda_v}} \left( e^x + \lambda_v - e^x \frac{I_{\nu}(x)}{I_{\nu+1}(x)} \right) \leq 0
\]
holds for all \( x > 0 \) and \( \nu \geq 0 \). But from part a of this lemma we know that the function \( f_v \) is decreasing on \((0, \rho_v)\) and is increasing on \([\rho_v, +\infty)\) for all \( \nu \geq 0 \). Hence for all \( x > 0 \) and \( \nu \geq 0 \) we have \( f_v(x) \geq f_v(\rho_v) = \lambda_v \), and with this the proof of this part is done. All that remains is just to observe that since in \( \rho_v \) the function \( f_v \) has a global minimum, the value \( \lambda_v = f_v(\rho_v) \) is the largest positive constant (depending on \( \nu \)) for which the function \( g_v \) is decreasing on \([0, +\infty)\). The positivity of \( \lambda_v \) easily follows from the well-known fact that due to Cochran [8] the function \( v \mapsto I_v(x) \) is strictly decreasing on \([0, +\infty)\) for all fixed \( x > 0 \), and consequently \( I_v(x) > I_{v+1}(x) \), i.e. \( f_v(x) > 0 \) for each \( v \geq 0 \) and \( x > 0 \).

c. Due to Gronwall [11] it is known that the function \( v \mapsto x I_v(x)/I_{v+1}(x) \) is increasing on \([0, +\infty)\) for each fixed \( x > 0 \). Hence we obtain that
\[
\frac{x I_v(x)}{I_{v+1}(x)} = 1 + \frac{x I'_v(x)}{I_v(x)} - x \coth x,
\]
which holds for all \( v \geq 0 \) and \( x > 0 \). Now consider the function \( \varphi : \mathbb{R} \to \mathbb{R} \), defined by
\[
\varphi(x) = 1 + \frac{x I'_v(x)}{I_v(x)} - x \coth x.
\]
In what follows we show that \( \varphi(x) \geq 0 \) for all \( x \in \mathbb{R} \), where equality holds if and only if \( x = 0 \). Since for arbitrary \( x \) we have \( I'_0(x) = I_1(x) \), it follows that
\[
\varphi(x) = x^2 \left[ \frac{1}{x} I_1(x) - \frac{1}{x} \coth x - 1 \right] = x^2 \left[ \sum_{n \geq 1} \frac{2}{x^2 + j_{0,n}^2} - \sum_{n \geq 1} \frac{2}{x^2 + (n \pi)^2} \right],
\]
where \( j_{0,n} \) is the \( n \)th positive zero of the Bessel function \( J_0 \). Here we have used the well-known Mittag-Leffler expansion [10, Eq. (7.9.3)]
\[
\frac{I_{v+1}(x)}{I_v(x)} = \sum_{n \geq 1} \frac{2x}{x^2 + j_{v,n}^2},
\]
where \( j_{v,n} \) is the \( n \)th positive zero of the Bessel function \( J_v \), and the Mittag-Leffler theorem [25, p. 192] to obtain the partial fraction decomposition
\[
\frac{x \coth x - 1}{x^2} = \sum_{n \geq 1} \frac{2}{x^2 + (n \pi)^2}.
\]
Now observe that \( j_{1/2,n} = n \pi \) for each \( n \geq 1 \) integer. This can be verified easily by using the formula [1, p. 438]
\[
\sqrt{\pi/2} \cdot x^{-1/2} J_{1/2}(x) = -\frac{\sin x}{x},
\]
the infinite product formula for the sine function [1, p. 75]
\[
\frac{\sin x}{x} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2\pi^2}\right)
\]
and the infinite product formula [29, p. 498] for the Bessel function of the first kind \(J_\nu\), i.e.
\[
J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} \prod_{n \geq 1} \left(1 - \frac{x^2}{\nu^2 n^2}\right),
\]
which is valid for arbitrary \(x\) and \(\nu \neq -1, -2, \ldots\). Hence we have
\[
\varphi(x) = \sum_{n \geq 1} \frac{2x^2(f_{1/2,n}^2 - J_{0,n}^2)}{(x^2 + f_{0,n}^2)(x^2 + f_{1/2,n}^2)}
\]
and this is clearly positive for all real \(x\) because, for fixed \(n \geq 1\), the function \(\nu \mapsto f_{\nu,n}\) is strictly increasing [6] on \([0, \infty)\), and consequently in particular we have \(f_{1/2,n} > f_{0,n}\) for each \(n \geq 1\) integer. We note that in fact there is another argument to prove the positivity of \(\varphi\). Namely, since
\[
I_{1/2}(x) = \sqrt{\frac{2x}{\pi}} \sinh x \quad \text{and} \quad I_{3/2}(x) = \sqrt{\frac{2x}{\pi}} \left(\frac{\cosh x}{x} - \frac{\sinh x}{x^2}\right),
\]
\(\varphi\) can be rewritten as follows:
\[
\varphi(x) = x \left[ \frac{I_1(x)}{I_0(x)} - \frac{x \coth x - 1}{x} \right] = x \left[ \frac{I_1(x)}{I_0(x)} - \frac{I_{3/2}(x)}{I_{1/2}(x)} \right].
\]
On the other hand it is known (see [14, Theorem 3] and [3, Lemma 1.4]) that for each fixed \(\beta \in (0, 2]\) and each \(x > 0\), the function \(\nu \mapsto I_{\nu+\beta}(x)/I_{\nu}(x)\) is decreasing, where \(\nu \geq (\beta + 1)/2\), \(\nu > -1\). This in particular implies that the function \(\nu \mapsto I_{\nu+1}(x)/I_{\nu}(x)\) is decreasing on \([0, \infty)\) for each fixed \(x > 0\). Thus for all \(x > 0\) we have
\[
I_{\nu+1}(x)/I_{\nu}(x) > I_{3/2}(x)/I_{1/2}(x),
\]
and consequently \(\varphi(x) > 0\) for each \(x > 0\) as well as for each \(x < 0\), since \(\varphi\) is an even function. Moreover, it can be shown that \(\varphi(x) < 1/2\) for each \(x \in \mathbb{R}\), and using the asymptotic formula (3) it is easy to see that if \(x\) tends to infinity, then \(\varphi(x)\) tends to 1/2. With other words, in fact \(\varphi\) maps \(\mathbb{R}\) into \([0, 1/2]\). Hence we have \(h_{\nu}(x) > 0\) for all \(\nu \geq 0\) and \(x > 0\), i.e. indeed \(h_{\nu}\) is strictly increasing, and with this the proof of this lemma is complete. \(\Box\)

