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J. Math. Anal. Appl. 312 (2005) 596–619
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Large time behavior and energy relaxation time limit of the solutions to an energy transport model in semiconductors

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Received 29 April 2004

Available online 24 May 2005

Submitted by G. Bluman

Abstract

In this paper, the global existence and the large time behavior of smooth solutions to the initial boundary value problem for the multi-dimensional energy transport model are studied. It is also proved that the solutions of the problem converge to an *isothermal* drift–diffusion model as energy relaxation time τ goes to 0 by compactness argument with the help of energy estimates and entropy inequality.

Keywords: Energy transport; Drift diffusion; Large time behavior; Energy relaxation time limit; Entropy inequality

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[1] Supported by the Young Scientists Funds of NSF of China (Grant No. 10401019).
[2] Supported by the Special Funds of State Major Basic Research Projects (Grant No. 1999075107), the Funds of NSAF of China (Grant No. 10276036) and the Funds of NSF of China (Grant No. 10431060).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.03.063

1. Introduction

In recent years, a class of strongly coupled parabolic systems with cross diffusion terms were derived from applied science. In real applications, due to more information included, such class of cross diffusion models describe the phenomena more clearly than the classical weakly coupled diffusion systems. But very few theoretical results have been obtained up to now. It is well known that if the system is not weakly coupled, no general theory like the results in [15] can be used directly. In fact, the structure is completely different from the weakly coupled case so that the usual method including the maximum principle and the regularity theory for parabolic equations cannot be used.

In the present paper, we will study a system, i.e. energy transport model, derived from semiconductor simulations. For more details of the energy transport model, we refer to [1,2,8,13,14,17]. The energy transport model is a degenerate quasi-linear cross diffusion parabolic system with principal part in divergence form. The common form of the energy transport model is governed by the system

$$\begin{aligned} &\frac{\partial}{\partial t}\rho(\mu,T)+\operatorname{div}J_1=0,\\ &\frac{\partial}{\partial t}U(\mu,T)+\operatorname{div}J_2=\nabla V\cdot J_1+W(\mu,T)\quad \text{in } \Omega,\\ &\lambda^2\Delta V=\rho-C(x), \end{aligned}\tag{1.1}$$

with

$$\begin{aligned} J_1&=-L_{11}\left(\nabla\left(\frac{\mu}{T}\right)-\frac{\nabla V}{T}\right)-L_{12}\nabla\left(-\frac{1}{T}\right),\\ J_2&=-L_{21}\left(\nabla\left(\frac{\mu}{T}\right)-\frac{\nabla V}{T}\right)-L_{22}\nabla\left(-\frac{1}{T}\right), \end{aligned}\tag{1.2}$$

where the parameters μ and T are chemical potential of the electrons and the electron temperature respectively, V is the electrostatic potential, $\rho(\mu,T)$ is the electron density, $U(\mu,T)$ is the density of the internal energy, $W(\mu,T)$ is the energy relaxation term satisfying $W(\mu,T)(T-T_0)\leqslant 0$, where the positive constant T_0 is the lattice temperature, J_1 is the carrier flux density, J_2 is the energy flux density, or heat flux, L is the diffusion matrix, λ is the scaled Debye length, and $C(x)$ is the doping profile which represents the background of the device. The expressions for ρ, U, L and W are constitutive relations. Various forms, corresponding to different models, are found in the literature.

In a parabolic band structure, the relations for $\rho(\mu,T)$ and $U(\mu,T)$ derived from the Boltzmann statistics are

$$\rho(\mu,T)=T^{\frac{3}{2}}\exp\left\{\frac{\mu}{T}\right\},\qquad U(\mu,T)=\frac{3}{2}\rho T.\tag{1.3}$$

Several authors have recently studied stationary energy transport models [4,6,10] and have obtained useful results. For the transient case, the first results on the existence of a weak solution and its large time behavior for a more general parabolic system were obtained by P. Degond et al. [7]. They employed semidiscretization in time and entropy function under the physically motivated Dirichlet–Neumann boundary conditions and initial conditions. Furthermore, A. Jüngel has established in [12] the regularity and uniqueness when the coefficient matrix L depends merely on x. However, in both [7] and [12] it is required that L is uniformly positive definite. However the situation in which the coefficient matrix is only positive definite may arise in physics.

One of the most commonly used model in real applications which discussed in [3,13], with coefficient matrix L and W as

$$L = \mu_0 \rho \begin{pmatrix} 1 & \frac{3}{2}T \\ \frac{3}{2}T & \frac{15}{4}T^2 \end{pmatrix}, \tag{1.4}$$

$$W(\mu, T) = \frac{3}{2}\rho \frac{T_0 - T}{\tau}, \tag{1.5}$$

is the system for which the diffusion matrix is only positive definite without the uniformity, where μ_0 is the mobility constant, τ is the energy relaxation time.

L. Chen et al. [5] have studied the existence and uniqueness of $W_2^{2,1}(Q_\tau)$ solution for this energy transport model. For the system (1.1)–(1.5) in 1D case, the global existence and large time behavior of the solutions were obtained by Y. Li and L. Chen in [16].

The macroscopic models derived from semiconductor simulations contain three classes: hydrodynamic models, energy transport models and drift–diffusion models [14]. Each of them has its own advantage. For instance, drift–diffusion models are easy to be analyzed mathematically, but are not able to describe the temperature effects which are important in applications. The hydrodynamic models have the property of hyperbolic systems, which include more information (conservation law of mass, balance laws of momentum and energy), but are hard to be analyzed. The energy transport models, which combine the conservation of mass and balance of energy, represent a reasonable compromise. Theoretically drift–diffusion models and energy transport models can both be derived from hydrodynamic models by different scale of relaxation limits [9]. A natural question is whether there are some relations between the energy transport models and the drift–diffusion models.

The first part of this paper is to establish the global existence and asymptotic behavior of the solutions for (1.1)–(1.5) in multi-dimension when the initial data is around an *isothermal* stationary solution. In the second part we first give an a priori estimates for any smooth solution, without any restriction on the initial data, by entropy inequality. This result will be used in our second result: To prove that the solutions obtained in the first part converge to a solution of the *isothermal* drift–diffusion model as τ goes to 0. As far as we know, this is the first result on discussing of the relation between the energy transport model and the drift–diffusion model by energy relaxation time limit.

Without loss of generality, we suppose that $\lambda = \mu_0 = 1$, and set $\mathcal{E} = \rho T$. Then the model (1.1)–(1.5) can be rewritten as the following form in which the unknowns are ρ, $\mathcal{E}$ and $E = \nabla V$:

$$\begin{cases} \rho_t + \operatorname{div}(j_1) = 0, \\ \mathcal{E}_t + \operatorname{div}(j_2) = \frac{2}{3}E \cdot j_1 - \frac{\mathcal{E} - T_0\rho}{\tau}, \\ j_1 = \frac{\rho^2}{\mathcal{E}}E - \nabla\rho, \\ j_2 = \rho E - \nabla\mathcal{E}, \\ \operatorname{div} E = \rho - C(x), \end{cases} \tag{1.6}$$

with the initial data

$$\rho(x,0)=\rho_I(x), \quad \mathcal{E}(x,0)=\mathcal{E}_I(x), \quad x\in\Omega, \tag{1.7}$$

where $Q_t=\Omega\times(0,t)$, $\Omega\in\mathbb{R}^3$, is a bounded uniformly cone domain which satisfies the requirement in Sobolev embedding theorem.

The Neumann boundary conditions which we will consider are

$$\nabla\rho\cdot\gamma|_{\partial\Omega}=\nabla\mathcal{E}\cdot\gamma|_{\partial\Omega}=E\cdot\gamma|_{\partial\Omega}=0, \tag{1.8}$$

where γ is the unit outer normal vector to $\partial\Omega$.

The energy relaxation time τ is a positive constant for which we assume that

$$0<\tau\leqslant\tau_0.$$

We consider a typical stationary solution $(\mathcal{N}, T_0\mathcal{N}, \bar{E}=\nabla\mathcal{V})$ for (1.6)–(1.8). The corresponding stationary problem is

$$\begin{cases} \nabla\mathcal{N}-\frac{\mathcal{N}}{T_0}\nabla\mathcal{V}=0, & x\in\Omega,\\ \Delta\mathcal{V}=\mathcal{N}-C(x), & x\in\Omega, \end{cases} \tag{1.9}$$

with boundary condition

$$\nabla\mathcal{V}\cdot\gamma|_{\partial\Omega}=0. \tag{1.10}$$

For the global existence and asymptotic behavior, we obtain

Theorem 1.1. *Suppose* $0<\underline{C}\leqslant C(x)\leqslant\bar{C}$, $\rho(x,0)\in H^2(\Omega)$, $\mathcal{E}(x,0)\in H^2(\Omega)$. *For any fixed* $\tau>0$, *if there exists a suitably small constant* $\delta_1>0$ *such that*

$$\|\rho(\cdot,0)-\mathcal{N}(\cdot)\|_{H^2}+\|\mathcal{E}(\cdot,0)-T_0\mathcal{N}(\cdot)\|_{H^2}+(\bar{C}-\underline{C})\leqslant\delta_1,$$

then the problem (1.6)–(1.8) *has a unique solution* $(\rho,\mathcal{E},E)$ *in* $\Omega\times[0,\infty)$ *satisfying*

$$\begin{aligned}&\|\rho(\cdot,t)-\mathcal{N}(\cdot)\|_{H^2}+\|\mathcal{E}(\cdot,t)-T_0\mathcal{N}(\cdot)\|_{H^2}+\|E(\cdot,t)-\bar{E}(\cdot)\|_{H^3}\\ &\quad\leqslant c_0\big(\|\rho(\cdot,0)-\mathcal{N}(\cdot)\|_{H^2}+\|\mathcal{E}(\cdot,0)-T_0\mathcal{N}(\cdot)\|_{H^2}\big)\exp(-a_1t),\end{aligned}$$

for some positive constants c_0 *and* a_1 *depending on* τ.

For the energy relaxation time limit problem as $\tau\to 0$, we have the following theorem for one space dimensional case.

