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An Application of Crystal Bases to Representations of Affine Lie Algebras

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It has been shown that up to degree shifts any integrable highest weight (or standard) module of level k for an affine Lie algebra $\mathfrak{g}$ can be imbedded in the tensor product of k copies of level one integrable highest weight modules. When the affine Lie algebra $\mathfrak{g}$ is of classical type the path realizations of the crystal bases for the level one $\mathscr{U}_q(\mathfrak{g})$-modules have been used to obtain these results. © 1995 Academic Press, Inc.

INTRODUCTION

The representation theory of affine Lie algebras (see [5]) has been a very important area of research during the last two decades because of its interactions with several other areas of mathematics and physics. One of the crucial features of affine Lie algebra representation theory is the existence of explicit constructions of certain level one representations in terms of vertex operators. The level of the representation is the number by which the canonical central element of the affine Lie algebra acts. One way to give explicit constructions of higher level representations is to realize it inside the tensor product of known level one representations and then use the explicit realizations of level one representations to obtain the desired representations. It has been a well-known conjecture for at least the last 10 years that any integrable highest weight (or standard) representation of level $k \in \mathbb{Z}^+$ of the affine Lie algebra is contained in the tensor product of k copies of level one integrable highest weight representations. For the affine Lie algebra $A_n^{(1)}$ and $C_n^{(1)}$ the conjecture is clearly true since all fundamental representations are of level one. To our knowledge, so far there is no published proof of this conjecture. Recently, in order to show that certain results on vertex operator algebras obtained by Dong–Lepowsky [2] are as general as the corresponding results obtained by

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Frenkel–Zhu [3] from a differential viewpoint, Xie [13] tried to prove this conjecture by direct computations. He succeeded in proving the conjecture for the affine Lie algebras $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$, and $G_2^{(1)}$. However, he realized that to prove the conjecture for arbitrary rank affine Lie algebras $B_n^{(1)}$ and $D_n^{(1)}$ using his computational techniques is almost impossible. In fact, he proved the conjecture in these two cases for $n \leq 5$ only. Analyzing Xie's work, we realized that for arbitrary rank affine Lie algebras of classical type it is more practical to use the crystal base approach.

In this paper we prove the conjecture for arbitrary rank affine Lie algebras $\mathfrak{g}$ $(= B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$, or $D_{n+1}^{(2)})$ using the path realizations of the crystal bases for the level one integrable highest weight representations of the quantum affine Lie algebras $\mathscr{U}_q(\mathfrak{g})$ obtained in [8, 9]. The crystal bases introduced by Kashiwara [6] can be thought of as a specialization of the global crystal base [7] or canonical base [11] at $q = 0$, where q denotes the quantum parameter. The crystal base theory provides a nice combinatorial tool to study tensor product decompositions which we use to prove the conjecture for the affine Lie algebras of classical type. We also prove the conjecture for the algebras $E_6^{(2)}$ and $D_4^{(3)}$ by using the computational techniques developed in [13]. Note that for the affine Lie algebra $A_2^{(2)}$ the result follows from [12] (see also [1]). Thus our results combined with those of Xie [12, 13] prove the conjecture for all affine Lie algebras.

1. Preliminaries and Notations

In this section we recall some basic concepts about affine Kac–Moody Lie algebras and their integrable highest weight (also known as standard) representations. For further details, see [5].

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Kac–Moody Lie algebra associated with the affine GCM $A = (a_{ij})_{i,j=0}^n$. Let $\mathfrak{g}' = \mathfrak{g}'(A) = [\mathfrak{g}(A), \mathfrak{g}(A)]$ be the derived algebra of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{g}' \oplus \mathbb{C}d$, where d is the scaling element of $\mathfrak{g}$. Let $\{e_i, f_i, h_i | 0 \leq i \leq n\}$ be the set of Chevalley generators of $\mathfrak{g}$. Then $\mathfrak{h}' = \oplus_{i=0}^n \mathbb{C}\mathfrak{h}_i$ and $\mathfrak{h} = \mathfrak{h}' \oplus \mathbb{C}\mathfrak{d}$ are the Cartan subalgebras of $\mathfrak{g}'$ and $\mathfrak{g}$, respectively. Let $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ denote the simple roots, with $\alpha_0(d) = 1$, $\alpha_i(d) = 0$, $i = 1, 2, \ldots, n$. There exist positive integers $a_0, a_1, \ldots, a_n$ and $a_0^\vee, a_1^\vee, \ldots, a_n^\vee$ with g.c.d. $(a_0, a_1, \ldots, a_n) = 1$ and g.c.d. $(a_0^\vee, a_1^\vee, \ldots, a_n^\vee) = 1$ such that

$$A \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{and} \quad (a_0^\vee, a_1^\vee, \ldots, a_n^\vee)A = (0, 0, \ldots, 0).$$

Then $c = \sum_{i=0}^n a_i^\vee h_i \in \mathfrak{g}$ and $\delta = \sum_{i=0}^n a_i \alpha_i \in \mathfrak{h}^*$ are the canonical central element and null root, respectively. Note that $\delta(d) = a_0$. Let $P \subset \mathfrak{h}^*$ denote the weight lattice. Then $P^+ = \{\lambda \in P | \lambda(h_i) \in \mathbb{Z}_{\geq 0},\ 0 \leq i \leq n\}$ is the set of dominant integral weights. It is known that for $\lambda \in P^+$ there exists a unique highest weight $\mathfrak{g}$-module $L(\lambda)$ called the integrable highest weight (or standard) module which has the following properties:

(1) There exists a unique (up to scalar multiple) nonzero vector $v_\lambda \in L(\lambda)$ such that

(i) $hv_\lambda = \lambda(h)$, for $i = 0, 1, \ldots, n$

(ii) $e_i v_\lambda = 0$, for $i = 0, 1, \ldots, n$

(iii) $f_i^{(\lambda(h_i)+1} v_\lambda = 0$, for $i = 0, 1, \ldots, n$

(iv) $L(\lambda) = \mathscr{U}(\mathfrak{g}) v_\lambda$, and

(2) the canonical central element c acts as the scalar $l = \lambda(c)$ which is called the level of $L(\lambda)$.

The integrable highest weight $\mathfrak{g}'(A)$-module is defined similarly. It is easy to see that for any dominant integral weight $\lambda \in P^+$ and $k \in \mathbb{C}$,

$$L(\lambda + k\delta) \cong L(\lambda) \otimes L(k\delta) \text{ as } \mathfrak{g}\text{-modules} \tag{1.1}$$

and

$$L(\lambda) \cong L(\lambda + k\delta) \text{ as } \mathfrak{g}'\text{-modules.} \tag{1.2}$$

The dominant integral weights $\Lambda_i \in P^+$ defined by $\Lambda_i(h_j) = \delta_{ij}$ and $\Lambda_i(d) = 0$ are called the fundamental weights and $L(\Lambda_i)$ are called the fundamental $\mathfrak{g}$-modules. It is easy to see that for $0 \leq i, j \leq n$,

$$L(\Lambda_i + \Lambda_j) \subset L(\Lambda_i) \otimes L(\Lambda_j). \tag{1.3}$$

Since any dominant integral weight $\lambda \in P^+$ can be written uniquely as

$$\lambda = \sum_{i=0}^{n} k_i \Lambda_i + k_\delta \delta, \qquad k_i \in \mathbb{Z}^+, k_\delta \in \mathbb{C}, \tag{1.4}$$

by (1.1) and (1.3) we have

$$L(\lambda) \subset \bigotimes_{i=0}^{n} \left(\otimes^{k_i} L(\Lambda_i) \right) \otimes L(k_\delta \delta) \tag{1.5}$$

as $\mathfrak{g}$-modules, where $\otimes^{k_i} L(\Lambda_i)$ denotes the k_i-fold tensor product of $L(\Lambda_i)$. Thus, to show that $L(\lambda)$ is contained in the tensor product of level one integrable highest weight modules, it suffices to show that the fundamental modules $L(\Lambda_i)$ have this property. Since the fundamental modules $L(\Lambda_i)$ for the affine Lie algebras $A_n^{(1)}$ and $C_n^{(1)}$ are all of level one there is nothing left to prove. As we mentioned before, this result has already been proved for the affine Lie algebras $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$, and $A_2^{(2)}$ (see [1, 12, 13]). We prove this result for the remaining affine Lie algebras $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $E_6^{(2)}$, and $D_4^{(3)}$ case by case. For the affine Lie algebras $E_6^{(2)}$ and $D_4^{(3)}$ we follow the computational technique of Xie [13]. To prove this result for the affine Lie algebras $B_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, and $D_{n+1}^{(2)}$, we will use the path realizations of the crystal bases for the level one fundamental modules of the corresponding quantum affine Lie algebras. For the general theory of crystal bases we refer the reader to [6, 7] and for path realizations we refer the reader to [4, 8, 9, 10].