**Remark 1.** Although Lemma 1 is derived to obtain new lower and upper bounds for the generalized Marcum Q-function, the results included in Lemma 1 are interesting in their own right. In 1974 Nasell [17] proved that if \(\nu > -1\) and \(x > 0\), then
\[
\left(1 + \frac{\nu}{x}\right)I_{\nu+1}(x) < I_\nu(x).
\]
From part a of Lemma 1 we know that for each \(\nu \geq 0\) the function \(f_\nu\) is decreasing on \((0, \rho_\nu]\) and increasing on \([\rho_\nu, \infty)\). Thus we have that if \(\nu \geq 0\) and \(x \in (0, \rho_\nu]\), then
\[
\left(1 + \frac{2\nu + 1}{x}\right)I_{\nu+1}(x) \leq I_\nu(x),
\]
which provides an improvement of (4). Moreover, when \(\nu \geq 0\) and \(x \geq \rho_\nu\) the inequality (5) is reversed, and this reversed inequality provides a counterpart of inequality (4).

Recall that in 1984 Simpson and Spector [23] have studied the function \(v_\nu\) and proved that this function is strictly increasing and convex on \((0, \infty)\). As a consequence of the above results the authors deduced the inequality
\[
v_\nu^2(x) > x^2 + (2\nu + 1)(2\nu + 2) + \nu + 1/2,
\]
which holds for all \(\nu \geq 0\) and \(x > 0\). For \(\nu = 0\) the inequality (6) was used to prove that a nonlinearily elastic cylinder eventually becomes unstable in uniaxial compression. Moreover, the authors proved that for any \(\nu > 0\) the function \(v_\nu\) has application in the buckling and necking of such cylinders. For more details the interested reader is referred to the paper of Simpson and Spector [24] and to the references therein. From the proof of Lemma 1 it is clear that for all \(\nu \geq 0\) and \(x \in (0, \rho_\nu]\) we have
\[
v_\nu(x) \geq x + 2\nu + 1,
\]
and consequently
\[ v_ν^2(x) ≥ x^2 + 2(2ν + 1)x + (2ν + 1)^2. \] (7)

Inequality (7) provides an improvement of (6) when \( x ∈ [3/4, ρ_ν] \). Moreover, it is also clear that from the proof of Lemma 1 we have for all \( ν > 0 \) and \( x ≥ ρ_ν \) that
\[ x + ν + 1/2 ≤ v_ν(x) ≤ x + 2ν + 1, \]
and consequently
\[ v_ν^2(x) - (2x + 3ν + 3/2)v_ν(x) + x^2 + (3ν + 3/2)x ≤ (2ν + 1)(v + 1/2). \]

For related results the interested reader is referred to the papers [4,18].

Remark 2. Table 1 contains for \( ν ∈ \{0, 1, ..., 10\} \) the explicit values of the positive roots \( ρ_ν \) of the equation \((x + 2ν + 1)I_{ν+1}(x) = xI_ν(x)\) and of the largest constants \( λ_ν = f_ν(ρ_ν) \), where the function \( f_ν \) is as in Lemma 1.