Theorem 1.2. *Under the assumptions of Theorem* 1.1, *if* $(\rho^\tau,\mathcal{E}^\tau,E^\tau)$ *is the sequence of solutions of* (3.18)–(3.20), *then there exists* $(\rho,\mathcal{E},E)$ *with* $\mathcal{E}=T_0\rho$, *such that, as* $\tau\to 0$,

$$\begin{aligned}
&\rho^\tau\to\rho && \text{strongly in } L^2\big(0,t;C^{1,\alpha}(\Omega)\big),\\
&\mathcal{E}^\tau\to\mathcal{E} && \text{strongly in } L^2\big(0,t;C^{\alpha}(\Omega)\big),\\
&\mathcal{E}^{\tau^2}\to\mathcal{E}^2 && \text{strongly in } L^2\big(0,t;L^{p}(\Omega)\big),\\
&E^\tau\to E && \text{strongly in } \big(L^2\big(0,t;C^{2,\alpha}(\Omega)\big)\cap C\big(0,T;C^{1,\alpha}(\Omega)\big)\big)^3,
\end{aligned}$$

for $\alpha \in (0, 1/2)$, $1 \leqslant p < \infty$. *Here,* (ρ, E) *is a weak solution to the following isothermal drift–diffusion equation*:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\frac{\rho}{T_0} E - \nabla \rho) = 0, \\ \operatorname{div} E = \rho - C(x), \end{cases}$$

in the sense of distributions.

2. Global existence and large time behavior

2.1. Isothermal stationary solution

In this section, we shall get some estimates to the *isothermal* stationary solution. We consider the corresponding stationary problem (1.9)–(1.10) to obtain the following theorem.

Theorem 2.1. *Assume that* $0 < \underline{C} \leqslant C(x) \leqslant \bar{C}$ *and* $C \in L^\infty(\Omega)$, *then the problem* (1.9)–(1.10) *has a solution* $(\mathcal{N}, \mathcal{V})$, *for which the following estimates hold*:

$$0 < \underline{C} \leqslant \mathcal{N}(x) \leqslant \bar{C}, \quad x \in \Omega, \tag{2.1}$$

$$\underline{c} \leqslant \mathcal{V}(x) \leqslant \bar{c}, \quad x \in \Omega, \tag{2.2}$$

$$\left|\Delta\mathcal{V}(x)\right|, \left|\nabla\mathcal{V}(x)\right|, \left|\nabla\mathcal{N}(x)\right|, \left\|\Delta\mathcal{N}(x)\right\|_{L^2} \leqslant c_1\delta_0, \quad x \in \Omega, \tag{2.3}$$

where $\delta_0 = (\bar{C} - \underline{C})$, c_1 *is a positive constant and* $\underline{c}, \bar{c}$ *are constants.*

Proof. We will only prove the inequality (2.3). The discussion on the existence for the stationary solution and the inequalities (2.1) and (2.2) can be found in [4].

Multiplying $(1.9)_2$ by $(\mathcal{V} - \bar{\mathcal{V}})$ with $\bar{\mathcal{V}} = \frac{1}{|\Omega|}\int_\Omega \mathcal{V}(x)\,dx$, and integrating the result over Ω, we have, with the help of integration by parts whenever it is necessary,

$$\begin{aligned} \int_\Omega |\nabla\mathcal{V}|^2\,dx &\leqslant \int_\Omega (\mathcal{N} - C)(\mathcal{V} - \bar{\mathcal{V}})\,dx \leqslant \epsilon \int_\Omega |\mathcal{V} - \bar{\mathcal{V}}|^2\,dx + \frac{1}{4\epsilon}\int_\Omega |\mathcal{N} - C|^2\,dx \\ &\leqslant \frac{1}{2}\int_\Omega |\nabla\mathcal{V}|^2\,dx + c_2\delta_0^2, \end{aligned}$$

where $\epsilon = \frac{1}{2}$ and the Hölder inequality and the Poincaré inequality have been used. Then we have

$$\int_\Omega |\nabla\mathcal{V}|^2\,dx \leqslant 2c_2\delta_0^2. \tag{2.4}$$

From $(1.9)_2$ and (2.1), we can directly obtain, for all $x \in \Omega$, that

$$\left|\Delta\mathcal{V}(x)\right| \leqslant c_3\delta_0. \tag{2.5}$$

This, with the help of the interpolation inequality, yields

$$\left|\nabla\mathcal{V}(x)\right|^2 \leqslant c_4\left(\|\Delta\mathcal{V}\|_{L^\infty}^2 + \|\nabla\mathcal{V}\|_{L^2}^2\right) \leqslant c_5\delta_0^2,$$

for all $x \in \Omega$. From the above inequality and $(1.9)_1$, it implies for all $x \in \Omega$,

$$\left|\nabla\mathcal{N}(x)\right| \leqslant c_6\delta_0.$$

Taking the divergence of $(1.9)_1$ with respect to x, and integrating it over Ω, with the help of $(1.9)_2$, (2.1) and the above inequalities, we can show that

$$\begin{aligned}\int_\Omega |\Delta\mathcal{N}|^2\,dx &= \int_\Omega \frac{\mathcal{N}^2}{T_0^4}\left(|\nabla\mathcal{V}|^2 + T_0\Delta\mathcal{V}\right)^2 dx \\ &\leqslant c_7\int_\Omega \left(|\Delta\mathcal{V}|^2 + |\nabla\mathcal{V}|^2\right)dx \leqslant c_8\delta_0^2, \end{aligned}\tag{2.6}$$

where c_i, $i = 1, \ldots, 8$, are generic positive constants which depend on the domain Ω. □

2.2. The local existence of solution

For any fixed τ, we will prove the local existence of the solution by Banach fixed point theorem in this section. For simplicity, we only give a sketch on this standard argument.

Theorem 2.2. *Assume that $C(x) \in L^\infty(\Omega)$, $(\rho_0(x), \mathcal{E}_0(x)) \in H^2(\Omega)$ with $(\rho_0(x), \mathcal{E}_0(x)) \geqslant 2\underline{D} > 0$, $\underline{D}$ is a positive constant. Then for any fixed $\tau > 0$, there exists a $T_1 > 0$, such that* (1.6)–(1.8) *has a unique smooth solution $(\rho(x,t), \mathcal{E}(x,t))$ satisfying*

$$\left(\rho(x,t), \mathcal{E}(x,t)\right) \in L^\infty\left([0,T_1); H^2(\Omega)\right),$$
$$\left(\rho_t(x,t), \mathcal{E}_t(x,t)\right) \in L^\infty\left([0,T_1); L^2(\Omega)\right).$$

Set $M_0 \equiv \|\rho_0\|_{H^2}^2 + \|\mathcal{E}_0\|_{H^2}^2$. We consider the following space:

$$\begin{aligned}\mathcal{D} := \Big\{(\rho,\mathcal{E}) \mid \sup_{0\leqslant t\leqslant T_1}\left(\|\rho\|_{H^2}^2 + \|\mathcal{E}\|_{H^2}^2 + \|\rho_t\|_{L^2}^2 + \|\mathcal{E}_t\|_{L^2}^2\right) \leqslant M, \\ M > M_0,\ \rho, \mathcal{E} \geqslant \underline{D}\Big\},\end{aligned}\tag{2.7}$$

where M and $\underline{D}$ are positive constants. We use the metric

$$|||(\rho,\mathcal{E})||| = \sup_{0\leqslant t\leqslant T_1}\left(\|\rho\|_{L^2}^2 + \|\mathcal{E}\|_{L^2}^2\right) + \int_0^{T_1}\left(\left\|\rho(\cdot,t)\right\|_{H^1}^2 + \left\|\mathcal{E}(\cdot,t)\right\|_{H^1}^2\right)dt. \tag{2.8}$$

Define a mapping $\mathcal{F} : (\rho,\mathcal{E}) \in \mathcal{D} \to (u,v)$ in the following way. First, for any fixed ρ, we solve the problem

$$\begin{cases}\Delta V = \rho - C(x), & x \in \Omega, \\ \nabla V \cdot \gamma|_{\partial\Omega} = 0,\end{cases}\tag{2.9}$$

and get a unique $E = \nabla V$. Then, we solve the linear parabolic system for (u, v),

$$u_t - \Delta u + \frac{2\rho}{\mathcal{E}} E \cdot \nabla u - \frac{\rho^2}{\mathcal{E}^2} E \cdot \nabla v + \frac{\rho u}{\mathcal{E}} \operatorname{div} E = 0, \tag{2.10}$$

$$v_t - \Delta v + \frac{v}{\tau} + \frac{5}{3} E \cdot \nabla u + \left(\operatorname{div} E - \frac{2\rho}{3\mathcal{E}} E \cdot E - \frac{T_0}{\tau}\right) u = 0, \tag{2.11}$$

$$\nabla u \cdot \gamma|_{\partial\Omega} = \nabla v \cdot \gamma|_{\partial\Omega} = 0, \tag{2.12}$$

$$u(x, 0) = \rho_I(x), \qquad v(x, 0) = \mathcal{E}_I(x), \tag{2.13}$$

where $(x, t) \in (\Omega \times [0, T))$. The solvability can be found in [15].

Thus, we have the following lemmas.

Lemma 2.3. *Assume that $C(x) \in L^\infty(\Omega)$, $(\rho_I(x), \mathcal{E}_I(x)) \in H^2(\Omega)$ with $(\rho_I(x), \mathcal{E}_I(x)) \geqslant 2\underline{D}$, $\underline{D} > 0$ is a constant. Then there exists a $T_1 > 0$ such that $\mathcal{F}$ maps $\mathcal{D}$ into itself.*

Remark 2.4. For any given $(\rho, \mathcal{E}) \in \mathcal{D}$, from the theory of linear parabolic system, we only need to prove that $(u, v) \in \mathcal{D}$. By Sobolev embedding theorem, $L^\infty(0, T_1; L^2(\Omega)) \cap L^2(0, T_1; H^1(\Omega)) \hookrightarrow L^{10/3}(0, T_1; L^{10/3}(\Omega))$. Thus we get $(\rho, \mathcal{E}) \in W_3^{2,1}(Q_{T_1})$. From the result $W_3^{2,1}(Q_{T_1}) \hookrightarrow C^\alpha(Q_{T_1})$ with $0 < \alpha < \frac{1}{2}$ (see [11]), we have $(\rho, \mathcal{E}) \in C^\alpha(Q_{T_1})$. Then Lemma 2.3 can be established by using energy estimates.