2. The Affine Lie Algebra $B_n^{(1)}$, $n \geq 3$

The canonical central element of the affine Lie algebra $B_n^{(1)}$ is $c = h_0 + h_1 + 2h_2 + \cdots + 2h_{n-1} + h_n$ and the null root is $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$. So the fundamental modules $L(\Lambda_0)$, $L(\Lambda_1)$, and $L(\Lambda_n)$ are of level one and the other fundamental modules $L(\Lambda_k)$, $k = 2, 3, \ldots, n-1$, are of level two. First we will show that $L(\Lambda_k)$, $k = 2, 3, \ldots, n-1$, is contained in the tensor product of the level one modules $L(\Lambda_0)$, $L(\Lambda_1)$ up to degree shift. We need the path realizations of the crystal bases of the $\mathcal{U}_q(\mathcal{B}_n^{(1)})$-modules $L(\Lambda_0)$ and $L(\Lambda_1)$ (see [8, 9] for more details) which we briefly recall. We will also recall some general results about crystal bases which we will use in this section as well as in Sections 3–6.

Consider the set

$$B = \left\{ b_j = (\delta_{ij})_{i=1}^{2n+1} \,\middle|\, j = 1, 2, \ldots, 2n+1 \right\} \subset \mathbb{Z}^{2n+1}. \tag{2.1}$$

The set B has a crystal graph structure for $\mathcal{U}_q(\mathcal{B}_n^{(1)})$ with the maps $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1, 2, \ldots, n)$, given by

$$\begin{cases} \tilde{e}_i b_{i+1} = b_i, & i = 1, 2, \ldots, n, \\ \tilde{e}_i b_{2n+2-i} = b_{2n+1-i}, & i = 1, 2, \ldots, n, \\ \tilde{e}_0 b_2 = b_{2n+1}, & \tilde{e}_0 b_1 = b_{2n}, \\ \tilde{e}_1 b_j = 0, & \text{otherwise}, \\ \tilde{f}_i b = b' & \text{if and only if } \tilde{e}_i b' = b \text{ for } b, b' \in B, \\ \tilde{f}_i b_j = 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

Also define

$$\begin{cases} \overline{\mathrm{wt}}(b_1) = \Lambda_1 - \Lambda_0, & \overline{\mathrm{wt}}(b_2) = \Lambda_2 - \Lambda_1 - \Lambda_0, \\ \overline{\mathrm{wt}}(b_j) = \Lambda_j - \Lambda_{j-1}, & j = 3, 4, \ldots, n \\ \overline{\mathrm{wt}}(b_{n+1}) = 0, & \overline{\mathrm{wt}}(b_n) = 2\Lambda_n - \Lambda_{n-1}, \overline{\mathrm{wt}}(b_{n+2}) = \Lambda_{n-1} - 2\Lambda_n, \\ \overline{\mathrm{wt}}(b_{2n+1-j}) = \Lambda_j - \Lambda_{j+1}, & j = 2, 3, \ldots, n-2, \\ \qquad \overline{\mathrm{wt}}(b_{2n}) = \Lambda_0 + \Lambda_1 - \Lambda_2 \text{ and } \overline{\mathrm{wt}}(b_{2n+1}) = \Lambda_0 - \Lambda_1. & \end{cases} \tag{2.3}$$

For $i = 0, 1$, let $\eta^{(i)} = (\eta_k^{(i)})_{k \geq 1} = (\dots \eta_3^{(i)}\eta_2^{(i)}\eta_1^{(i)})$ denote the sequence (with period two) in B defined by

$$\eta_k^{(i)} = \begin{cases} b_{2n+1} & \text{if } k + i \text{ is odd}, \\ b_1 & \text{if } k + i \text{ is even}. \end{cases} \tag{2.4}$$

The sequence $\eta^{(i)}$ $(i = 0, 1)$ is called a ground-state path of weight Λ_i. A Λ_i-path, by definition, is a sequence $p = (p_k)_{k \geq 1} = (\dots p_3 p_2 p_1)$ such that $p_k \in B$ and $p_k = \eta_k^{(i)}$ for $k \gg 0$. The set $\mathscr{P}(\Lambda_i)$ $(i = 0, 1)$ of Λ_i-paths is a crystal graph for the $\mathscr{U}_q(\mathscr{B}_n^{(1)})$-module $L(\Lambda_i)$ (see [9]). The actions of $\tilde{e}_j$ and $\tilde{f}_j$ on a Λ_i-path in $\mathscr{P}(\Lambda_i)$ can be described by the following combinatorial rule.

For $b \in B$ and $j = 0, 1, 2, \dots, n$, define $s_j(b) = (\underbrace{11\dots1}_{r}\underbrace{00\dots0}_{t})$, where $r = \max\{k \mid \tilde{e}_j^k b \neq 0\}$ and $t = \max\{b \mid \tilde{f}_j^b b \neq 0\}$. Write $s_j(b) = (\cdot)$ if r and t are both zero. For $p = (\dots p_3 p_2 p_1) \in \mathscr{P}(\Lambda_i)$, define the j-signature $s_j(p)$ by

$$s_j(p) = (\dots s_j(p_3) s_j(p_2) s_j(p_1)). \tag{2.5}$$

We define the reduced j-signature $\hat{s}_j(p)$ of the path $p = (p_k)_{k \geq 1}$ to be the sequence of 1 and 0 obtained from $s_j(p)$ by deleting the (01) pairs in succession and also neglecting the dots. Then the reduced j-signature of $p = (p_k)_{k \geq 1}$ is of the form

$$\hat{s}_j(p) = (\underbrace{11\dots1}_{r}\underbrace{00\dots0}_{t})$$

with the corresponding terms in $p = (p_k)_{k \geq 1}$ being, say, p_{i_1}, $p_{i_2}, \dots, p_{i_r}, p_{i_{r+1}}, p_{i_{r+2}}, \dots, p_{i_{r+t}}$ respectively.