The values of \( ρ_ν \) were computed with the Matlab function fminbnd. The above computations suggest that the function \( ν ↦ ρ_ν \) is decreasing from \([0, ∞)\) onto \((1, ρ_0)\), while the function \( ν ↦ λ_ν \) is increasing from \([0, ∞)\) onto \([λ_0, ∞)\). We note here that the monotonicity of the function \( ν ↦ ρ_ν \) can be proved with the aid of the monotonicity of the function \( ν ↦ ρ_ν \). For this let us recall (see [14, Theorem 3] and [3, Lemma 1.4]) that for each fixed \( β ∈ (0, 2) \) and each \( x > 0 \), the function \( ν ↦ I_{ν+β}(x)/I_ν(x) \) is decreasing, where \( ν ≥ -(β + 1)/2, ν > 1 \). From this it follows that for each \( x > 0 \) and each \( ν > 0 \), the function \( ν ↦ f_ν(x) \) is increasing on \([0, ∞)\). Hence for each \( x > 0 \) and \( ν_1 ≥ ν_2 ≥ 0 \) we have \( f_{ν_1}(x) ≥ f_{ν_2}(x) \). Now, since \( f_ν \) is decreasing, we get \( λ_ν_1 = f_{ν_1}(ρ_ν_1) ≥ f_{ν_2}(ρ_ν_1) = f_{ν_2}(ρ_ν_2) = λ_ν_2 \).

2. Lower and upper bounds for the generalized Marcum \( Q \)-function

In this section we establish some new lower and upper bounds for the generalized Marcum function by using the results of Lemma 1. These bounds are natural extensions of the bounds stated in [28] and the proofs of Theorems 1 and 2 are similar to those given in [5]. In the followings \( \text{erfc} : R → (0, 2) \) stands, as usual, for the complementary error function, which is defined by
\[ \text{erfc}(x) = \frac{2}{\sqrt{π}} \int_x^∞ e^{-t^2} \, dt. \]

Our first main result, which improves [5, Theorem 1], reads as follows.

Theorem 1. If \( ν ≥ 1 \) and \( b ≥ a > 0 \), then the following inequalities hold
\[ Q_ν(a, b) ≥ \frac{\sqrt{π}}{2} \frac{b^νI_{ν-1}(ab)}{a^{ν-1}(e^{ab} - e^{-ab})} \left[ \text{erfc} \left( \frac{b-a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b+a}{\sqrt{2}} \right) \right], \] (8)
\[ Q_ν(a, b) ≤ \frac{I_{ν-1}(ab)}{(ab)^{ν-1}(e^{ab} + λ_{ν-1})} \left[ \int_{b-a}^∞ (u + a)^{2ν-1}e^{-u^2/2} \, du + λ_{ν-1} \int_{b-a}^∞ u^{2ν-1}e^{-u^2/2} \, du \right], \] (9)
where \( λ_{ν-1} = f_{ν-1}(ρ_{ν-1}) \) is the best possible constant in (9), i.e. cannot be replaced by a larger constant, and \( ρ_{ν-1} \) is the unique simple positive root of the equation \((x + 2ν - 1)I_ν(x) = xI_{ν-1}(x)\).
In order to deduce the lower bound in (12), we used that the function \( I_\nu \) is increasing on \((0, \infty)\), it follows that for all \( t \geqslant b \) and \( \nu \geqslant 0 \) we have

\[
I_\nu(t) \geq \frac{b^{\nu+1}}{t^{\nu+1}} \frac{e^t - e^{-t}}{e^{ab} - e^{-ab}} I_\nu(b).
\]

Changing in (10) \( t \) with \( at \), \( b \) with \( ab \) and \( \nu \) with \( \nu - 1 \), from (1) we obtain

\[
Q_\nu(a, b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^{\nu} e^{-\frac{t^2}{2}} I_{\nu-1}(at) \, dt
\]

\[
\leq \frac{1}{a^{\nu-1}} \int_b^\infty t^{\nu} e^{-\frac{t^2}{2}} \left( e^{at} + \lambda_{\nu-1} \right) (at)^{\nu-1} \left( e^{ab} + \lambda_{\nu-1} \right) (ab)^{\nu-1} I_{\nu-1}(ab) \, dt
\]

\[
= \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}(e^{ab} + \lambda_{\nu-1})} \int_b^\infty t^{2\nu-1} e^{-\frac{t^2}{2}} \left( e^{at} + \lambda_{\nu-1} \right) \, dt
\]

\[
= \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}(e^{ab} + \lambda_{\nu-1})} \left[ \int_b^\infty t^{2\nu-1} e^{-\frac{t^2}{2}} \, dt + \lambda_{\nu-1} \int_b^\infty t^{2\nu-1} e^{-\frac{t^2}{2}} \, dt \right]
\]

\[
= \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}(e^{ab} + \lambda_{\nu-1})} \left[ \int_{b-a}^\infty (u + a)^{2\nu-1} e^{-\frac{u^2}{2}} \, du + \lambda_{\nu-1} \int_b^\infty t^{2\nu-1} e^{-\frac{t^2}{2}} \, dt \right].
\]