Lemma 2.5. *Assume that $C(x) \in L^\infty(\Omega)$, $(\rho_I(x), \mathcal{E}_I(x)) \in H^2(\Omega)$ with $(\rho_I(x), \mathcal{E}_I(x)) \geqslant 2\underline{D} > 0$. Then there exists a $T_1 > 0$, such that $|||(\delta u, \delta v)||| \leqslant \frac{1}{2} |||(\delta\rho, \delta\mathcal{E})|||$. Namely the map $\mathcal{F} : \mathcal{D} \to \mathcal{D}$ is a contraction with metric* (2.8)*.*

The proof of Theorem 2.2. By the Banach fixed point theorem and with the help of Lemmas 2.3 and 2.5, we can show that there exists a small $T_1 > 0$, such that there exists exactly one fixed point $(\rho, \mathcal{E})$ with $(\rho, \mathcal{E}) = \mathcal{F}(\rho, \mathcal{E})$ in the corresponding space $\mathcal{D}$, and the fixed point is the unique solution of (2.10)–(2.13). □

2.3. Asymptotic behavior of smooth solution

Let $(\rho, \mathcal{E}, E)$ be a solution to (1.6)–(1.8), and set $\psi = V - \mathcal{V}$, $\varphi = \rho - \mathcal{N}$, $f = \mathcal{E} - T_0\mathcal{N}$, where $(\mathcal{N}, \mathcal{V})$ is a solution to (1.9)–(1.10). To prove Theorem 1.1, we first establish the following a priori estimate.

Lemma 2.6. *For any fixed $\tau > 0$, if there exists sufficiently small constant δ_0, such that for any $T > 0$,*

$$\sup_{0 \leqslant t \leqslant T} \left(\|\varphi(\cdot, t)\|_{H^2} + \|f(\cdot, t)\|_{H^2} \right) + (\bar{C} - \underline{C}) \leqslant \delta_0, \tag{2.14}$$

then it holds

$$\|\varphi(\cdot, t)\|_{H^2}^2 + \|f(\cdot, t)\|_{H^2}^2 \leqslant c\left(\|\varphi(\cdot, 0)\|_{H^2}^2 + \|f(\cdot, 0)\|_{H^2} \right) \exp(-a_1 t), \tag{2.15}$$

for any $t \in [0, T]$, where a_1 and c are dependent on τ.

Proof. From (1.8) and (1.10), we get

$$\nabla\psi\cdot\gamma|_{\partial\Omega}=\nabla\varphi\cdot\gamma|_{\partial\Omega}=\nabla f\cdot\gamma|_{\partial\Omega}=0. \tag{2.16}$$

By using (1.6) and (1.9), we have the following equations for φ and f:

$$\varphi_t-\Delta\varphi+\frac{\mathcal{N}^2}{f+T_0\mathcal{N}}\varphi+\frac{(2f+T_0\mathcal{N})\mathcal{N}}{(f+T_0\mathcal{N})^2}\nabla\mathcal{N}\cdot\nabla\psi-\frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}\nabla\psi\cdot\nabla f$$
$$+\operatorname{div}\left(\frac{(\varphi+2\mathcal{N})(\nabla\psi+\nabla\mathcal{V})}{f+T_0\mathcal{N}}\varphi-\frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})}f\right)=0. \tag{2.17}$$

$$f_t-\Delta f+\frac{f}{\tau}+\frac{5}{3}(\nabla\psi+\nabla\mathcal{V})\cdot\nabla\varphi+\left(\varphi+\Delta\mathcal{V}+\mathcal{N}-\frac{T_0}{\tau}\right)\varphi+\nabla\mathcal{N}\cdot\nabla\psi$$
$$-\frac{2}{3}(\nabla\psi+\nabla\mathcal{V})\left(\frac{(\varphi+\mathcal{N})^2}{f+T_0\mathcal{N}}\nabla\psi+\frac{(\varphi+2\mathcal{N})\nabla\mathcal{V}}{f+T_0\mathcal{N}}\varphi-\frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})}f\right)=0. \tag{2.18}$$

From $(1.6)_3$, we obtain that $\Delta\psi=\varphi$. Multiplying it by $(\psi-\bar\psi)$, where $\bar\psi$ is the mean value of ψ in Ω, and using the integration by parts with the help of (2.16) and Poincaré inequality, we deduce

$$\int_\Omega|\nabla\psi|^2\,dx\leqslant O(1)\int_\Omega\varphi^2\,dx. \tag{2.19}$$

Then taking gradient to the equation $\Delta\psi=\varphi$, and noting (2.14), we have the following estimates:

$$\int_\Omega|D^i\psi|^2\,dx\leqslant O(1)\delta_0,\quad i=1,2,3,\ \forall t\in[0,T].$$

The above estimates and (2.14) give, with the help of Sobolev embedding theorem, that

$$\sup_{0\leqslant t\leqslant T}\big(|\nabla\psi|,|\varphi|,|f|\big)\leqslant O(1)\delta_0. \tag{2.20}$$

Multiplying (2.17) by φ, integrating the result over Ω, and making the integration by parts with the help of (2.16) and (1.10) whenever necessary, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\varphi^2\,dx+\int_\Omega\left(|\nabla\varphi|^2+\frac{\mathcal{N}^2}{f+T_0\mathcal{N}}\varphi^2\right)dx$$
$$=-\int_\Omega\frac{(2f+T_0\mathcal{N})\mathcal{N}}{(f+T_0\mathcal{N})^2}\varphi\nabla\mathcal{N}\cdot\nabla\psi\,dx$$
$$+\int_\Omega\left\{\frac{(\varphi+2\mathcal{N})}{f+T_0\mathcal{N}}\varphi\nabla\mathcal{V}\cdot\nabla\varphi-\frac{\mathcal{N}}{T_0(f+T_0\mathcal{N})}f\nabla\mathcal{V}\cdot\nabla\varphi\right\}dx$$
$$+\int_\Omega\frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}\varphi\nabla\psi\cdot\nabla f\,dx+\int_\Omega\frac{(\varphi+2\mathcal{N})}{f+T_0\mathcal{N}}\varphi\nabla\psi\cdot\nabla\varphi\,dx$$
$$=I_1+I_2+I_3.$$

With the help of (2.19), I_1 and the second term of I_3 can be controlled by $O(\delta_0)\int_\Omega(|\nabla\varphi|^2+\varphi^2+f^2)\,dx$ by using the Gagliardo–Nirenberg inequality, Hölder inequality and the smallness of δ_0. Using (2.16) and integration by parts, we estimate the first term of I_3 as follows:

$$\begin{aligned}
&\int\limits_\Omega \frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}\varphi\nabla\psi\cdot\nabla f\,dx\\
&\quad=-\int\limits_\Omega \frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}f\nabla\psi\cdot\nabla\varphi\,dx-\int\limits_\Omega \frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}f\varphi\Delta\psi\,dx\\
&\qquad-\int\limits_\Omega \frac{2\mathcal{N}}{(f+T_0\mathcal{N})^2}f\varphi\nabla\psi\cdot\nabla\mathcal{N}\,dx\\
&\qquad+\int\limits_\Omega \frac{2\mathcal{N}^2}{(f+T_0\mathcal{N})^3}f\varphi\nabla\psi\cdot(\nabla f+T_0\nabla\mathcal{N})\,dx,
\end{aligned}$$

which with the choosing $2\delta_0\leqslant T_0\underline{C}$, implies

$$\int\limits_\Omega \frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}\varphi\nabla\psi\cdot\nabla f\,dx\leqslant O(\delta_0)\int\limits_\Omega\big(|\nabla\varphi|^2+\varphi^2+f^2\big)\,dx.$$

Then we are able to show that

$$\frac{1}{2}\frac{d}{dt}\int\limits_\Omega\varphi^2\,dx+\int\limits_\Omega\big(|\nabla\varphi|^2+c_1\varphi^2\big)\,dx\leqslant O(\delta_0)\int\limits_\Omega\big(|\nabla\varphi|^2+f^2+\varphi^2\big)\,dx,\tag{2.21}$$

where $c_1=\frac{\underline{C}^2}{2T_0\bar{C}}>0$.

Multiplying (2.18) by τf and integrating it over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_\Omega\tau f^2\,dx+\int\limits_\Omega\big(\tau|\nabla f|^2+f^2\big)\,dx\\
&\quad=-\tau\int\limits_\Omega\left\{\frac{5}{3}(\nabla\psi+\nabla\mathcal{V})\cdot\nabla\varphi f+(\varphi+\Delta\mathcal{V})\varphi f+\nabla\mathcal{N}\cdot\nabla\psi f\right\}dx\\
&\qquad-\int\limits_\Omega(\tau\mathcal{N}-T_0)\varphi f\,dx\\
&\qquad+\tau\int\limits_\Omega\frac{2}{3}(\nabla\psi+\nabla\mathcal{V})\cdot\left(\frac{(\varphi+\mathcal{N})^2}{f+T_0\mathcal{N}}\nabla\psi f+\frac{(\varphi+2\mathcal{N})\nabla\mathcal{V}}{f+T_0\mathcal{N}}\varphi f\right.\\
&\qquad\left.-\frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})}f^2\right)dx\\
&\quad=I_4+I_5+I_6,
\end{aligned}$$

where I_4, I_6 can be controlled by $O(\delta_0)\tau\int_\Omega(|\nabla\varphi|^2+\varphi^2+f^2)\,dx$, due to the smallness of (2.3) and (2.20). By using Hölder inequality, we get

$$I_5=-\int_\Omega(\tau\mathcal{N}-T_0)\varphi f\,dx\leqslant\frac{1}{4}\int_\Omega f^2\,dx+c_2(1+\tau^2)\int_\Omega\varphi^2\,dx,$$

where $c_2=(\bar{C}+T_0)^2>0$. Then we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\tau f^2\,dx+\int_\Omega\left(\tau|\nabla f|^2+\frac{3}{4}f^2\right)dx \leqslant O(\delta_0)\tau\int_\Omega\left(|\nabla\varphi|^2+\varphi^2+f^2\right)dx+c_2(1+\tau^2)\int_\Omega\varphi^2\,dx. \tag{2.22}$$

Multiplying (2.21) by $(1+\frac{c_2}{c_1}(1+\tau))$ and adding it to (2.22), we conclude by choosing δ_0 suitable small, that

$$\frac{d}{dt}\int_\Omega\left[(1+\tau)\varphi^2+\tau f^2\right]dx+\int_\Omega\left[(1+\tau)|\nabla\varphi|^2+\tau|\nabla f|^2\right]dx+a_1\int_\Omega(\varphi^2+f^2)\,dx \leqslant 0, \tag{2.23}$$

where $a_1=\min(\frac{3}{2},2c_1)>0$.