Then, by definition,

$$\begin{cases} \tilde{e}_j(p) = (\dots p_{i_{r-1}} \dots \tilde{e}_j(p_{i_r}) \dots p_{i_{r+1}} \dots p_1) \\ \tilde{f}_j(p) = (\dots p_{i_r} \dots \tilde{f}_j(p_{i_{r+1}}) \dots p_{i_{r+2}} \dots p_1). \end{cases} \tag{2.6}$$

For example, consider the path

$$p = (\dots b_{2n+1} b_1 b_{2n+1} b_1 b_2 b_3 \dots b_k) \in \mathscr{P}(*)$$

where k is even and $k < n$, say. Then the 0-signature is

$$s_0(p) = (\dots 01011 \cdot \cdots \cdot).$$

Therefore, the reduced 0-signature is

$$\hat{s}_0(p) = (1),$$

the corresponding term in p being b_2. Hence, using (2.2), we have

$$\begin{cases} \tilde{f}_0(p) = 0 \\ \tilde{e}_0(p) = (\dots b_{2n+1} b_1 b_{2n+1} b_1 \tilde{e}_0(b_2) b_3 \dots b_k) \\ \qquad = (\dots b_{2n+1} b_1 b_{2n+1} b_1 b_{2n+1} b_2 \dots b_k). \end{cases} \tag{2.7}$$

Also, observe that repeating this process we get

$$\begin{aligned} &\tilde{e}_0^2(\dots b_{2n+1} b_1 b_{2n+1} b_1 b_2 b_3 \dots b_k) \\ &\quad = \tilde{e}_0(\dots b_{2n+1} b_1 b_{2n+1} b_1 b_{2n+1} b_3 \dots b_k) = 0. \end{aligned} \tag{2.8}$$

In the rest of the paper, without further reference, when we need to, we will use the above description to calculate the $\tilde{e}_j$ and $\tilde{f}_j$ action on a path $p = (p_k)_{k \geq 1}$. We also recall that (see [8]) the weight of a Λ_i-path $p = (p_k)_{k \geq 1} \in \mathscr{P}(\Lambda_i)$ is given by the formula

$$\begin{aligned} \mathrm{wt}(p) = \Lambda_i &+ \sum_{k=1}^{\infty} \left(\overline{\mathrm{wt}}(p_k) - \overline{\mathrm{wt}}(\eta_k^{(i)})\right) \\ &- \left(\sum_{k=1}^{\infty} k\left(H\left(p_{k+1} \otimes p_k\right) - H\left(\eta_{k+1}^{(i)} \otimes \eta_k^{(i)}\right)\right)\right) d^{-1}\delta, \end{aligned} \tag{2.9}$$

where $d = 1$ (except for $A_{2n}^{(2)}$ when $d = 2$) and H is a $\mathbb{Z}$-valued function on $B \otimes B$ called the "energy function." Explicit formulas for the energy functions H are given in [10]. For example, using the formula for the energy function H for $B_n^{(1)}$ given in [10, Section 5.3] we have

$$\begin{cases} H\left(b_1 \otimes b_{2n+1}\right) = -1, & H\left(b_{2n+1} \otimes b_1\right) = 1 \\ H\left(b_{2n+1} \otimes b_2\right) = 1, & \\ H\left(b_i \otimes b_{i+1}\right) = 0, & \text{for } i = 1, 2, 3, \dots, n-1. \end{cases} \tag{2.10}$$

In order to prove that the level two fundamental modules occur as direct summands in the tensor products of two level one fundamental modules we will need to show that $\mathscr{P}(\lambda) \otimes \mathscr{P}(\mu)$ $(\lambda, \mu \in \{\Lambda_0, \Lambda_1\}$ in this case) contains a highest weight element with the appropriate highest weight and hence the connected component of this highest weight element in $\mathscr{P}(\lambda) \otimes \mathscr{P}(\mu)$ is the crystal graph for the desired representation. We will use the following known result to determine the highest weight elements in $\mathscr{P}(\lambda) \otimes \mathscr{P}(\mu)$.

PROPOSITION 2.11 ([4, Lemma 5.1]). *For* λ, $\mu \in P^+$ *and* $p' \otimes p \in \mathscr{P}(\lambda) \otimes \mathscr{P}(\mu)$ *the following are equivalent*:

(1) $\tilde{e}_i(p' \otimes p) = 0$ *for all* $i = 0, 1, 2, \ldots, n$ (*i.e.*, $p' \otimes p$ *is a highest weight element in* $\mathscr{P}(\lambda) \otimes \mathscr{P}(\mu)$).

(2) $p' = \eta^{(\lambda)}$, $\tilde{e}_i^{\lambda(h_i)+1} p = 0$, *for all* $i = 0, 1, \ldots, n$ *where* $\eta^{(\lambda)}$ *denotes the ground-state path in* $\mathscr{P}(\lambda)$ *with weight* λ.

THEOREM 2.12. *For the affine Lie algebra* $\mathfrak{g} = B_n^{(1)}$, *the integrable highest weight* $\mathfrak{g}$-*module* $L(\Lambda_k - [k/2]\delta)$, $k = 2, 3, \ldots, n-1$, *occurs as a direct summand in* $L(\Lambda_i) \otimes L(\Lambda_0)$ *where* $k \equiv i \bmod 2$. *In particular, the level two fundamental* $\mathfrak{g}$-*modules* $L(\Lambda_k)$, $k = 2, 3, \ldots, n-1$, *are contained in the tensor product of two level one fundamental modules up to degree shifts.*

Proof. Consider the elements $p = (p_j)_{j \geq 1} \in \mathscr{P}(\Lambda_0)$ such that $p_j = b_{k-j+1} \in B$ for $j = 1, 2, \ldots, k-1$ and $p_j = \eta_j^{(0)}$ for $j \geq k$. Note that

$$p = \begin{cases} (\ldots b_1 b_{2n+1} b_1 b_{2n+1} b_2 b_3 \ldots b_k), & \text{if } k \text{ is odd} \\ (\ldots b_1 b_{2n+1} b_1 b_2 b_3 \ldots b_k), & \text{if } k \text{ is even.} \end{cases} \tag{2.13}$$

By formula (2.9) we have

$$\begin{aligned} \mathrm{wt}(p) &= \Lambda_0 + \sum_{j=1}^{k-1} \left(\overline{\mathrm{wt}}(p_j) - \overline{\mathrm{wt}}\left(\eta_j^{(0)}\right)\right) \\ &\quad - \delta\left(\sum_{j=1}^{k-1} j\left(H\left(p_{j+1} \otimes p_j\right) - H\left(\eta_{j+1}^{(0)} \otimes \eta_j^{(0)}\right)\right)\right) \\ &= \Lambda_0 + \sum_{j=1}^{[k/2]} \left\{\left(\overline{\mathrm{wt}}(b_{k-2j+2}) - \overline{\mathrm{wt}}(b_{2n+1})\right)\right. \\ &\qquad\qquad \left. + \left(\overline{\mathrm{wt}}(b_{k-2j+1}) - \overline{\mathrm{wt}}(b_i)\right)\right\} \\ &\quad - \delta\left[\sum_{j=1}^{[k/2]} (2j-1)\left(H\left(b_{k-2j+1} \otimes b_{k-2j+2}\right) - H\left(b_1 \otimes b_{2n+1}\right)\right)\right. \\ &\qquad + \sum_{j=1}^{[k/2]-1} (2j)\left(H\left(b_{k-2j} \otimes b_{k-2j+1}\right) - H\left(b_{2n+1} \otimes b_1\right)\right) \\ &\qquad \left. + e(k)(k-1)\left(H\left(b_{2n+1} \otimes b_2\right) - H\left(b_{2n+1} \otimes b_1\right)\right)\right] \end{aligned}$$

where

$$e(k) = \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k \text{ is odd.} \end{cases} \tag{2.14}$$