Remark 3. Recently, in order to extend the results of Corazza and Ferrari [9], we proved [5] the followings:

\[
\sqrt{\frac{\pi}{2}} \frac{b^{\nu} I_{\nu-1}(ab)}{a^{\nu-1} e^{ab}} \text{erfc} \left( \frac{b-a}{\sqrt{2}} \right) \leq Q_\nu(a, b) \leq \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1} e^{ab}} \int_{b-a}^\infty (u + a)^{2\nu-1} e^{-\frac{u^2}{2}} \, du,
\]

where \( \nu > 1 \) and \( b > a > 0 \). Moreover, here the right-hand side inequality in (12) holds true for all \( \nu \geqslant 1/2 \). We recall that in order to deduce the lower bound in (12), we used that the function \( x \mapsto x^{\nu+1} I_\nu(x) e^{-x} \) is increasing on \((0, \infty)\) for each \( \nu \geqslant 0 \), and consequently it follows that for all \( t \geqslant b \) and \( \nu \geqslant 0 \) we have
\[ I_v(t) \geq \frac{e^t b^{v+1}}{e^{t v^{v+1}} t^v} I_v(b). \]  

(13)

It is easy to verify that for all \( t \geq b \) and \( v \geq 0 \) we have

\[
b^{v+1} - e^{-t} \frac{t^{v+1}}{e^{t v^{v+1}}} b^v I_v(b) \geq \frac{e^t b^{v+1}}{e^{t v^{v+1}} t^v} I_v(b),
\]

and from this it is clear that (10) improves (13), and consequently for \( v \geq 1 \) the lower bound in (8) is tighter than the lower bound in (12). The situation is similar with the upper bounds. Namely, in [5], in order to deduce the upper bound in (12), we used that the function \( x \mapsto x^{-v} e^{-x} I_v(x) \) is decreasing on \((0, \infty)\) for each \( v \geq -1/2 \), and thus for all \( t \geq b \) and \( v \geq -1/2 \) we have

\[ I_v(t) \leq \frac{e^t t^v}{e^{t b^v} t^v} I_v(b). \]  

(14)

It is easy to verify that for all \( k \), as usual, for \( k \in \{0, 1, 2, \ldots, n\} \) we denote by

\[ C_n^k = \frac{n!}{(n-k)!k!} \]

the binomial coefficient and we use the familiar notations

\[ (2k)!! = 2 \cdot 4 \cdot \cdots \cdot (2k) \quad \text{and} \quad (2k-1)!! = 1 \cdot 3 \cdot \cdots \cdot (2k-1). \]

**Corollary 1.** If \( n \in \{1, 2, 3, \ldots\} \) and \( b \geq a > 0 \), then the inequalities

\[
Q_n(a, b) \geq \frac{b^n I_{n-1}(ab)}{a^{n-1}(e^{ab} - e^{-ab})} \left[ A_0(b-a) - A_0(b+a) \right],
\]

and

\[
Q_n(a, b) \leq \frac{I_{n-1}(ab)}{(ab)^{n-1}(e^{ab} + e^{-ab})} \left[ \sum_{j=0}^{2n-1} C_{2n-1}^j a^j A_{2n-j-1}(b-a) + \lambda_{n-1} \cdot e^{-\frac{a^2}{2}} A_{2n-1}(b) \right]
\]

(16)

hold. Here \( \lambda_{n-1} \) is the minimal value of the function \( f_{n-1} : (0, \infty) \to \mathbb{R} \), defined by

\[ f_{n-1}(x) = e^x \left[ \frac{I_{n-1}(x)}{I_n(x)} - 1 \right], \]

i.e. \( \lambda_{n-1} = f_{n-1}(\rho_{n-1}) \), where \( \rho_{n-1} \) is the unique simple positive root of the transcendent equation \( (x + 2n-1)I_n(x) = xI_{n-1}(x) \). The coefficients occurring in the above lower and upper bounds are defined for all \( m \in \{0, 1, 2, \ldots\} \) as

\[ A_m(\alpha) = \int_{\alpha}^{\infty} u^m e^{-u^2} \, du, \]

which can be rewritten for all \( k \in \{1, 2, 3, \ldots\} \) as follows

\[ A_{2k}(\alpha) = e^{-\frac{\alpha^2}{2}} \sum_{i=1}^{k} \alpha^{2i-1} \frac{(2k-1)!!}{(2i-1)!!} + (2k-1)!! A_0(\alpha), \]

\[ A_{2k+1}(\alpha) = e^{-\frac{\alpha^2}{2}} \sum_{i=1}^{k} \alpha^{2i} \frac{(2k)!!}{(2i)!!} + (2k)!! A_1(\alpha), \]
where
\[ A_0(\alpha) = \sqrt{\frac{\pi}{2}} \text{erfc}\left(\frac{\alpha}{\sqrt{2}}\right) \quad \text{and} \quad A_1(\alpha) = e^{-\frac{\alpha^2}{2}}. \]