Taking the gradient of (2.17) with respect to x, multiplying it by $\nabla\varphi$, and then integrating the result over Ω and making the integration by parts with the help of (2.16), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\varphi|^2\,dx+\int_\Omega\left(|\Delta\varphi|^2+\frac{\mathcal{N}^2}{f+T_0\mathcal{N}}|\nabla\varphi|^2\right)dx\\
&\quad=-\int_\Omega\nabla\varphi\cdot\nabla\left(\frac{\mathcal{N}^2}{f+T_0\mathcal{N}}\right)\varphi\,dx\\
&\qquad+\int_\Omega\left\{\frac{(2f+T_0\mathcal{N})\mathcal{N}}{(f+T_0\mathcal{N})^2}\nabla\mathcal{N}\cdot\nabla\psi-\frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}\nabla\psi\cdot\nabla f\right\}\Delta\varphi\,dx\\
&\qquad+\int_\Omega\operatorname{div}\left\{\frac{(\nabla\psi+\nabla\mathcal{V})(\varphi+2\mathcal{N})}{f+T_0\mathcal{N}}\varphi-\frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})}f\right\}\Delta\varphi\,dx\\
&\quad=I_7+I_8+I_9.
\end{aligned}$$

By the smallness of (2.3), (2.20) and using (2.19), I_8 can be bounded by $O(\delta_0)\int_\Omega(|\nabla\varphi|^2+|\nabla f|^2+\varphi^2)\,dx$, and I_7 and I_9 can be estimated as follows:

$$\begin{aligned}
I_7&=-\int_\Omega\left\{\frac{2\mathcal{N}\varphi}{f+T_0\mathcal{N}}\nabla\mathcal{N}\cdot\nabla\varphi-\frac{\mathcal{N}^2\varphi}{(f+T_0\mathcal{N})^2}(\nabla\varphi\cdot\nabla f+T_0\nabla\varphi\cdot\nabla\mathcal{N})\right\}dx\\
&\leqslant O(\delta_0)\int_\Omega\left(|\nabla\varphi|^2+|\nabla f|^2+\varphi^2+f^2\right)dx,
\end{aligned}$$

$$
\begin{aligned}
I_9 &= \int_\Omega \bigg\{ \frac{2(\varphi+\mathcal{N})}{f+T_0\mathcal{N}}(\nabla\psi+\nabla\mathcal{V})\cdot\nabla\varphi \\
&\quad + \frac{2\nabla\mathcal{N}\cdot(\nabla\psi+\nabla\mathcal{V})+(\varphi+\Delta\mathcal{V})(\varphi+2\mathcal{N})}{f+T_0\mathcal{N}}\varphi \\
&\quad - \frac{\mathcal{N}}{T_0(f+T_0\mathcal{N})}\nabla\mathcal{V}\cdot\nabla f - \frac{(\nabla\psi+\nabla\mathcal{V})(\varphi+2\mathcal{N})}{(f+T_0\mathcal{N})^2}(T_0\nabla\mathcal{N}+\nabla f)\varphi \\
&\quad + \frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})^2}(T_0\nabla\mathcal{N}+\nabla f)f - \frac{\mathcal{N}\Delta\mathcal{V}+\nabla\mathcal{V}\cdot\nabla\mathcal{N}}{T_0(f+T_0\mathcal{N})}f \bigg\}\Delta\varphi\,dx \\
&= \int_\Omega \bigg\{ \frac{2(\varphi+\mathcal{N})}{f+T_0\mathcal{N}}\nabla\psi\cdot\nabla\varphi + \frac{(\varphi+\Delta\mathcal{V})(\varphi+2\mathcal{N})}{f+T_0\mathcal{N}}\varphi - \frac{\mathcal{N}\Delta\mathcal{V}}{T_0(f+T_0\mathcal{N})}f \bigg\}\Delta\varphi\,dx \\
&\quad + \int_\Omega \bigg\{ \frac{2(\varphi+\mathcal{N})}{f+T_0\mathcal{N}}\nabla\mathcal{V}\cdot\nabla\varphi - \frac{\mathcal{N}}{T_0(f+T_0\mathcal{N})}\nabla\mathcal{V}\cdot\nabla f \bigg\}\Delta\varphi\,dx \\
&\quad + \int_\Omega \bigg\{ \frac{2\nabla\mathcal{N}\cdot(\nabla\psi+\nabla\mathcal{V})}{f+T_0\mathcal{N}}\varphi - \frac{(\nabla\psi+\nabla\mathcal{V})(\varphi+2\mathcal{N})}{(f+T_0\mathcal{N})^2}(T_0\nabla\mathcal{N}+\nabla f)\varphi \bigg\}\Delta\varphi\,dx \\
&\quad + \int_\Omega \bigg\{ \frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})^2}(T_0\nabla\mathcal{N}+\nabla f)f - \frac{\nabla\mathcal{V}\cdot\nabla\mathcal{N}}{T_0(f+T_0\mathcal{N})}f \bigg\}\Delta\varphi\,dx \\
&\leqslant O(\delta_0)\int_\Omega \big(|\Delta\varphi|^2+|\nabla\varphi|^2+|\nabla f|^2+\varphi^2+f^2\big)\,dx.
\end{aligned}
$$

Thus we have the following estimate for $\nabla\varphi$:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega \big(|\Delta\varphi|^2 + c_1|\nabla\varphi|^2\big)\,dx \\
&\quad \leqslant O(\delta_0)\int_\Omega \big(|\Delta\varphi|^2+|\nabla\varphi|^2+|\nabla f|^2+\varphi^2+f^2\big)\,dx.
\end{aligned}
\tag{2.24}
$$

Taking the gradient of (2.18), multiplying it by $\tau\nabla f$, integrating the result over Ω, and using the integration by parts and the smallness of $|f|$, $|\varphi|$, $|\nabla\psi|$ and (2.3), similar to the estimates of before, one gets

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \tau|\nabla f|^2\,dx + \int_\Omega \big(\tau|\Delta f|^2 + |\nabla f|^2\big)\,dx \\
&\quad = -\tau\int_\Omega \frac{2}{3}(\nabla\psi+\nabla\mathcal{V})\cdot\bigg\{ \frac{(\varphi+\mathcal{N})^2}{f+T_0\mathcal{N}}\nabla\psi + \frac{(\varphi+2\mathcal{N})\nabla\mathcal{V}}{f+T_0\mathcal{N}}\varphi \\
&\qquad - \frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})}f \bigg\}\Delta f\,dx
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \tau \int_\Omega \left\{ \frac{5}{3}(\nabla\psi + \nabla\mathcal{V})\cdot\nabla\varphi + (\varphi + \Delta\mathcal{V})\varphi + \nabla\mathcal{N}\cdot\nabla\psi \right\} \Delta f\,dx \\
&\quad + \int_\Omega (\tau\mathcal{N} - T_0)\varphi\Delta f\,dx \\
&= -\tau \int_\Omega \left\{ \frac{2}{3}\frac{(\varphi+\mathcal{N})^2}{f + T_0\mathcal{N}}\nabla\psi\cdot\nabla\psi + \frac{5}{3}\nabla\psi\cdot\nabla\varphi + (\varphi + \Delta\mathcal{V})\varphi \right\} \Delta f\,dx \\
&\quad - \tau \int_\Omega \frac{2}{3}(\nabla\psi + \nabla\mathcal{V})\cdot \left\{ \frac{(\varphi + 2\mathcal{N})\nabla\mathcal{V}}{f + T_0\mathcal{N}}\varphi - \frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f + T_0\mathcal{N})} f \right\} \Delta f\,dx \\
&\quad - \tau \int_\Omega \left\{ \frac{2}{3}\frac{(\varphi+\mathcal{N})^2}{f + T_0\mathcal{N}}\nabla\mathcal{V}\cdot\nabla\psi - \nabla\mathcal{N}\cdot\nabla\psi \right\} \Delta f\,dx \\
&\quad + \int_\Omega (\tau\mathcal{N} - T_0)\varphi\Delta f\,dx \qquad (2.25) \\
&\leqslant O(\delta_0)\tau \int_\Omega \left(|\Delta f|^2 + |\nabla f|^2 + |\nabla\varphi|^2 + f^2 + \varphi^2 \right) dx \\
&\quad + \frac{1}{4}\int_\Omega |\nabla f|^2\,dx + c_2(1+\tau^2)\int_\Omega |\nabla\varphi|^2\,dx, \qquad (2.26)
\end{aligned}
$$

where the last integral on the left-hand side of (2.25) can be controlled by

$$
\begin{aligned}
\int_\Omega (\tau\mathcal{N} - T_0)\varphi\Delta f\,dx &= -\tau\int_\Omega \nabla\mathcal{N}\cdot\nabla f\varphi\,dx - \int_\Omega (\tau\mathcal{N} - T_0)\nabla\varphi\cdot\nabla f\,dx \\
&\leqslant O(\delta_0)\tau\int_\Omega \left(|\nabla f|^2 + \varphi^2 \right) dx + \frac{1}{4}\int_\Omega |\nabla f|^2\,dx + c_2(1+\tau^2)\int_\Omega |\nabla\varphi|^2\,dx.
\end{aligned}
$$

Multiplying (2.24) by $(1 + \frac{c_2}{c_1}(1+\tau))$, adding it to (2.26), and choosing δ_0 small enough and combining with (2.23), we obtain the following estimate:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \left[(1+\tau)\left(|\nabla\varphi|^2 + \varphi^2 \right) + \tau\left(|\nabla f|^2 + f^2 \right) \right] dx \\
&\quad + \int_\Omega \left[(1+\tau)\left(|\Delta\varphi|^2 + |\nabla\varphi|^2 \right) + \tau\left(|\Delta f|^2 + |\nabla f|^2 \right) \right] dx \\
&\quad + a_1\int_\Omega \left(|\nabla\varphi|^2 + \varphi^2 + |\nabla f|^2 + f^2 \right) dx \leqslant 0. \qquad (2.27)
\end{aligned}
$$

Now we turn to estimate the higher order derivatives.