Now, using (2.3) and (2.10) we get

$$\begin{aligned} \mathrm{wt}(p) &= \Lambda_k - \Lambda_i - \delta\left(\sum_{j=1}^{[k/2]} (2j-1) - \sum_{j=1}^{[k/2]-1} (2j) \right) \\ &= \Lambda_k - \Lambda_i - \delta\left(\left(\left[\frac{k}{2} \right] \right)^2 - \left[\frac{k}{2} \right] \left(\left[\frac{k}{2} \right] - 1 \right) \right) \\ &= \Lambda_k - \Lambda_i - \left[\frac{k}{2} \right] \delta \end{aligned} \tag{2.15}$$

where $k \equiv i \mod 2$. Hence for the element $\eta^{(i)} \otimes p \in \mathscr{P}(\Lambda_i) \otimes \mathscr{P}(\Lambda_0)$, $i = 0, 1$, we have

$$\begin{aligned} \mathrm{wt}(\eta^{(i)} \otimes p) &= \mathrm{wt}(\eta^{(i)}) + \mathrm{wt}(p) \\ &= \Lambda_i + \Lambda_k - \Lambda_i - \left[\frac{k}{2} \right] \delta \\ &= \Lambda_k - \left[\frac{k}{2} \right] \delta. \end{aligned} \tag{2.16}$$

When k is odd, looking at the signature of the path $p = (p_k)_{k \geq 1}$ (see 2.5 for definition) we observe that

$$\tilde{e}_i(p) = \tilde{e}_i(\ldots b_1 b_{2n+1} b_1 b_{2n+1} b_2 b_3 \ldots b_k) = 0, \qquad i \neq 1$$
$$\tilde{e}_1(\ldots b_1 b_{2n+1} b_1 b_{2n+1} b_2 b_3 \ldots b_k) = (\ldots b_1 b_{2n+1} b_1 b_{2n+1} b_1 b_3 \ldots b_k)$$

and

$$\tilde{e}_1^2(\ldots b_1 b_{n+1} b_1 b_{2n+1} b_2 b_3 \ldots b_k) = 0.$$

Thus,

$$\tilde{e}_i^{\Lambda_1(h_i)+1}(p) = 0 \qquad \text{for } i = 0, 1, 2, \ldots, n.$$

Similarly, when k is even, we observe that

$$\tilde{e}_i(p) = \tilde{e}_i(\ldots b_i b_{2n+1} b_1 b_{2n+1} b_1 b_2 \ldots b_k) = 0, \qquad i \neq 0,$$

and as seen in (2.8), $\tilde{e}_0^2(\ldots b_1 b_{2n+1} b_1 b_2 b_3 \ldots b_k) = 0$. Thus,

$$\tilde{e}_i^{\Lambda_0(h_i)+1}(p) = 0 \qquad \text{for } i = 0, 1, 2, \ldots, n.$$

Hence, by Proposition 2.11 and (2.16) the connected component of

$$\eta^{(i)} \otimes p \in \mathscr{P}(\Lambda_i) \otimes \mathscr{P}(\Lambda_0), i = 0, 1$$

is the crystal graph of $L(\Lambda_k - [k/2]\delta)$ and $L(\Lambda_k - [k/2]\delta)$ occurs as a direct summand in $L(\Lambda_i) \otimes L(\Lambda_0)$ where $k \equiv i \bmod 2$. ∎

Now from (1.1), (1.5), and the above theorem we have

THEOREM 2.17. *Any integrable highest weight $B_n^{(1)}$-module $L(\Lambda)$ is contained in the tensor product of $k = \Lambda(c)$ number of level one fundamental modules up to degree shift.*

3. THE AFFINE LIE ALGEBRA $D_n^{(1)}$, $n \geq 4$

For the affine Lie algebra $D_n^{(1)}$, the canonical central element and the null root are $c = h_0 + h_1 + 2h_2 + \cdots + + 2h_{n-2} + h_{n-1} + h_n$ and $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$, respectively. Hence the fundamental modules $L(\Lambda_0)$, $L(\Lambda_1)$, $(L(\Lambda_{n-1})$, and $L(\Lambda_n)$ are of level one in this case. The other fundamental modules $L(\Lambda_k)$, $k = 2, 3, \ldots, n-2$, are of level two. In order to show that any integrable highest weight $D_n^{(1)}$-module is contained in the tensor product of the level one fundamental modules it suffices to show that $L(\Lambda_k)$, $k = 2, 3, \ldots, n-2$, is contained in the tensor product of the level one modules $L(\Lambda_0)$, $L(\Lambda_1)$ up to degree shift. To prove this we will use the path realizations of the crystal bases of the $\mathscr{U}_q(\mathscr{D}_n^{(1)})$-modules $L(\Lambda_0)$ and $L(\Lambda_1)$, which we briefly recall below (see [8, 9] for more details).

Consider the set

$$B = \left\{ b_j = (\delta_{ij})_{i=1}^{2n} \,\middle|\, j = 1, 2, \ldots, 2n \right\} \subset \mathbb{Z}^{2n}. \tag{3.1}$$

The set B has a crystal graph structure for $\mathscr{U}_q(\mathscr{D}_n^{(1)})$ with the maps $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1, 2, \ldots, n)$, given by

$$\begin{cases} \tilde{e}_i b_{i+1} = b_i, & i = 1, 2, \ldots, n-1, \\ \tilde{e}_i b_{2n+1-i} = b_{2n-i}, & i = 1, 2, \ldots, n-1, \\ \tilde{e}_n b_{n+1} = b_{n-1}, & \tilde{e}_n b_{n+2} = b_n, \\ \tilde{e}_0 b_2 = b_{2n}, & \tilde{e}_0 b_1 = b_{2n-1}, \\ \tilde{e}_i b_j = 0, & \text{otherwise}, \\ \tilde{f}_i b = b' & \text{if and only if } \tilde{e}_i b' = b \text{ for } b, b' \in B, \\ \tilde{f}_i b_j = 0 & \text{otherwise}. \end{cases} \tag{3.2}$$

Also, define

$$\begin{cases} \overline{\mathrm{wt}}(b_1) = \Lambda_1 - \Lambda_0, & \overline{\mathrm{wt}}(b_2) = \Lambda_2 - \Lambda_1 - \Lambda_0, \\ \overline{\mathrm{wt}}(b_j) = \Lambda_j - \Lambda_{j-1}, & j = 3,4,\ldots,n-2, \\ \overline{\mathrm{wt}}(b_{n-1}) = \Lambda_n + \Lambda_{n-1} - \Lambda_{n-2}, & \overline{\mathrm{wt}}(b_n) = \Lambda_n - \Lambda_{n-1}, \\ \overline{\mathrm{wt}}(b_{n+1}) = \Lambda_{n-1} - \Lambda_n, & \overline{\mathrm{wt}}(b_{n+2}) = \Lambda_{n-2} - \Lambda_{n-1} - \Lambda_n, \\ \overline{\mathrm{wt}}(b_{2n-j}) = \Lambda_j - \Lambda_{j+1}, & j = 2,3,\ldots,n-3, \\ \overline{\mathrm{wt}}(b_{2n-1}) = \Lambda_0 + \Lambda_1 - \Lambda_2, \quad \text{and} & \overline{\mathrm{wt}}(b_{2n}) = \Lambda_0 - \Lambda_1. \end{cases} \tag{3.3}$$

For $i = 0,1$, let $\eta^{(i)} = (\eta_k^{(i)})_{k \geq 1} = (\ldots \eta_3^{(i)} \eta_2^{(i)} \eta_1^{(i)})$ denote the sequence (with period two) in B defined by

$$\eta_k^{(i)} = \begin{cases} b_{2n} & \text{if } k+i \text{ is odd}, \\ b_1 & \text{if } k+i \text{ is even}, \end{cases} \tag{3.4}$$