**Proof.** Choosing \( \nu = n \) in (8) the lower bound in (15) is clear. For the upper bound in (16) we use for \( \nu = n \) the inequality (9) and thus we need to evaluate the integrals
\[ \int_{b-a}^{\infty} (u+a)^{2n-1} e^{-\frac{u^2}{2}} du \quad \text{and} \quad \int_{b}^{\infty} u^{2n-1} e^{-\frac{u^2+a^2}{2}} du. \]

The first integral, by using the well-known Newton binomial formula, has been already computed in [5] and is as follows:
\[ \int_{b-a}^{\infty} (u+a)^{2n-1} e^{-\frac{u^2}{2}} du = \sum_{j=0}^{2n-1} C_{2n-1}^j a^j A_{2n-j-1}(b-a). \]

The second integral can be rewritten as follows
\[ \int_{b}^{\infty} u^{2n-1} e^{-\frac{u^2+a^2}{2}} du = e^{-\frac{a^2}{2}} \int_{b}^{\infty} u^{2n-1} e^{-\frac{u^2}{2}} du = e^{-\frac{a^2}{2}} A_{2n-1}(b), \]
and with this the proof is complete. \( \square \)

**Remark 4.** First note that, based on the argument presented in Remark 3, the lower and upper bounds in (15) and (16) for the generalized Marcum function of integer order are tighter than the bounds presented in [5, Corollary 1]. Secondly, we note that from (8) for \( \nu = 1 \) or from (15) for \( n = 1 \) we reobtain the result of Wang [28, Eq. (11)]
\[ Q_1(a, b) \geq \sqrt{\frac{\pi}{2}} \frac{bI_0(ab)}{e^{ab} - e^{-ab}} \left[ \text{erfc}\left(\frac{b-a}{\sqrt{2}}\right) - \text{erfc}\left(\frac{b+a}{\sqrt{2}}\right) \right], \]
which holds for all \( b > a > 0 \). Now from (16) for \( n = 1 \) we obtain that
\[ Q_1(a, b) \leq \frac{I_0(ab)}{e^{ab} + \lambda_0} \left[ \sum_{j=0}^{1} C_{2n-1}^j a^j A_{1-j}(b-a) + \lambda_0 \cdot e^{-\frac{a^2}{2}} A_1(b) \right] \]
\[ = \frac{I_0(ab)}{e^{ab} + \lambda_0} \left[ e^{-\frac{a^2}{2}} A_0(b) + \frac{\pi}{\sqrt{2}} \text{erfc}\left(\frac{b-a}{\sqrt{2}}\right) + \lambda_0 \cdot e^{-\frac{a^2}{2}} A_1(b) \right], \]
where \( b > a > 0 \), \( \lambda_0 = I_0(\rho_0) \simeq 3.03442206626763 \) and \( \rho_0 \simeq 1.54512596391949 \) is the unique simple positive root of the equation \( (x+1)I_1(x) = xI_0(x) \). A similar result has been stated recently by Wang [28, Eq. (8)], who proved that
\[ Q_1(a, b) \leq \frac{I_0(ab)}{e^{ab} + 3} \left[ e^{-\frac{a^2}{2}} + \frac{\pi}{\sqrt{2}} \text{erfc}\left(\frac{b-a}{\sqrt{2}}\right) + 3 \cdot e^{-\frac{a^2}{2}} A_1(b) \right]. \]

Our result improves Wang’s result, moreover, the constant \( \lambda_0 \) is the best possible. It is important to note that, in order to deduce the above lower bound, Wang [28, Eq. (11)] used, taking into account some numerical experiments, the fact that the function \( x \mapsto xI_0(x)/(e^x - e^{-x}) \) is increasing on \((0, \infty)\), i.e. for all \( t \geq b > 0 \) the inequality \( tI_0(t)/(e^t - e^{-t}) \geq bI_0(b)/(e^b - e^{-b}) \) holds. Analogously, in order to deduce the above upper bound, Wang [28, Eq. (8)] used the fact that the function \( x \mapsto I_0(x)/(e^x + 3) \) is decreasing on \((0, \infty)\), i.e. for all \( t \geq b \) the inequality \( I_0(t)/(e^t + 3) \leq I_0(b)/(e^b + 3) \) holds. But, the above inequalities are stated in [28] without analytical proofs. However, parts c and b of Lemma 1 guarantees that indeed the function \( x \mapsto xI_0(x)/(e^x - e^{-x}) \) is increasing and the function \( x \mapsto I_0(x)/(e^x + 3) \) is decreasing on \((0, \infty)\).