From $(1.6)_3$, we have $\Delta\psi_t = \varphi_t$. Then by (2.16), we get $\nabla\psi_t\cdot\gamma|_{\partial\Omega} = 0$, and similar to the former discussion, we obtain that

$$\int_\Omega |\nabla\psi_t|^2\,dx \leqslant \int_0^1 \varphi_t^2\,dx. \tag{2.28}$$

Differentiating (2.17) with respect to t, we get

$$\begin{aligned}
&\varphi_{tt} - \Delta\varphi_t + \frac{\mathcal{N}^2}{f+T_0\mathcal{N}}\varphi_t - \frac{\mathcal{N}^2\varphi}{(f+T_0\mathcal{N})^2} f_t + \frac{(2f+T_0\mathcal{N})\mathcal{N}}{(f+T_0\mathcal{N})^2}\nabla\mathcal{N}\cdot\nabla\psi_t \\
&\quad - \frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}(\nabla\psi\cdot\nabla f_t + \nabla f\cdot\nabla\psi_t) - \frac{2\mathcal{N}f\nabla\mathcal{N}\cdot\nabla\psi}{(f+T_0\mathcal{N})^3} f_t \\
&\quad + \frac{2\mathcal{N}^2}{(f+T_0\mathcal{N})^3}\nabla\psi\cdot\nabla f f_t \\
&\quad + \operatorname{div}\left\{\frac{(\nabla\psi+\nabla\mathcal{V})(\varphi+2\mathcal{N})}{f+T_0\mathcal{N}}\varphi - \frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})} f\right\}_t = 0.
\end{aligned} \tag{2.29}$$

Multiplying (2.29) by φ_t, integrating it over Ω, and using the fact $\nabla\varphi_t\cdot\gamma|_{\partial\Omega} = 0$, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \varphi_t^2\,dx + \int_\Omega \left(|\nabla\varphi_t|^2 + \frac{\mathcal{N}^2}{f+T_0\mathcal{N}}\varphi_t^2\right)dx = I_{10} + I_{11} + I_{12}. \tag{2.30}$$

By the smallness of $|f|$, $|\varphi|$, $|\nabla\psi|$, and (2.3), with the help of (2.28), we obtain the following estimates:

$$\begin{aligned}
I_{10} &= \int_\Omega \left\{\frac{\mathcal{N}^2\varphi}{(f+T_0\mathcal{N})^2} f_t - \frac{(2f+T_0\mathcal{N})\mathcal{N}}{(f+T_0\mathcal{N})^2}\nabla\mathcal{N}\cdot\nabla\psi_t \right.\\
&\qquad \left. - \frac{2\mathcal{N}\nabla\mathcal{N}\cdot\nabla\psi}{(f+T_0\mathcal{N})^3} f f_t + \frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}\nabla\psi\cdot\nabla f_t\right\}\varphi_t\,dx \\
&\leqslant O(\delta_0)\int_\Omega \left(|\nabla\varphi_t|^2 + \varphi_t^2 + f_t^2\right)dx, \\
I_{11} &= \int_\Omega \left\{\frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}\nabla f\cdot\nabla\psi_t + \frac{2\mathcal{N}^2}{(f+T_0\mathcal{N})^3}\nabla\psi\cdot\nabla f f_t\right\}\varphi_t\,dx \\
&\leqslant O(\delta_0)\int_\Omega \left(|\nabla\varphi_t|^2 + |\nabla f_t|^2 + f_t^2 + \varphi_t^2\right)dx,
\end{aligned}$$

where the last term of I_{10} is treated similarly to I_3, and we only need to deal with the two terms in I_{11}, by using the Gagliardo–Nirenberg inequality and (2.14), as follows:

$$\begin{aligned}
\int_\Omega \frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}\nabla f\cdot\nabla\psi_t\varphi_t\,dx &\leqslant O(1)\|\nabla f\|_{L^2}\|\nabla\psi_t\|_{L^4}\|\varphi_t\|_{L^4} \\
&\leqslant O(\delta_0)\int_\Omega \left(|\nabla\varphi_t|^2 + \varphi_t^2\right)dx,
\end{aligned}$$

$$\int_\Omega \frac{2\mathcal{N}^2}{(f+T_0\mathcal{N})^3}\nabla\psi\cdot\nabla f f_t\varphi_t\,dx \leqslant O(\delta_0)\|f_t\|_{L^2}\|\nabla f\|_{L^4}\|\varphi_t\|_{L^4}$$
$$\leqslant O(\delta_0)\int_\Omega\left(|\nabla\varphi_t|^2+\varphi_t^2+f_t^2\right)dx.$$

The part I_{12} in (2.30) is estimated by

$$\begin{aligned}I_{12}&=\int_\Omega\left\{\frac{(\nabla\psi+\nabla\mathcal{V})(\varphi+2\mathcal{N})}{f+T_0\mathcal{N}}\varphi-\frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})}f\right\}_t\nabla\varphi_t\,dx\\&=\int_\Omega\left\{\frac{2(\nabla\psi+\nabla\mathcal{V})(\varphi+\mathcal{N})}{f+T_0\mathcal{N}}\varphi_t+\frac{(\varphi+2\mathcal{N})\varphi}{f+T_0\mathcal{N}}\nabla\psi_t\right.\\&\qquad\left.-\frac{\mathcal{N}^2\nabla\mathcal{V}}{(f+T_0\mathcal{N})^2}f_t-\frac{(\nabla\psi+\nabla\mathcal{V})(\varphi+2\mathcal{N})}{(f+T_0\mathcal{N})^2}\varphi f_t\right\}\cdot\nabla\varphi_t\,dx\\&\leqslant O(\delta_0)\int_\Omega\left(|\nabla\varphi_t|^2+\varphi_t^2+f_t^2\right)dx.\end{aligned}$$

Thus we can get the following estimate for φ_t:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\varphi_t^2\,dx+\int_\Omega\left(|\nabla\varphi_t|^2+c_1\varphi_t^2\right)dx\leqslant O(\delta_0)\int_\Omega\left(|\nabla\varphi_t|^2+\varphi_t^2+f_t^2\right)dx. \tag{2.31}$$

Differentiating (2.18) with respect to t, we obtain

$$\begin{aligned}&f_{tt}-\Delta f_t+\frac{f_t}{\tau}+\frac{5}{3}(\nabla\psi+\nabla\mathcal{V})\cdot\nabla\varphi_t+(2\varphi+\Delta\mathcal{V})\varphi_t+\left(\frac{5}{3}\nabla\varphi+\nabla\mathcal{N}\right)\cdot\nabla\psi_t\\&\quad+\left(\mathcal{N}-\frac{T_0}{\tau}\right)\varphi_t-\frac{2}{3}(\nabla\psi+\nabla\mathcal{V})\cdot\left\{\frac{(\varphi+\mathcal{N})^2}{f+T_0\mathcal{N}}\nabla\psi_t+\frac{2(\varphi+\mathcal{N})(\nabla\psi+\nabla\mathcal{V})}{f+T_0\mathcal{N}}\varphi_t\right.\\&\quad\left.-\frac{\mathcal{N}^2\nabla\mathcal{V}}{(f+T_0\mathcal{N})^2}f_t-\frac{(\varphi+\mathcal{N})^2}{(f+T_0\mathcal{N})^2}\nabla\psi f_t-\frac{(\varphi+2\mathcal{N})\nabla\mathcal{V}}{(f+T_0\mathcal{N})^2}\varphi f_t\right\}\\&\quad-\frac{2}{3}\left\{\frac{(\varphi+\mathcal{N})^2}{f+T_0\mathcal{N}}\nabla\psi+\frac{(\varphi+2\mathcal{N})\nabla\mathcal{N}}{f+T_0\mathcal{N}}\varphi-\frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})}f\right\}\cdot\nabla\psi_t=0.\end{aligned}\tag{2.32}$$

Multiplying (2.32) by τf_t, integrating it over Ω, and noting $\nabla f_t\cdot\gamma|_{\partial\Omega}=0$ and (2.28), it can be shown that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\tau f_t^2\,dx+\int_\Omega\left(\tau|\nabla f_t|^2+f_t^2\right)dx=I_{13}+I_{14}+I_{15}+I_{16}+I_{17},$$

where

$$\begin{aligned}I_{13}=\tau\int_\Omega&\left\{\frac{5}{3}\nabla\psi\cdot\nabla\varphi_t+(2\varphi+\Delta\mathcal{V})\varphi_t+\frac{5}{3}\nabla\varphi\cdot\nabla\psi_t\right.\\&\left.-\frac{2}{3}\frac{(\varphi+\mathcal{N})^2}{f+T_0\mathcal{N}}\nabla\psi\cdot\nabla\psi_t\right\}f_t\,dx\end{aligned}$$

$$
\begin{aligned}
&\quad -\tau\int_\Omega \frac{2}{3}\nabla\psi\cdot\left\{\frac{(\varphi+\mathcal{N})^2}{f+T_0\mathcal{N}}\nabla\psi_t+\frac{2(\varphi+\mathcal{N})}{f+T_0\mathcal{N}}\nabla\psi\varphi_t\right.\\
&\quad\left.-\frac{(\varphi+\mathcal{N})^2}{(f+T_0\mathcal{N})^2}\nabla\psi f_t\right\}f_t\,dx,\\
I_{14}&=\tau\int_\Omega\left\{\frac{5}{3}\nabla\mathcal{V}\cdot\nabla\varphi_t+\nabla\mathcal{N}\cdot\nabla\psi_t\right\}f_t\,dx\\
&\quad-\tau\int_\Omega\frac{2}{3}\left\{\frac{(\varphi+2\mathcal{N})\nabla\mathcal{N}}{f+T_0\mathcal{N}}\varphi-\frac{\mathcal{N}\nabla\mathcal{V}}{T_0(f+T_0\mathcal{N})}f\right\}\cdot\nabla\psi_t f_t\,dx,\\
I_{15}&=-\tau\int_\Omega\frac{2}{3}\nabla\mathcal{V}\cdot\left\{\frac{(\varphi+\mathcal{N})^2}{f+T_0\mathcal{N}}\nabla\psi_t+\frac{2(\varphi+\mathcal{N})(\nabla\psi+\nabla\mathcal{V})}{f+T_0\mathcal{N}}\varphi_t\right.\\
&\quad\left.-\frac{\mathcal{N}^2\nabla\mathcal{V}}{(f+T_0\mathcal{N})^2}f_t-\frac{(\varphi+\mathcal{N})^2}{(f+T_0\mathcal{N})^2}\nabla\psi f_t-\frac{(\varphi+2\mathcal{N})\nabla\mathcal{V}}{(f+T_0\mathcal{N})^2}\varphi f_t\right\}f_t\,dx,\\
I_{16}&=-\tau\int_\Omega\frac{2}{3}\nabla\psi\cdot\nabla\mathcal{V}\left\{\frac{2(\varphi+\mathcal{N})}{f+T_0\mathcal{N}}\varphi_t-\frac{\mathcal{N}^2}{(f+T_0\mathcal{N})^2}f_t-\frac{(\varphi+2\mathcal{N})}{(f+T_0\mathcal{N})^2}\varphi f_t\right\}f_t\,dx,\\
I_{17}&=-\int_\Omega(\tau\mathcal{N}-T_0)\varphi_t f_t\,dx.
\end{aligned}
$$