The sequence $\eta^{(i)}$ ($i = 0,1$) is called a ground-state path of weight Λ_i. A Λ_i-path, by definition, is a sequence $p = (p_k)_{k \geq 1} = (\ldots p_3 p_2 p_1)$ such that $p_k \in B$ and $p_k = \eta_k^{(i)}$ for $k \gg 0$. The set $\mathscr{P}(\Lambda_i)$ ($i = 0,1$) of Λ_i-paths is a crystal graph for the $\mathscr{U}_q(\mathscr{D}_n^{(1)})$-module $L(\Lambda_i)$. In this realization the weight of a Λ_i-path $p = (p_k)_{k \geq 1}$ is given by the formula (2.9). In this case, using the explicit formula for the energy function H given in [10, Section 5.4] we have

$$\begin{cases} H(b_1 \otimes b_{2n}) = -1, & H(b_{2n} \otimes b_1) = 1 \\ H(b_{2n} \otimes b_2) = 1, & \\ H(b_i \otimes b_{i+1}) = 0, & \text{for } i = 1,2,3,\ldots,n-2. \end{cases} \tag{3.5}$$

THEOREM 3.6. *For the affine Lie algebra $\mathfrak{g} = D_n^{(1)}$, the integrable highest weight $\mathfrak{g}$-module $L(\Lambda_k - [k/2]\delta)$, $k = 2,3,\ldots,n-2$, occurs as a direct summand in $L(\Lambda_i) \otimes L(\Lambda_0)$ where $k \equiv i \bmod 2$. In particular, the level two fundamental $\mathfrak{g}$-modules $L(\Lambda_k)$, $k = 2,3,\ldots,n-2$, are contained in the tensor product of two level one fundamental modules up to degree shifts.*

Proof. Consider the elements $p = (p_j)_{j \geq 1} \in \mathscr{P}(\Lambda_0)$ such that $p_j = b_{k-j+1} \in B$ for $j = 1,2,\ldots,k-1$ and $p_j = \eta_j^{(0)}$ for $j \geq k$. Note that

$$p = \begin{cases} (\ldots b_1 b_{2n} b_1 b_{2n} b_2 b_3 \ldots b_k), & \text{if } k \text{ is odd} \\ (\ldots b_1 b_{2n} b_1 b_2 b_3 \ldots b_k), & \text{if } k \text{ is even}. \end{cases} \tag{3.7}$$

By formula (2.9) we have

$$\begin{aligned}
\mathrm{wt}(p) &= \Lambda_0 + \sum_{j=1}^{k-1} \left(\overline{\mathrm{wt}}(p_j) - \overline{\mathrm{wt}}\left(\eta_j^{(0)}\right)\right) \\
&\quad - \delta\left(\sum_{j=1}^{k-1} j\left(H\left(p_{j+1} \otimes p_j\right) - H\left(\eta_{j+1}^{(0)} \otimes \eta_j^{(0)}\right)\right)\right) \\
&= \Lambda_0 + \sum_{j=1}^{[k/2]} \left\{\left(\overline{\mathrm{wt}}(b_{k-2j+1}) - \overline{\mathrm{wt}}(b_{2n+1})\right)\right. \\
&\qquad\qquad \left. + \left(\overline{\mathrm{wt}}(b_{k-2j+1}) - \overline{\mathrm{wt}}(b_1)\right)\right\} \\
&\quad - \delta\left[\sum_{j=1}^{[k/2]} (2j-1)\left(H\left(b_{k-2j+1} \otimes b_{k-2j+1}\right) - H\left(b_1 \otimes b_{2n}\right)\right)\right. \\
&\qquad + \sum_{j=1}^{[k/2]-1} (2j)\left(H\left(b_{k-2j} \otimes b_{k-2j+1}\right) - H\left(b_{2n} \otimes b_1\right)\right) \\
&\qquad \left. + e(k)(k-1)\left(H\left(b_{2n} \otimes b_2\right) - H\left(b_{2n} \otimes b_1\right)\right)\right]
\end{aligned}$$

where $e(k)$ is as given in (2.14).

Now, using (3.3) and (3.5), for $k \leq n-2$ we get, as in (2.15),

$$\begin{aligned}
\mathrm{wt}(p) &= \Lambda_k - \Lambda_i - \delta\left(\sum_{j=1}^{[k/2]} (2j-1) - \sum_{j=1}^{[k/2]-1} (2j)\right) \\
&= \Lambda_k - \Lambda_i - \left[\frac{k}{2}\right]\delta
\end{aligned}$$

where $k \equiv i \bmod 2$. Hence for the element $\eta^{(i)} \otimes p \in \mathscr{P}(\Lambda_i) \otimes \mathscr{P}(\Lambda_0)$, $i = 0, 1$, we have

$$\begin{aligned}
\mathrm{wt}\left(\eta^{(i)} \otimes p\right) &= \mathrm{wt}\left(\eta^{(i)}\right) + \mathrm{wt}(p) \\
&= \Lambda_i + \Lambda_k - \Lambda_i - \left[\frac{k}{2}\right]\delta \\
&= \Lambda_k - \left[\frac{k}{2}\right]\delta. \qquad (3.8)
\end{aligned}$$

When k is odd, looking at the signature of the path $p = (p_k)_{k \geq 1}$ (see (2.5) for definition) we observe that

$$\tilde{e}_i(p) = \tilde{e}_i(\ldots b_1 b_{2n} b_1 b_{2n} b_2 b_3 \ldots b_k) = 0, \qquad i \neq 1,$$
$$\tilde{e}_1(\ldots b_1 b_{2n} b_1 b_{2n} b_2 b_3 \ldots b_k) = (\ldots b_1 b_{2n} b_1 b_{2n} b_1 b_3 \ldots b_k),$$

and

$$\tilde{e}_1^2(\ldots b_1 b_{2n} b_1 b_{2n} b_2 b_3 \ldots b_k) = 0.$$

Thus,

$$\tilde{e}_i^{\Lambda_1(h_i)+1}(p) = 0 \qquad \text{for } i = 0, 1, 2, \ldots, n.$$

Similarly, when k is even, we observe that

$$\tilde{e}_i(p) = \tilde{e}_i(\ldots b_1 b_{2n} b_1 b_{2n} b_1 b_2 \ldots b_k) = 0, \qquad i \neq 0$$
$$\tilde{e}_0(\ldots b_1 b_{2n} b_1 b_2 b_3 \ldots b_k) = (\ldots b_1 b_{2n} b_1 b_{2n} b_3 \ldots b_k)$$

and

$$\tilde{e}_0^2(\ldots b_1 b_{2n} b_1 b_2 b_3 \ldots b_k) = 0.$$

Thus,

$$\tilde{e}_i^{\Lambda_0(h_i)+1}(p) = 0 \qquad \text{for } i = 0, 1, 2, \ldots, n.$$

Hence, by Proposition 2.11 and (3.8) the connected component of

$$\eta^{(i)} \otimes p \in \mathscr{P}(\Lambda_i) \otimes \mathscr{P}(\Lambda_0), i = 0, 1$$

is the crystal graph of $L(\Lambda_k - [k/2]\delta)$ and $L(\Lambda_k - [k/2]\delta)$ occurs as a direct summand in $L(\Lambda_i) \otimes L(\Lambda_0)$ where $k \equiv i \bmod 2$. ∎

Now from (1.1), (1.5), and the above theorem we have the following.

THEOREM 3.9. *Any integrable highest weight $D_n^{(1)}$-module $L(\Lambda)$ is contained in the tensor product of $k = \Lambda(c)$ number of level one fundamental modules up to degree shift.*

4. THE AFFINE LIE ALGEBRA $A_{2n-1}^{(2)}$, $n \geq 3$

In this case, the canonical central element is $c = h_0 + h_1 + 2h_2 + \cdots + 2h_n$ and the null root is $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$. Hence, $L(\Lambda_0)$ and $L(\Lambda_1)$ are the only level one fundamental modules in this case. The other fundamental modules $L(\Lambda_k)$, $k = 2, 3, \ldots, n$, are of level two. Now we briefly recall the path realizations of the crystal bases for the $\mathscr{U}_q(\mathscr{A}^{(2)}_{2n-1})$-modules for $L(\Lambda_0)$ and $L(\Lambda_1)$ (see [8, 9] for more details).