The following theorem complements and improves [5, Theorem 2].

**Theorem 2.** If \( \nu \geq 1 \) and \( a > b > 0 \), then the following inequalities hold
\[ Q_1(a, b) \geq 1 - \frac{1}{\sqrt{2\pi} b^\nu I_{\nu-1}(ab)} \left[ \frac{1}{2} \text{erfc}\left(\frac{a-b}{\sqrt{2}}\right) + \frac{1}{2} \text{erfc}\left(\frac{a+b}{\sqrt{2}}\right) - \text{erfc}\left(\frac{a}{\sqrt{2}}\right) \right]. \]
\[ Q_1(a, b) \leq 1 - \frac{(ab)^{1-\nu} I_{\nu-1}(ab)}{e^{ab} + \lambda_{\nu-1}} \left[ \int_{-a}^{b} (u+a)^{2\nu-1} e^{-\frac{u^2}{2}} du + \lambda_{\nu-1} \int_{b}^{\infty} u^{2\nu-1} e^{-\frac{u^2+a^2}{2}} du \right]. \]
where $\lambda_{\nu-1} = f_{\nu-1}(\rho_{\nu-1})$ is the best possible constant in (18), i.e. cannot be replaced by a larger constant, and $\rho_{\nu-1}$ is the unique simple positive root of the equation $(x + 2\nu - 1)I_{\nu}(x) = xI_{\nu-1}(x)$.

**Proof.** In order to prove (17), using part c of Lemma 1, we conclude that for all $t \in (0, b]$ and $\nu \geq 0$ we have

$$I_{\nu}(t) \leq \frac{b^{\nu+1} e^t - e^{-t}}{t^{\nu+1}} I_{\nu}(b). \quad (19)$$

Changing in (19) $t$ with $at$, $b$ with $ab$ and $\nu$ with $\nu - 1$, in view of (2) we have

$$Q_{\nu}(a, b) = 1 - \frac{1}{a^{\nu-1}} \int_0^b t^\nu e^{-\frac{t^2+4a^2}{2}} I_{\nu-1}(at) \, dt$$

$$\geq 1 - \frac{1}{a^{\nu-1}} \int_0^b t^\nu e^{-\frac{t^2}{2}} \frac{(ab)^\nu}{(at)^\nu} e^{at} - e^{-at} I_{\nu-1}(ab) \, dt$$

$$= 1 - \frac{b^{\nu} I_{\nu-1}(ab)}{a^{\nu-1}(e^{ab} - e^{-ab})} \int_0^b e^{-\frac{t^2}{2}} (e^{at} - e^{-at}) \, dt$$

$$= 1 - \frac{b^{\nu} I_{\nu-1}(ab)}{a^{\nu-1}(e^{ab} - e^{-ab})} \left[ \int_0^b e^{-\frac{t^2}{2}} \, dt - \int_0^b e^{-\frac{a-\nu t^2}{2}} \, dt \right]$$

$$= 1 - \frac{b^{\nu} I_{\nu-1}(ab)}{a^{\nu-1}(e^{ab} - e^{-ab})} \left[ \int_0^b e^{-\frac{t^2}{2}} \, dt - \int_0^b e^{-\frac{a-\nu t^2}{2}} \, dt \right]$$

$$= 1 - \frac{b^{\nu} I_{\nu-1}(ab)}{a^{\nu-1}(e^{ab} - e^{-ab})} \left[ \int_0^b e^{-\frac{t^2}{2}} \, dt - \int_0^b e^{-\frac{a-\nu t^2}{2}} \, dt \right]$$

$$= 1 - \frac{\sqrt{2\pi} b^{\nu} I_{\nu-1}(ab)}{2 a^{\nu-1}(e^{ab} - e^{-ab})} \left[ \text{erfc} \left( -\frac{a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b-a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{a}{\sqrt{2}} \right) + \text{erfc} \left( \frac{b+a}{\sqrt{2}} \right) \right]$$

$$= 1 - \frac{\sqrt{2\pi} b^{\nu} I_{\nu-1}(ab)}{2 a^{\nu-1}(e^{ab} - e^{-ab})} \left[ \frac{1}{2} \text{erfc} \left( \frac{a-b}{\sqrt{2}} \right) + \frac{1}{2} \text{erfc} \left( \frac{a+b}{\sqrt{2}} \right) - \text{erfc} \left( \frac{a}{\sqrt{2}} \right) \right],$$

where we have used the relation $\text{erfc}(x) + \text{erfc}(-x) = 2$, which holds for each real $x$.