By the smallness of (2.3), (2.20), due to the Gagliardo–Nirenberg inequality and Hölder inequality, I_i, $i=13,\dots,17$, can be governed by

$$
\begin{aligned}
I_{13}&\leqslant O(\delta_0)\tau\int_\Omega\left(|\nabla\varphi_t|^2+\varphi_t^2+f_t^2\right)dx,\\
I_{14}&\leqslant O(\delta_0)\tau\int_\Omega\left(|\nabla\varphi_t|^2+\varphi_t^2+f_t^2\right)dx,\\
I_{15}&\leqslant O(\delta_0)\tau\int_\Omega\left(|\nabla\varphi_t|^2+\varphi_t^2+f_t^2\right)dx,\\
I_{16}&\leqslant O(\delta_0)\tau\int_\Omega\left(\varphi_t^2+f_t^2\right)dx,\\
I_{17}&\leqslant\frac{1}{4}\int_\Omega f_t^2\,dx+c_2(1+\tau^2)\int_\Omega\varphi_t^2\,dx.
\end{aligned}
$$

From the above inequalities, it gives

$$
\frac{1}{2}\frac{d}{dt}\int_\Omega\tau f_t^2\,dx+\int_\Omega\left(\tau|\nabla f_t|^2+\frac{3}{4}f_t^2\right)dx
$$

$$\leqslant O(\delta_0)\tau \int_\Omega \left(|\nabla\varphi_t|^2 + \varphi_t^2 + f_t^2\right) dx + c_2(1+\tau^2) \int_\Omega \varphi_t^2\, dx. \tag{2.33}$$

Multiplying (2.31) by $(1 + \frac{c_2}{c_1}(1+\tau))$, adding it to (2.33), and choosing δ_0 suitable small, we obtain

$$\frac{d}{dt}\int_\Omega \left[(1+\tau)\varphi_t^2 + \tau f_t^2\right] dx + \int_\Omega \left[(1+\tau)|\nabla\varphi_t|^2 + \tau|\nabla f_t|^2\right] dx$$
$$+ a_1 \int_\Omega \left(\varphi_t^2 + f_t^2\right) dx \leqslant 0. \tag{2.34}$$

Combining (2.27) with (2.34), we obtain the following estimate:

$$\frac{d}{dt}\int_\Omega \left[(1+\tau)\left(\varphi_t^2 + |\nabla\varphi|^2 + \varphi^2\right) + \tau\left(f_t^2 + |\nabla f|^2 + f^2\right)\right] dx$$
$$+ \int_\Omega \left[(1+\tau)\left(|\nabla\varphi_t|^2 + |\Delta\varphi|^2 + |\nabla\varphi|^2\right) + \tau\left(|\nabla f_t|^2 + |\Delta f|^2 + |\nabla f|^2\right)\right] dx$$
$$+ a_1 \int_\Omega \left(\varphi_t^2 + f_t^2 + |\nabla\varphi|^2 + |\nabla f|^2 + \varphi^2 + f^2\right) dx \leqslant 0. \tag{2.35}$$

Integrating (2.35) from 0 to t with respect to t, we get

$$(1+\tau)\int_\Omega \left(\varphi_t^2 + |\nabla\varphi|^2 + \varphi^2\right) dx + (1+\tau)\int_0^t \int_\Omega \left(|\nabla\varphi_t|^2 + |\Delta\varphi|^2 + |\nabla\varphi|^2\right) dx\, ds$$
$$+ \tau \int_\Omega \left(f_t^2 + |\nabla f|^2 + f^2\right) dx + \tau \int_0^t \int_\Omega \left(|\nabla f_t|^2 + |\Delta f|^2 + |\nabla f|^2\right) dx\, ds$$
$$+ a_1 \int_0^t \int_\Omega \left(\varphi_t^2 + f_t^2 + |\nabla\varphi|^2 + |\nabla f|^2 + \varphi^2 + f^2\right) dx\, ds$$
$$\leqslant c_0(1+\tau)\left(\left\|\varphi(\cdot,0)\right\|_{H^2}^2 + \left\|f(\cdot,0)\right\|_{H^2}^2\right), \tag{2.36}$$

where c_0 is a positive constant independent of τ.

From Eqs. (2.17) and (2.18), we can get the estimate for $\Delta\varphi$ and Δf,

$$\int_\Omega \left(|\Delta\varphi|^2 + |\Delta f|^2\right) dx \leqslant O(1) \int_\Omega \left(\varphi_t^2 + |\nabla\varphi|^2 + f_t^2 + |\nabla f|^2\right) dx$$
$$+ O(1)\left(1 + \frac{1}{\tau}\right) \int_\Omega (\varphi^2 + f^2)\, dx. \qquad \square \tag{2.37}$$

Remark 2.7. By the standard argument, Theorem 1.1 can be proved with the help of Theorem 2.2 and Lemma 2.6.

3. Relaxation time limit

3.1. A priori estimates for arbitrary large initial data

In this section, we will first get more a priori estimates on the system (1.6) for arbitrary large initial data. We rewrite (1.6) in the following form, where the unknowns are ρ, T and V:

$$\rho_t - \operatorname{div}\left(\nabla\rho - \frac{\rho}{T}\nabla V\right) = 0, \tag{3.1}$$

$$(\rho T)_t - \operatorname{div}\big(\nabla(\rho T) - \rho\nabla V\big) = -\frac{2}{3}\nabla V\cdot\left(\nabla\rho - \frac{\rho}{T}\nabla V\right) - \frac{(T-T_0)\rho}{\tau}, \tag{3.2}$$

$$\Delta V = \rho - C(x). \tag{3.3}$$

Theorem 3.1. *If (ρ, T) is a smooth positive solution to the problem* (3.1)–(3.3), (1.7)–(1.8), *then we have the following a priori estimate*:

$$\eta(t) + \int_0^t\int_\Omega\left[\frac{3\rho}{2T^2}|\nabla T|^2 + \left(\frac{\nabla\rho}{\sqrt{\rho}} - \frac{\sqrt{\rho}}{T}\nabla V\right)^2\right]dx\,dt \leqslant \eta(0), \tag{3.4}$$

where

$$\eta(t) = \int_\Omega\left[\rho\ln\rho - \frac{5}{2}\rho + \eta_0 + \frac{3}{2}\rho\left(\frac{T}{T_0} - \ln T\right) + \frac{1}{T_0}|\nabla V|^2\right]dx, \tag{3.5}$$

$\eta_0 = e^{\frac{5}{2}}$ *is a positive constant, and the following quantities are bounded by the initial entropy $\eta(0)$, $|\Omega|$ and T_0 (uniformly in τ),*

$$\|\rho\|_{L^\infty(0,\infty;L^1(\Omega))},\quad \|\rho T\|_{L^\infty(0,\infty;L^1(\Omega))},\quad \|\nabla V\|_{L^\infty(0,\infty;L^2(\Omega))}.$$

Remark 3.2. This entropy estimates also hold for mixed Dirichlet and Neumann boundary problem.

Remark 3.3. To prove the relaxation time limit, we only need the bounds of

$$\|\rho\|_{L^\infty(0,\infty;L^1(\Omega))},\qquad \|\rho T\|_{L^\infty(0,\infty;L^1(\Omega))}\quad\text{and}\quad \|\nabla V\|_{L^\infty(0,\infty;L^2(\Omega))}.$$

In fact, by entropy inequality and certain technical calculations, we can also get some estimates for the currents J_1 and J_2, which would be helpful in getting the existence for large data.

Proof. First we will show the entropy inequality (3.4). Multiplying (3.1) by $\ln\frac{\rho}{T^{3/2}} - \frac{1}{T_0}V$ and making the integration by parts, we have

$$\int_0^t\int_\Omega\left[\rho_t\left(\ln\rho - \frac{3}{2}\ln T - \frac{1}{T_0}V\right)\right]dx\,dt$$

$$+\int_0^t\int_\Omega\left(\nabla\rho-\frac{\rho}{T}\nabla V\right)\cdot\nabla\left[\ln\frac{\rho}{T^{\frac{3}{2}}}-\frac{1}{T_0}V\right]dx\,dt=0. \tag{3.6}$$

We denote the two integrals as I_1 and I_2. By $\rho_t=\Delta V_t$, which can be obtained from (3.3), we have

$$\begin{aligned}I_1&=\int_0^t\int_\Omega\left[(\rho\ln\rho)_t-\rho_t-\frac{3}{2}(\rho\ln T)_t+\frac{3}{2}\frac{\rho}{T}T_t+\frac{1}{T_0}\big(|\nabla V|^2\big)_t\right]dx\,dt\\&=\eta_1(t)+\int_0^t\int_\Omega\frac{3}{2}\frac{\rho}{T}T_t\,dx\,dt-\eta_1(0),\end{aligned} \tag{3.7}$$

where

$$\eta_1(t)=\int_\Omega\left[\rho\ln\rho-\rho+\eta_0-\frac{3}{2}\rho\ln T+\frac{1}{T_0}|\nabla V|^2\right]dx.$$

The calculation on I_2 shows

$$\begin{aligned}I_2&=\int_0^t\int_\Omega\left(\nabla\rho-\frac{\rho}{T}\nabla V\right)\cdot\left(\frac{\nabla\rho}{\rho}-\frac{3}{2T}\nabla T-\frac{1}{T_0}\nabla V\right)dx\,dt\\&=\int_0^t\int_\Omega\left[\frac{|\nabla\rho|^2}{\rho}-\frac{\nabla\rho\cdot\nabla V}{T}-\frac{3\nabla\rho\cdot\nabla T}{2T}+\frac{3\rho\nabla T\cdot\nabla V}{2T^2}-\frac{\nabla\rho\cdot\nabla V}{T_0}\right.\\&\quad\left.+\frac{\rho|\nabla V|^2}{T_0T}\right]dx\,dt.\end{aligned} \tag{3.8}$$