Consider the set

$$B = \left\{b_j = (\delta_{ij})_{i=1}^{2n} \middle| j = 1, 2, \ldots, 2n\right\} \subset \mathbb{Z}^{2n}. \tag{4.1}$$

The set B has a crystal graph structure for $\mathscr{U}_q(\mathscr{A}^{(2)}_{2n-1})$ with the maps $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1, 2, \ldots, n)$ given by

$$\begin{cases} \tilde{e}_i b_{i+1} = b_i, & i = 1, 2, \ldots, n, \\ \tilde{e}_i b_{2n+1-i} = b_{2n-i}, & i = 1, 2, \ldots, n-1, \\ \tilde{e}_0 b_2 = b_{2n}, & \tilde{e}_0 b_1 = b_{2n-1}, \\ \tilde{e}_i b_j = 0, & \text{otherwise}, \\ \tilde{f}_i b = b' & \text{if and only if } \tilde{e}_i b' = b \text{ for } b, b' \in B, \\ \tilde{f}_i b_j = 0 & \text{otherwise}. \end{cases} \tag{4.2}$$

Also, define

$$\begin{cases} \overline{\mathrm{wt}}(b_1) = \Lambda_1 - \Lambda_0, & \overline{\mathrm{wt}}(b_2) = \Lambda_2 - \Lambda_1 - \Lambda_0, \\ \overline{\mathrm{wt}}(b_j) = \Lambda_j - \Lambda_{j-1}, & j = 3, 4, \ldots, n, \\ \overline{\mathrm{wt}}(b_{2n-j}) = \Lambda_j - \Lambda_{j+1}, & j = 2, 3, \ldots, n-1, \\ \overline{\mathrm{wt}}(b_{2n-1}) = \Lambda_0 + \Lambda_1 - \Lambda_2 & \text{and} \quad \overline{\mathrm{wt}}(b_{2n}) = \Lambda_0 - \Lambda_1. \end{cases} \tag{4.3}$$

For $i = 0, 1$, let $\eta^{(i)} = (\eta_k^{(i)})_{k \geq 1} = (\ldots \eta_3^{(i)} \eta_2^{(i)} \eta_1^{(i)})$ denote the sequence (with period two) in B defined by

$$\eta_k^{(i)} = \begin{cases} b_{2n} & \text{if } k + i \text{ is odd}, \\ b_1 & \text{if } k + i \text{ is even}. \end{cases} \tag{4.4}$$

The sequence $\eta^{(i)}$ $(i = 0, 1)$ is called a ground-state path of weight Λ_i. A Λ_i path, by definition, is a sequence $p = (p_k)_{k \geq 1} = (\ldots p_3 p_2 p_1)$ such that $p_k \in B$ and $p_k = \eta_k^{(i)}$ for $k \gg 0$. The set $\mathscr{P}(\Lambda_i)$ $(i = 0, 1)$ of Λ_i-paths is a crystal graph for the $\mathscr{U}_q(\mathscr{A}^{(2)}_{2n-1})$-module $L(\Lambda_i)$. In this realization the weight of a Λ_i-path $p = (p_k)_{k \geq 1}$ is given by the formula (2.9). In this case, using the explicit formula for the energy function H given in [10, Section 5.2] we have

$$\begin{cases} H(b_1 \otimes b_{2n}) = -1, & H(b_{2n} \otimes b_1) = 1 \\ H(b_{2n} \otimes b_2) = 1, & \\ H(b_i \otimes b_{i+1}) = 0, & \text{for } i = 1, 2, 3, \ldots, n-1. \end{cases} \tag{4.5}$$

THEOREM 4.6. *For the affine Lie algebra* $\mathfrak{g} = A_{2n-1}^{(2)}$, *the integrable highest weight* $\mathfrak{g}$*-module* $L(\Lambda_k - [k/2]\delta)$, $k = 2, 3, \ldots, n$, *occurs as a direct summand in* $L(\Lambda_i) \otimes L(\Lambda_0)$ *where* $k \equiv i \bmod 2$. *In particular, the level two fundamental* $\mathfrak{g}$*-modules* $L(\Lambda_k)$, $k = 2, 3, \ldots, n$, *are contained in the tensor product of two level one fundamental modules up to degree shifts.*

Proof. Consider the elements $p = (p_j)_{j \geq 1} \in \mathscr{P}(\Lambda_0)$ such that $p_j = b_{k-j+1} \in B$ for $j = 1, 2, \ldots, k-1$ and $p_j = \eta_j^{(0)}$ for $j \geq k$. Note that

$$p = \begin{cases} (\ldots b_1 b_{2n} b_1 b_{2n} b_2 b_3 \ldots b_k), & \text{if } k \text{ is odd} \\ (\ldots b_1 b_{2n} b_1 b_2 b_3 \ldots b_k), & \text{if } k \text{ is even.} \end{cases} \tag{4.7}$$

As in the proof of Theorem 3.6, using (2.9), (4.3), and (4.5) we have

$$\begin{aligned} \mathrm{wt}(p) &= \Lambda_0 + \sum_{j=1}^{k-1} \left(\overline{\mathrm{wt}}(p_j) - \overline{\mathrm{wt}}\left(\eta_j^{(0)}\right)\right) \\ &\quad - \delta \sum_{j=1}^{k-1} j\left(H\left(p_{j+1} \otimes p_j\right) - H\left(\eta_{j+1}^{(0)} \otimes \eta_j^{(0)}\right)\right) \\ &= \Lambda_k - \Lambda_i - \delta\left(\sum_{j=1}^{[k/2]} (2j-1) - \sum_{j=1}^{[k/2]-1} (2j)\right) \\ &= \Lambda_k - \Lambda_i - \left[\frac{k}{2}\right]\delta \end{aligned}$$

where $k \equiv i \bmod 2$. Hence for the element $\eta^{(i)} \otimes p \in \mathscr{P}(\Lambda_i) \otimes \mathscr{P}(\Lambda_0)$, $i = 0, 1$, we have

$$\begin{aligned} \mathrm{wt}\left(\eta^{(i)} \otimes p\right) &= \mathrm{wt}\left(\eta^{(i)}\right) + \mathrm{wt}(p) \\ &= \Lambda_i + \Lambda_k - \Lambda_i - \left[\frac{k}{2}\right]\delta \\ &= \Lambda_k - \left[\frac{k}{2}\right]\delta. \end{aligned} \tag{4.8}$$

When k is odd, looking at the signature of the path $p = (p_k)_{k \geq 1}$ (see Eq. (2.5) for definition), we observe that

$$\tilde{e}_i(p) = \tilde{e}_i(\ldots b_1 b_{2n} b_1 b_{2n} b_2 b_3 \ldots b_k) = 0, \qquad i \neq 1$$

$$\tilde{e}_1(\ldots b_1 b_{2n} b_1 b_{2n} b_2 b_3 \ldots b_k) = (\ldots b_1 b_{2n} b_1 b_{2n} b_1 b_3 \ldots b_k)$$

and

$$\tilde{e}_1^2(\dots b_1 b_{2n} b_1 b_{2n} b_2 b_3 \dots b_k) = 0.$$

Thus,

$$\tilde{e}_i^{\Lambda_1(h_i)+1}(p) = 0 \qquad \text{for } i = 0, 1, 2, \dots, n.$$

Similarly, when k is even, we observe that

$$\tilde{e}_i(p) = \tilde{e}_i(\dots b_1 b_{2n} b_1 b_{2n} b_1 b_2 \dots b_k) = 0, \quad i \neq 0$$
$$\tilde{e}_0(\dots b_1 b_{2n} b_1 b_2 b_3 \dots b_k) = (\dots b_1 b_{2n} b_1 b_{2n} b_3 \dots b_k)$$

and

$$\tilde{e}_0^2(\dots b_1 b_{2n} b_1 b_2 b_3 \dots b_n) = 0.$$

Thus,

$$\tilde{e}_i^{\Lambda_0(h_i)+1}(p) = 0 \qquad \text{for } i = 0, 1, 2, \dots, n.$$

Hence, by Proposition 2.11 and (4.8) the connected component of

$$\eta^{(i)} \otimes p \in \mathscr{P}(\Lambda_i) \otimes \mathscr{P}(\Lambda_0), \qquad i = 0, 1$$

is the crystal graph of $L(\Lambda_k = [k/2]\delta)$ and $L(\Lambda_k - [k/2]\delta)$ occurs as a direct summand in $L(\Lambda_i) \otimes L(\Lambda_0)$ where $k \equiv i \bmod 2$. ∎

Now from (1.1), (1.5), and the above theorem we have the following.