To prove (18) we proceed as in the proof of Theorem 1. Using again the fact that the function $g_{\nu}$ is decreasing on $(0, \infty)$ for each $\nu > 0$, it follows that for all $t \in [0, b]$ and $\nu \geq 0$ we have

$$I_{\nu}(t) \geq \frac{e^t + \lambda_{\nu}}{e^t + \lambda_{\nu}} \frac{t^\nu}{b^{\nu} I_{\nu}(b)} \quad (20)$$

Changing in (20) $t$ with $at$, $b$ with $ab$ and $\nu$ with $\nu - 1$ in view of (2) we get

$$Q_{\nu}(a, b) = 1 - \frac{1}{a^{\nu-1}} \int_0^b t^\nu e^{-\frac{t^2+4a^2}{2}} I_{\nu-1}(at) \, dt$$

$$\leq 1 - \frac{1}{a^{\nu-1}} \int_0^b t^\nu e^{-\frac{t^2+4a^2}{2}} \frac{e^{at} + \lambda_{\nu-1}}{e^{at} + \lambda_{\nu-1}} (at)^{\nu-1} I_{\nu-1}(ab) \, dt$$

$$= 1 - \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}(e^{ab} + \lambda_{\nu-1})} \int_0^b t^{2\nu-1} e^{-\frac{t^2+4a^2}{2}} (e^{at} + \lambda_{\nu-1}) \, dt$$

$$= 1 - \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}(e^{ab} + \lambda_{\nu-1})} \left[ \int_0^b t^{2\nu-1} e^{-\frac{u^2}{2}} \, du + \lambda_{\nu-1} \int_0^b t^{2\nu-1} e^{-\frac{u^2}{2}} \, du \right]$$

$$= 1 - \frac{I_{\nu-1}(ab)}{(ab)^{\nu-1}(e^{ab} + \lambda_{\nu-1})} \left[ \int_{-a}^b (u + a)^{2\nu-1} e^{-\frac{u^2}{2}} \, du + \lambda_{\nu-1} \int_0^b u^{2\nu-1} e^{-\frac{u^2}{2}} \, du \right]. \quad \square
Remark 5. Recently, in order to deduce the counterpart of (12), we proved [5] that
\[
Q_v(a, b) \leq 1 - \frac{I_{v-1}(ab)}{(ab)^{v-1}e^{ab}} \int_{-a}^{b} (u + a)^{2v-1}e^{-u^2/2} \, du,
\]
which holds for all \( v \geq 1/2 \) and \( a > b > 0 \). The key tool in the proof of (21) was the fact the function \( x \mapsto x^{-v}e^{-x}I_v(x) \) is decreasing on \((0, \infty)\) for each \( v \geq -1/2 \), and thus we used that for all \( t \in [0, b] \) and \( v \geq -1/2 \)
\[
I_v(t) \geq \frac{e^{t^v}}{e^b b^v} I_v(b).
\]
Since for each \( v \geq 0 \) and \( t \in [0, b] \) we have
\[
\frac{e^{t^v}}{e^b + b^v} I_v(b) \geq \frac{e^{t^v}}{e^b b^v} I_v(b),
\]
it follows that (20) improves (22), and this in turn implies that for \( v \geq 1 \) the upper bound in (18) is tighter than the upper bound in (21). However, the lower bound in (17) cannot be compared with the lower bound stated in [5, Theorem 2] on the whole interval of validity.

The following result is a particular case of Theorem 2 and provides a generalization of the results of Wang [28, Eqs. (17), (27)].

**Corollary 2.** If \( n \in \{1, 2, 3, \ldots\} \) and \( a > b > 0 \), then the inequalities
\[
Q_n(a, b) \geq 1 - \frac{b^n I_{n-1}(ab)}{a^{n-1}(e^b - e^{-a})} \left[ B_0(a) - B_0(-a) \right],
\]
\[
Q_n(a, b) \leq 1 - \frac{(ab)^{n-1} I_{n-1}(ab)}{e^{ab} + \lambda_{n-1}} \left[ \sum_{j=0}^{2n-1} \frac{(ab)^{2j} B_{2n-j-1}(a) + \lambda_{n-1} \cdot e^{-\frac{b^2}{2}} B_{2n-j-1}(0)}{C_{2n-1}^j a^j} \right]
\]
hold, where \( \lambda_{n-1} \) is as in Corollary 1. The coefficients occurring in the above lower and upper bounds are defined for all \( m \in \{0, 1, 2, \ldots\} \) as
\[
B_m(\alpha) = \int_{-\alpha}^{b-\alpha} u^m e^{-\frac{u^2}{2}} \, du,
\]
which for all \( k \in \{1, 2, 3, \ldots\} \) can be rewritten as
\[
B_{2k}(\alpha) = -\sum_{i=1}^{k} \left[ e^{-\frac{\alpha^2}{2} \alpha^{2i-1}} + e^{-\frac{(b-\alpha)^2}{2}} (b - \alpha)^{2i-1} \right] (2k - 1)!! (2i - 1)!! + (2k - 1)!! B_0(\alpha),
\]
\[
B_{2k+1}(\alpha) = \sum_{i=1}^{k} \left[ e^{-\frac{\alpha^2}{2} \alpha^{2i}} - e^{-\frac{(b-\alpha)^2}{2}} (b - \alpha)^{2i} \right] (2k)!! (2i)!! + (2k)!! B_1(\alpha),
\]
where
\[
B_0(\alpha) = \sqrt{\frac{\pi}{2}} \left[ \text{erfc} \left( -\frac{\alpha}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b - \alpha}{\sqrt{2}} \right) \right] \quad \text{and} \quad B_1(\alpha) = e^{-\frac{\alpha^2}{2}} - e^{-\frac{(b-\alpha)^2}{2}}.
\]