Multiplying (3.2) by $(-\frac{3}{2T})-(-\frac{3}{2T_0})$, we have

$$\begin{aligned}&\int_0^t\int_\Omega\frac{3}{2}(\rho T)_t\left[\left(-\frac{1}{T}\right)-\left(-\frac{1}{T_0}\right)\right]dx\,dt\\&\quad+\int_0^t\int_\Omega\frac{3}{2}\big(\nabla(\rho T)-\rho\nabla V\big)\cdot\nabla\left(-\frac{1}{T}\right)dx\,dt\\&\quad+\int_0^t\int_\Omega\left[\nabla V\cdot\left(\nabla\rho-\frac{\rho}{T}\nabla V\right)\left(\frac{1}{T_0}-\frac{1}{T}\right)\right]dx\,dt\\&\quad-\frac{3}{2\tau}\int_0^t\int_\Omega\rho(T_0-T)\left(\frac{1}{T_0}-\frac{1}{T}\right)dx\,dt=0.\end{aligned} \tag{3.9}$$

We denote the three integrals in the above equation by I_3, I_4 and I_5, which can be treated as follows:

$$I_3 = \int_0^t \int_\Omega \left[-\frac{3}{2}\rho_t - \frac{3}{2}\frac{\rho}{T}T_t + \frac{3}{2}\frac{1}{T_0}(\rho T)_t \right] dx\,dt$$
$$= \eta_2(t) - \int_0^t \int_\Omega \frac{3}{2}\frac{\rho}{T}T_t\,dx\,dt - \eta_2(0), \tag{3.10}$$

where

$$\eta_2(t) = \int_\Omega \left[-\frac{3}{2}\rho + \frac{3}{2T_0}\rho T \right] dx.$$

Since $-\frac{3\rho}{2\tau}(T_0 - T)(\frac{1}{T_0} - \frac{1}{T}) \geqslant 0$ for a nonnegative solution, we have

$$I_4 + I_5 \geqslant \int_0^t \int_\Omega \left[\frac{3}{2}\frac{\nabla T}{T^2}\big(\nabla(\rho T) - \rho\nabla V\big) + \nabla V \cdot \left(\nabla\rho - \frac{\rho}{T}\nabla V\right)\left(\frac{1}{T_0} - \frac{1}{T}\right) \right] dx\,dt$$
$$= \int_0^t \int_\Omega \left[\frac{3\rho}{2T^2}|\nabla T|^2 + \frac{3\nabla\rho\cdot\nabla T}{2T} - \frac{3\rho\nabla T\cdot\nabla V}{2T^2} \right.$$
$$\left. + \frac{\rho}{T^2}|\nabla V|^2 - \frac{\nabla\rho\cdot\nabla V}{T} - \frac{\rho|\nabla V|^2}{T_0 T} + \frac{\nabla V\cdot\nabla\rho}{T_0} \right] dx\,dt. \tag{3.11}$$

Combined (3.6) to (3.9) together, with the calculation in (3.7), (3.8), (3.10) and (3.11), we have, for $\eta(t) = \eta_1(t) + \eta_2(t)$, that

$$\eta(t) + \int_0^t \int_\Omega \left[\frac{3\rho}{2T^2}|\nabla T|^2 + \frac{|\nabla\rho|^2}{\rho} - 2\frac{\nabla\rho\cdot\nabla V}{T} + \frac{\rho}{T^2}|\nabla V|^2 \right] dx\,dt \leqslant \eta(0).$$

This gives the entropy inequality (3.4).

Let $E(t)$ be the part of the entropy which does not contain the electric potential part, i.e.

$$E(t) = \eta(t) - \int_\Omega \frac{1}{T_0}|\nabla V|^2\,dx,$$

then $E(t) \leqslant \eta(0)$ for all $t \geqslant 0$. We can use the fundamental inequality $\frac{T}{T_0} - \ln T \geqslant 1 - \ln T_0 \geqslant 0$ for any $T > 0$, where, without loss of generality, we assume $0 < T_0 \leqslant 1$, and the one $\rho\ln\rho - \frac{5}{2}\rho + \eta_0 \geqslant 0$ for any $\rho > 0$, to get some estimates on ρ from the entropy. This gives that

$$\rho \in L^\infty\big(0,\infty; L^1(\Omega)\big), \qquad \nabla V \in L^\infty\big(0,\infty; L^2(\Omega)\big). \tag{3.12}$$

Then from the a priori estimates obtained above, we have

$$\int_\Omega \frac{3}{2}\rho\left(\frac{T}{T_0}-\ln T\right)dx \leqslant C. \tag{3.13}$$

On the other hand, by $\frac{T}{2T_0}-\ln T \geqslant 1-\ln(2T_0)$, it holds that

$$\begin{aligned}\int_\Omega \frac{3}{2}\rho\left(\frac{T}{T_0}-\ln T\right)dx &= \int_\Omega \frac{3}{2}\rho\left(\frac{T}{2T_0}+\frac{T}{2T_0}-\ln T\right)dx\\ &\geqslant \int_\Omega \frac{3\rho T}{4T_0}\,dx+\int_\Omega \frac{3}{2}\big(1-\ln(2T_0)\big)\rho\,dx.\end{aligned} \tag{3.14}$$

Thus by (3.12)–(3.14), we have

$$\rho T \in L^\infty\big(0,\infty;L^1(\Omega)\big). \tag{3.15}$$

In the following, we get further estimates from the boundedness of $E(t)$ in another point of view, i.e. the convex property of the entropy functional. Let $u_1=\ln\frac{\rho}{T^{3/2}}$, $u_2=-\frac{1}{T}$,

$$\begin{aligned}E(t) &= \int_\Omega \left[\rho\ln\frac{\rho}{T^{\frac{3}{2}}}+\frac{3}{2}\rho T\left(-\frac{1}{T}+\frac{1}{T_0}\right)-\rho+T_0^{\frac{3}{2}}\right]dx\\
&= \int_\Omega \left[(-u_2)^{-\frac{3}{2}}e^{u_1}u_1+\frac{3}{2}(-u_2)^{-\frac{5}{2}}e^{u_1}\left(u_2+\frac{1}{T_0}\right)\right.\\
&\qquad \left.-\left((-u_2)^{-\frac{3}{2}}e^{u_1}-T_0^{\frac{3}{2}}\right)\right]dx\\
&\geqslant \int_\Omega \left[u_1,\ u_2+\frac{1}{T_0}\right]\begin{bmatrix}(-u_2)^{-\frac{3}{2}}e^{u_1} & \frac{3}{2}(-u_2)^{-\frac{5}{2}}e^{u_1}\\ \frac{3}{2}(-u_2)^{-\frac{5}{2}}e^{u_1} & \frac{15}{4}(-u_2)^{-\frac{7}{2}}e^{u_1}\end{bmatrix}\begin{bmatrix}u_1\\ u_2+\frac{1}{T_0}\end{bmatrix}dx\\
&= \int_\Omega \left[u_1,\ u_2+\frac{1}{T_0}\right]\begin{bmatrix}\rho & \frac{3}{2}\rho T\\ \frac{3}{2}\rho T & \frac{15}{4}\rho T^2\end{bmatrix}\begin{bmatrix}u_1\\ u_2+\frac{1}{T_0}\end{bmatrix}dx\\
&= \int_\Omega \left[\frac{1}{4}\rho u_1^2+\frac{3}{4}\rho\left(\frac{T}{T_0}-1\right)^2+3\rho\left(\frac{1}{2}u_1+T\left(u_2+\frac{1}{T_0}\right)\right)^2\right]dx\\
&= \int_\Omega \left[\frac{1}{4}\rho u_1^2+\frac{3}{4T_0^2}\rho(T-T_0)^2+3\rho\left(\frac{1}{2}u_1+\frac{T-T_0}{T_0}\right)^2\right]dx,\end{aligned} \tag{3.16}$$

where we have used the convexity of functional $F(u_1,u_2)=(-u_2)^{-\frac{3}{2}}e^{u_1}$. It follows from (3.16) and (3.4) that

$$\sqrt{\rho}\,T \in L^\infty\big(0,\infty;L^2(\Omega)\big). \qquad \square \tag{3.17}$$

3.2. The convergence in one space dimensional case

Now we go to obtain the convergence to drift–diffusion model for one-dimensional case. This restriction is caused by the Sobolev compact embedding theorem. We consider the solution $(\rho, \mathcal{E}, E)$ to (1.6)–(1.8). For any fixed τ and t, we introduce the following variables:

$$\rho^\tau(x,t) = \rho(x,t), \qquad \mathcal{E}^\tau(x,t) = \mathcal{E}(x,t),$$
$$T^\tau(x,t) = T(x,t), \qquad E^\tau(x,t) = E(x,t),$$

which satisfy

$$\rho_t^\tau + \operatorname{div} j_1^\tau = 0, \qquad j_1^\tau = \frac{(\rho^\tau)^2}{\mathcal{E}^\tau} E^\tau - \nabla \rho^\tau, \tag{3.18}$$

$$\tau \mathcal{E}_t^\tau + \tau \operatorname{div} j_2^\tau = \tau \frac{2}{3} E^\tau \cdot j_1^\tau - (\mathcal{E}^\tau - T_0 \rho^\tau), \qquad j_2^\tau = \rho^\tau E^\tau - \nabla \mathcal{E}^\tau, \tag{3.19}$$

$$\operatorname{div} E^\tau = \rho^\tau - C(x). \tag{3.20}$$

By using (2.3) and (2.36), we obtain the following estimates uniformly in τ:

$$\rho_t^\tau \in L^2\big(0,t; H^1(\Omega)\big), \qquad \rho^\tau \in L^\infty\big(0,t; H^1(\Omega)\big) \cap L^2\big(0,t; H^2(\Omega)\big),$$
$$\mathcal{E}_t^\tau \in L^2\big(0,t; L^2(\Omega)\big), \qquad \mathcal{E}^\tau \in L^2\big(0,t; H^1(\Omega)\big). \tag{3.21}$$

From the above estimates and (3.17), with the help of the compact embedding of H^1, it holds that

$$\|\mathcal{E}^\tau\|_{L^\infty(0,t;L^2(\Omega))} \leqslant \|\sqrt{\rho^\tau}\, T^\tau\|_{L^\infty(0,t;L^2(\Omega))} \cdot \|\sqrt{\rho^\tau}\|_{L^\infty(Q_t)}$$

by the interpolation inequality.