THEOREM 4.9. *Any integrable highest weight $A_{2n-1}^{(2)}$-module $L(\Lambda)$ is contained in the tensor product of $k = \Lambda(c)$ number of level one fundamental modules up to degree shift.*

5. THE AFFINE LIE ALGEBRA $A_{2n}^{(2)}$, $n \geq 2$

In this case, the canonical central element is $c = h_0 + 2h_1 + \cdots + 2h_n$ and the null root is $\delta = 2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n$. Hence, $L(\Lambda_0)$ is the only level one fundamental module in this case. The other fundamental modules $L(\Lambda_k)$, $k = 1, 2, 3, \dots, n$, are of level two. To show that $L(\Lambda_k)$ is contained in $L(\Lambda_0) \otimes L(\Lambda_0)$ we use the path realization of the crystal base of the $\mathscr{U}_q(\mathscr{A}_{2n}^{(2)})$-module for $L(\Lambda_0)$ which we recall briefly below (see [8, 9] for details).

Consider the set

$$B = \{b_0, b_1, b_2, \dots, b_{2n}\} \subset \mathbb{Z}^{2n} \tag{5.1}$$

where $b_0 = (0, 0, 0, \dots, 0)$ and $b_j = (\delta_{ij})_{i=1}^{2n}$ for $j = 1, 2, \dots, 2n$. The set B has a crystal graph structure for $\mathcal{U}_q(\mathcal{A}_{2n}^{(2)})$ with the maps $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1, 2, \dots, n)$ given by

$$\begin{cases} \tilde{e}_i b_{i+1} = b_i, & i = 1, 2, \dots, n, \\ \tilde{e}_i b_{2n+1-i} = b_{2n-i}, & i = 1, 2, \dots, n-1, \\ \tilde{e}_0 b_1 = b_0, & \tilde{e}_0 b_0 = b_{2n}, \\ \tilde{e}_i b_j = 0, & \text{otherwise}, \\ \tilde{f}_i b = b' & \text{if and only if } \tilde{e}_i b' = b \text{ for } b, b' \in B, \\ \tilde{f}_i b_j = 0 & \text{otherwise}. \end{cases} \tag{5.2}$$

Also define

$$\begin{cases} \overline{\mathrm{wt}}(b_0) = 0 & \overline{\mathrm{wt}}(b_1) = \Lambda_1 - 2\Lambda_0, \\ \overline{\mathrm{wt}}(b_j) = \Lambda_j - \Lambda_{j-1}, & j = 2, 3, \dots, n, \\ \overline{\mathrm{wt}}(b_{2n-j}) = \Lambda_j - \Lambda_{j+1}, & j = 1, 2, \dots, n-1, \\ \overline{\mathrm{wt}}(b_{2n}) = 2\Lambda_0 - \Lambda_1. \end{cases} \tag{5.3}$$

Note that the sequence $\eta = (\eta_k)_{k \geq 1} = (\dots b_0 b_0 b_0)$ (i.e., $\eta_k = b_0$ for all $k \geq 1$) in B is called a ground-state path of weight Λ_0. By definition, a Λ_0-path is a sequence $p = (p_k)_{k \geq 1} = (\dots p_3 p_2 p_1)$ such that $p_k \in B$ and $p_k = \eta_k = b_0$ for $k \gg 0$. The set $\mathcal{P}(\Lambda_0)$ of Λ_0-paths is a crystal graph for the $\mathcal{U}_q(\mathcal{A}_{2n}^{(2)})$-module $L(\Lambda_0)$. In this realization the weight of a Λ_0-path $p = (p_k)_{k \geq 1}$ is given by the formula (2.9). In this case, using the explicit formula for the energy function H given in [10, Section 5.5] we have

$$\begin{cases} H(b_0 \otimes b_0) = 0, & H(b_0 \otimes b_1) = 1, \\ H(b_i \otimes b_{i+1}) = 0, & \text{for } i = 1, 2, 3, \dots, n. \end{cases} \tag{5.4}$$

THEOREM 5.5. *For the affine Lie algebra $\mathfrak{g} = A_{2n}^{(2)}$, the integrable highest weight $\mathfrak{g}$-module $L(\Lambda_k - (k/2)\delta)$, $k = 1, 2, 3, \dots, n$, occurs as a direct summand in $L(\Lambda_0) \otimes L(\Lambda_0)$. In particular, the level two fundamental $\mathfrak{g}$-modules $L(\Lambda_k)$, $k = 1, 2, 3, \dots, n$, are contained in the tensor product of two level one fundamental modules up to degree shifts.*

Proof. Consider the elements $p = (p_j)_{j \geq 1} \in \mathscr{P}(\Lambda_0)$ such that $p_j = b_{k-j+1} \in B$ for $j = 1, 2, \ldots, k$ and $p_j = \eta_j$ for $j \geq k$. Note that $p = (\ldots b_0 b_0 b_1 b_2 \ldots b_k)$. By formula (2.9) we have

$$\begin{aligned} \mathrm{wt}(p) &= \Lambda_0 + \sum_{j=1}^{k} \left(\overline{\mathrm{wt}}(p_j) - \overline{\mathrm{wt}}(\eta_j)\right) \\ &\quad - \frac{1}{2}\delta\left(\sum_{j=1}^{k} j\left(H\left(p_{j+1} \otimes p_j\right) - H\left(\eta_{j+1} \otimes \eta_j\right)\right)\right) \\ &= \Lambda_0 + \sum_{j=1}^{k} \left(\overline{\mathrm{wt}}(b_j) - \overline{\mathrm{wt}}(b_0)\right) \\ &\quad - \frac{1}{2}\delta\left(\sum_{j=1}^{k} j\left(H\left(b_{k-j} \otimes b_{k-j+1}\right) - H\left(b_0 \otimes b_0\right)\right)\right). \end{aligned}$$

Now, using (5.3) and (5.4) we get

$$\begin{aligned} \mathrm{wt}(p) &= \Lambda_0 + \Lambda_1 - 2\Lambda_0 + \sum_{j=2}^{k} \left(\Lambda_j - \Lambda_{j-1}\right) - \frac{k}{2}\delta \\ &= -\Lambda_0 + \Lambda_k - \frac{k}{2}\delta. \end{aligned}$$

Hence, the weight of the element $\eta \otimes p \in \mathscr{P}(\Lambda_0) \otimes \mathscr{P}(\Lambda_0)$ is

$$\begin{aligned} \mathrm{wt}\left(\eta \otimes p\right) &= \mathrm{wt}(\eta) + \mathrm{wt}(p) \\ &= \Lambda_0 - \Lambda_0 + \Lambda_k - \frac{k}{2}\delta \\ &= \Lambda_k - \frac{k}{2}\delta. \end{aligned} \tag{5.6}$$