**Proof.** Clearly for \( v = n \) (17) becomes (23). On the other hand, choosing \( v = n \) in (18), we just need to compute the integrals
\[
\int_{-a}^{b} (u + a)^{2n-1}e^{-\frac{u^2}{2}} \, du \quad \text{and} \quad \int_{0}^{b} u^{2n-1}e^{-\frac{u^2+2u^2}{2}} \, du.
\]
The first integral has been computed already in [5]:
\[
\int_{-a}^{b} (u + a)^{2n-1}e^{-\frac{u^2}{2}} \, du = \sum_{j=0}^{2n-1} C_{2n-1}^j a^j B_{2n-j-1}(a).
\]
Now, it remains to observe that
\[
\int_0^b u^{2n-1} e^{-u^2/2} \, du = e^{-b^2/2} \int_0^b u^{2n-1} e^{-u^2/2} \, B_{2n-1}(0). \quad \square
\]

**Remark 6.** First note that, in view of Remark 5, the upper bound in (24) is tighter than the upper bound in [5, Eq. (17)]. It is also worth mentioning that, in particular for \( n = 1 \), from (23) we get for all \( a > b > 0 \)
\[
Q_1(a, b) \geq 1 - \frac{b I_0(ab)}{e^{ab} - e^{-ab}} [B_0(a) - B_0(-a)]
\]
\[
= 1 - \sqrt{\pi} \frac{b I_0(ab)}{2 e^{ab} - e^{-ab}} \left[ \text{erfc} \left( \frac{a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b - a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{a}{\sqrt{2}} \right) + \text{erfc} \left( \frac{b}{\sqrt{2}} \right) \right]
\]
\[
= 1 - \sqrt{2\pi} I_0(ab) \left[ \frac{1}{2} \text{erfc} \left( \frac{a}{\sqrt{2}} \right) + \frac{1}{2} \text{erfc} \left( \frac{a + b}{\sqrt{2}} \right) - \text{erfc} \left( \frac{a}{\sqrt{2}} \right) \right],
\]
which was proved recently by Wang [28, Eq. (27)]. Finally, choosing \( n = 1 \) in (24), we easily get for each \( a > b > 0 \) that
\[
Q_1(a, b) \leq 1 - \frac{I_0(ab)}{e^{ab} + \lambda_0} \left[ \sum_{j=0}^{\infty} C_j a^j B_{1-j}(a) + \lambda_0 e^{-\frac{a^2}{2}} \sqrt{\pi} B_1(0) \right]
\]
\[
= 1 - \frac{I_0(ab)}{e^{ab} + \lambda_0} \left[ (\lambda_0 + 1) e^{-\frac{a^2}{2}} - e^{-\frac{a^2}{2}} - \lambda_0 e^{-\frac{a^2}{2}} + a \sqrt{\frac{\pi}{2}} \left[ \text{erfc} \left( \frac{a}{\sqrt{2}} \right) - \text{erfc} \left( \frac{b-a}{\sqrt{2}} \right) \right] \right],
\]
which improves [28, Eq. (17)]. This can be verified easily by taking into account that [28, Eq. (17)] is deduced from the inequality \( I_0(t) \geq (e^t + 3) I_0(b)/(e^b + 3) \), which holds for each \( t \in [0, b] \). But from (20) we know that in fact the following stronger inequality holds \( I_0(t) \geq (e^t + \lambda_0) I_0(b)/(e^b + \lambda_0) \) for all \( t \in [0, b] \), which was used to deduce the above upper bound for the Marcum \( Q \)-function. It is important to note here that for the case \( b < a \) the first upper bound for the Marcum \( Q \)-function appears in the paper of Corazza and Ferrari [9, Eq. (12)], which was generalized in [5, Eq. (17)]. Wang’s upper bound [28, Eq. (17)] improves Corazza’s and Ferrari’s bound [9, Eq. (12)] and thus our upper bound presented above in the case \( a > b \) provides the best possible upper bound of this kind, i.e. due to part b of Lemma 1 the constant \( \lambda_0 \) cannot be replaced with any larger constant.

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**References**