From the Aubin's lemma [18] on compactness, we have

$$L^2\big(0,t; H^2(\Omega)\big) \cap H^1\big(0,t; H^1(\Omega)\big) \hookrightarrow L^2\big(0,t; C^{1,\alpha}(\Omega)\big),$$

for $\alpha \in (0, \frac{1}{2})$. Then as $\tau \to 0$, for a subsequence of ρ^τ,

$$\rho^\tau \to \rho \quad \text{strongly in } L^2\big(0,t; C^{1,\alpha}(\Omega)\big) \cap C\big(0,t; C^\alpha(\Omega)\big),$$
$$\rho^\tau \rightharpoonup \rho \quad \text{weakly in } L^2\big(0,t; H^2(\Omega)\big).$$

Similarly, we obtain, for a subsequence of $\mathcal{E}^\tau$,

$$\mathcal{E}^\tau \to \mathcal{E} \quad \text{strongly in } L^2\big(0,t; C^\alpha(\Omega)\big),$$
$$\mathcal{E}^\tau \rightharpoonup \mathcal{E} \quad \text{weakly in } L^2\big(0,t; H^1(\Omega)\big).$$

By elliptic theory, we obtain the boundedness of E^τ in $(L^\infty(0,t; H^2(\Omega)) \cap L^2(0,t; H^3(\Omega)) \cap H^1(0,t; H^2(\Omega)))^3$ uniformly in τ, and for a subsequence

$$E^\tau \to E \quad \text{strongly in } \big(L^2\big(0,t; C^{2,\alpha}(\Omega)\big) \cap C\big(0,t; C^{1,\alpha}(\Omega)\big)\big)^3.$$

Denote $\mathcal{W}^\tau = \mathcal{E}^{\tau 2}$. Then we have

$$\|\nabla\mathcal{W}^\tau\|_{L^2(0,t;L^1(\Omega))} \leqslant \|\mathcal{E}^\tau\|_{L^\infty(0,t;L^2(\Omega))}\cdot\|\nabla\mathcal{E}^\tau\|_{L^2(Q_t)},$$
$$\|\mathcal{W}_t^\tau\|_{L^2(0,t;L^1(\Omega))} \leqslant 2\|\mathcal{E}^\tau\|_{L^\infty(0,t;L^2(\Omega))}\cdot\|\mathcal{E}_t^\tau\|_{L^2(Q_t)}.$$

Thus it yields that $\mathcal{W}^\tau\in L^2(0,t;W^{1,1}(\Omega))\cap H^1(0,t;L^1(\Omega))$ uniformly in τ. By Aubin's lemma, we obtain for a subsequence

$$\mathcal{E}^{\tau 2}\to\mathcal{E}^2\quad\text{strongly in } L^2\big(0,t;L^p(\Omega)\big),\ \text{for } 1\leqslant p<\infty.$$

For the positive solutions, (3.18)–(3.19) are equivalent to the following equations:

$$(\mathcal{E}^\tau)^2\rho_t^\tau-\mathcal{E}^\tau\operatorname{div}\big(\mathcal{E}^\tau\nabla\rho^\tau-(\rho^\tau)^2E^\tau\big)+\big[\mathcal{E}^\tau\nabla\rho^\tau-(\rho^\tau)^2E^\tau\big]\cdot\nabla\mathcal{E}^\tau=0, \tag{3.22}$$

$$\tau\mathcal{E}^\tau\mathcal{E}_t^\tau-\tau\mathcal{E}^\tau\operatorname{div}(\nabla\mathcal{E}^\tau-\rho^\tau E^\tau)+\tau\frac{2}{3}E^\tau\cdot\big[\mathcal{E}^\tau\nabla\rho^\tau-(\rho^\tau)^2E^\tau\big]+\mathcal{E}^\tau(\mathcal{E}^\tau-T_0\rho^\tau)$$
$$=0. \tag{3.23}$$

On the one hand, as $\tau\to 0$, we get, for all $\phi\in C_0^\infty(Q_t)$, that

$$\tau\int_0^t\int_\Omega\mathcal{E}^\tau\mathcal{E}_t^\tau\phi\,dx\,dt\leqslant\tau\|\mathcal{E}^\tau\|_{L^2(Q_t)}\|\mathcal{E}_t^\tau\|_{L^2(Q_t)}\|\phi\|_{L^\infty(Q_t)}\leqslant C_0\tau\to 0,$$

$$\tau\int_0^t\int_\Omega(\nabla\mathcal{E}^\tau-\rho^\tau E^\tau)\cdot\nabla(\mathcal{E}^\tau\phi)\,dx\,dt$$
$$=\tau\int_0^t\int_\Omega\big(|\nabla\mathcal{E}^\tau|^2\phi-\rho^\tau E^\tau\cdot\nabla\mathcal{E}^\tau\phi-\rho^\tau\mathcal{E}^\tau E^\tau\cdot\nabla\phi-\mathcal{E}^{\tau 2}\Delta\phi\big)\,dx\,dt$$
$$\leqslant\tau\big(\|\nabla\mathcal{E}^\tau\|^2_{L^2(Q_t)}\|\phi\|_{L^\infty(Q_t)}+\|\phi\|_{L^\infty(Q_t)}\|E^\tau\|_{L^\infty(Q_t)}\|\nabla\mathcal{E}^\tau\|_{L^2(Q_t)}\|\rho^\tau\|_{L^2(Q_t)}$$
$$+\|\nabla\phi\|_{L^\infty(Q_t)}\|E^\tau\|_{L^\infty(Q_t)}\|\mathcal{E}^\tau\|_{L^2(Q_t)}\|\rho^\tau\|_{L^2(Q_t)}+\|\mathcal{E}^\tau\|^2_{L^2(Q_t)}\|\Delta\phi\|_{L^\infty(Q_t)}\big)$$
$$\leqslant C_0\tau\to 0,$$

$$\frac{2\tau}{3}\int_0^t\int_\Omega\big[\mathcal{E}^\tau\nabla\rho^\tau-(\rho^\tau)^2E^\tau\big]\cdot E^\tau\phi\,dx\,dt$$
$$\leqslant\tau\big(\|E^\tau\|_{L^\infty(Q_t)}\|\phi\|_{L^\infty(Q_t)}\|\mathcal{E}^\tau\|_{L^2(Q_t)}\|\nabla\rho^\tau\|_{L^2(Q_t)}$$
$$+\|E^\tau\|_{L^\infty(Q_t)}\|\phi\|_{L^\infty(Q_t)}\|\rho^\tau\|^2_{L^2(Q_t)}\big)$$
$$\leqslant C_0\tau\to 0,$$

where the positive constant C_0 is independent of τ.

On the other hand, by the above convergence, we have

$$\int_0^t\int_\Omega\mathcal{E}^{\tau 2}\rho_t^\tau\phi\,dx\,dt\to\int_0^t\int_\Omega\mathcal{E}^2\rho_t\phi\,dx\,dt,$$

$$\int_0^t\int_\Omega \rho^{\tau 2}E^\tau\cdot\nabla\mathcal{E}^\tau\phi\,dx\,dt\to\int_0^t\int_\Omega \rho^2 E\cdot\nabla\mathcal{E}\phi\,dx\,dt,$$

$$\int_0^t\int_\Omega \rho^{\tau 2}\mathcal{E}^\tau E^\tau\cdot\nabla\phi\,dx\,dt\to\int_0^t\int_\Omega \rho^2\mathcal{E}E\cdot\nabla\phi\,dx\,dt,$$

$$\int_0^t\int_\Omega \mathcal{E}^{\tau 2}\Delta\rho^\tau\phi\,dx\,dt\to\int_0^t\int_\Omega \mathcal{E}^2\Delta\rho\phi\,dx\,dt,$$

for all $\phi\in C_0^\infty(Q_t)$ as $\tau\to 0$.

From (3.20), (3.22) and (3.23), we have

$$\int_0^t\int_\Omega \mathcal{E}^{\tau 2}\rho_t^\tau\phi\,dx\,dt-\int_0^t\int_\Omega \rho^{\tau 2}E^\tau\cdot(2\nabla\mathcal{E}^\tau\phi+\mathcal{E}^\tau\nabla\phi)\,dx\,dt$$
$$-\int_0^t\int_\Omega \mathcal{E}^{\tau 2}\Delta\rho^\tau\phi\,dx\,dt=0,$$

$$\tau\int_0^t\int_\Omega \mathcal{E}^\tau\mathcal{E}_t^\tau\phi\,dx\,dt+\tau\int_0^t\int_\Omega(\nabla\mathcal{E}^\tau-\rho^\tau E^\tau)\cdot\nabla(\mathcal{E}^\tau\phi)\,dx\,dt$$
$$+\tau\int_0^t\int_\Omega\frac{2}{3}\big[\mathcal{E}^\tau\nabla\rho^\tau-(\rho^\tau)^2E^\tau\big]\cdot E^\tau\phi\,dx\,dt+\int_0^t\int_\Omega\mathcal{E}^\tau(\mathcal{E}^\tau-T_0\rho^\tau)\phi\,dx\,dt=0,$$

$$-\int_0^t\int_\Omega E^\tau\cdot\nabla\phi\,dx\,dt=\int_0^t\int_\Omega\big(\rho^\tau-C(x)\big)\phi\,dx\,dt,$$

for $(x,t)\in(Q_t)$ and for all $\phi\in C_0^\infty(Q_t)$.

Let $\tau\to 0$. Then we obtain

$$\int_0^t\int_\Omega \mathcal{E}^2\rho_t\phi\,dx\,dt-\int_0^t\int_\Omega \rho^2E\cdot(2\nabla\mathcal{E}\phi+\mathcal{E}\nabla\phi)\,dx\,dt-\int_0^t\int_\Omega \mathcal{E}^2\Delta\rho\phi\,dx\,dt=0,$$

$$-\int_0^t\int_\Omega \mathcal{E}(\mathcal{E}-T_0\rho)\phi\,dx\,dt=0,$$

$$-\int_0^t\int_\Omega E\cdot\nabla\phi\,dx\,dt=\int_0^t\int_\Omega\big(\rho-C(x)\big)\phi\,dx\,dt,$$

for all $\phi\in C_0^\infty(Q_t)$. From the second equality, we have $\mathcal{E}=T_0\rho$. Then we can prove Theorem 1.2.

Remark 3.4. We could obtain Theorem 1.2 for the three-dimensional case if we assumed $\mathcal{E}^\tau \in L^\infty(0, t; H^1(\Omega))$.

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