Also, using the signature of the path $p = (\ldots b_0 b_0 b_1 b_2 \ldots b_k)$ we observe that

$$\begin{cases} \tilde{e}_i(p) = \tilde{e}_i(\ldots b_0 b_0 b_1 b_2 \ldots b_k) = 0, & i \neq 0 \\ \tilde{e}_0(p) = (\ldots b_0 b_0 b_0 b_2 \ldots b_k), & \\ \tilde{e}_0^2(p) = \tilde{e}_0(\ldots b_0 b_0 b_0 b_2 \ldots b_k) = 0. & \end{cases} \tag{5.7}$$

Thus, $\tilde{e}_i^{\Lambda_0(h_i)+1}(p) = 0$ for $i = 0, 1, 2, \ldots, n$. Hence, by Proposition 2.11

and (5.6) the connected component of $\eta \otimes p \in \mathscr{P}(\Lambda_0) \otimes \mathscr{P}(\Lambda_0)$ is the crystal graph of $L(\Lambda_k - (k/2)\delta)$, and $L(\Lambda_k - (k/2)\delta)$, $k = 1, 2, \ldots, n$, occurs as a direct summand in $L(\Lambda_0) \otimes L(\Lambda_0)$. ∎

Now from (1.1), (1.5), and the above theorem we have the following.

THEOREM 5.8. *Any integrable highest weight $A_{2n}^{(2)}$-module $L(\Lambda)$ is contained in the tensor product of a $\Lambda(c) = k$ - fold tensor product of the level one fundamental module $L(\Lambda_0)$ up to degree shift.*

6. THE AFFINE LIE ALGEBRA $D_{n+1}^{(2)}$, $n \geq 2$

For the affine Lie algebra $D_{n+1}^{(2)}$, the canonical central element is $c = h_0 + 2h_1 + 2h_2 + \cdots + 2h_{n-1} + h_n$ and the null root is $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n$. Hence, the fundamental modules $L(\Lambda_0)$ and $L(\Lambda_n)$ are of level one. The other fundamental modules $L(\Lambda_k)$, $k = 1, 2, 3, \ldots, n-1$, are of level two. To show that $L(\Lambda_k)$, $k = 1, 2, 3, \ldots, n-1$, is contained in $L(\Lambda_0) \otimes L(\Lambda_0)$ we use the path realization of the crystal base of the $\mathscr{U}_q(\mathscr{D}_{n+1}^{(2)})$-module $L(\Lambda_0)$ which we briefly recall below (see [8, 9] for details).

Consider the set

$$B = \{b_0, b_1, b_2, \ldots, b_{2n+1}\} \subset \mathbb{Z}^{2n+1} \tag{6.1}$$

where $b_0 = (0, 0, 0, \ldots, 0)$ and $b_j = (\delta_{ij})_{i=1}^{2n+1}$ for $j = 1, 2, \ldots, 2n+1$. The set B has a crystal graph structure for $\mathscr{U}_q(\mathscr{D}_{n+1}^{(2)})$ with the maps $\tilde{e}_i, \tilde{f}_i$ $(i = 0, 1, 2, \ldots, n)$ given by

$$\begin{cases} \tilde{e}_i b_{i+1} = b_i, & i = 1, 2, \ldots, n, \\ \tilde{e}_i b_{2n+2-i} = b_{2n+1-i}, & i = 1, 2, \ldots, n, \\ \tilde{e}_0 b_1 = b_0, & \tilde{e}_0 b_0 = b_{2n+1}, \\ \tilde{e}_i b_j = 0, & \text{otherwise}, \\ \tilde{f}_i b = b' & \text{if and only if } \tilde{e}_i b' = b \text{ for } b, b' \in B, \\ \tilde{f}_i b_j = 0 & \text{otherwise}. \end{cases} \tag{6.2}$$

Also define

$$\begin{cases} \overline{\mathrm{wt}}(b_0) = 0, & \overline{\mathrm{wt}}(b_1) = \Lambda_1 - 2\Lambda_0, \\ \overline{\mathrm{wt}}(b_j) = \Lambda_j - \Lambda_{j-1}, & j = 2, 3, \ldots, n-1, \quad \overline{\mathrm{wt}}(b_n) = 2\Lambda_n - \Lambda_{n-1}, \\ \overline{\mathrm{wt}}(b_{2n+1-j}) = \Lambda_j - \Lambda_{j+1}, & j = 1, 2, \ldots, n-2, \quad \overline{\mathrm{wt}}(b_{n+2}) = \Lambda_{n-1} - 2\Lambda_n, \\ \overline{\mathrm{wt}}(b_{n+1}) = 0 \quad \text{and} \quad \overline{\mathrm{wt}}(b_{2n+1}) = 2\Lambda_0 - \Lambda_1. \end{cases} \tag{6.3}$$

Note that the sequence $\eta = (\eta_k)_{k\geq 1} = (\ldots b_0 b_0 b_0)$ (i.e., $h_k = b_0$ for all $k \geq 1$) in B is called a ground-state path of weight Λ_0. By definition, a Λ_0-path is a sequence $p = (p_k)_{k\geq 1} = (\ldots p_3 p_2 p_1)$ such that $p_k \in B$ and $p_k = \eta_k = b_0$ for $k \gg 0$. The set $\mathscr{P}(\Lambda_0)$ of Λ_0-paths is a crystal graph for the $\mathscr{U}_q(\mathscr{D}_{n+1}^{(2)})$-module $L(\Lambda_0)$. In this realization the weight of a Λ_0-path $p = (p_k)_{k\geq 1}$ is given by the formula (2.9). In this case, using the explicit formula for the energy function H given in [10, Section 5.6] we have

$$\begin{cases} H(b_0 \otimes b_0) = 0, & H(b_0 \otimes b_1) = 1, \\ H(b_i \otimes b_{i+1}) = 0, & \text{for } i = 1,2,3,\ldots,n-1. \end{cases} \tag{6.4}$$

THEOREM 6.5. *For the affine Lie algebra* $\mathfrak{g} = D_{n+1}^{(2)}$, *the integrable highest weight* $\mathfrak{g}$-*module* $L(\Lambda_k - k\delta)$, $k = 1,2,3,\ldots,n-1$, *occurs as a direct summand in* $L(\Lambda_0) \otimes L(\Lambda_0)$. *In particular, the level two fundamental* $\mathfrak{g}$-*modules* $L(\Lambda_k)$, $k = 1,2,3,\ldots,n-1$, *are contained in the tensor product of two level one fundamental modules up to degree shifts.*

Proof. Consider the elements $p = (p_j)_{j\geq 1} \in \mathscr{P}(\Lambda_0)$ such that $p_j = b_{k-j+1} \in B$ for $j = 1,2,\ldots,k$ and $p_j = \eta_j$ for $j \geq k$. Note that $p = (\ldots b_0 b_0 b_1 b_2 \ldots b_k)$. Now as in the proof of Theorem 5.5, using (6.3), (6.4), and formula (2.9) we have

$$\begin{aligned} \mathrm{wt}(p) &= \Lambda_0 + \sum_{j=1}^{k} \left(\overline{\mathrm{wt}}(b_j) - \overline{\mathrm{wt}}(b_0)\right) \\ &\quad - \delta\left(\sum_{j=1}^{k} j\left(H\left(b_{k-j} \otimes b_{k-j+1}\right) - H\left(b_0 \otimes b_0\right)\right)\right) \\ &= \Lambda_0 + \Lambda_1 - 2\Lambda_0 + \sum_{j=2}^{k} (\Lambda_j - \Lambda_{j-1}) - k\delta \\ &= -\Lambda_0 + \Lambda_k - k\delta. \end{aligned}$$

Hence, the weight of the element $\eta \otimes p \in \mathscr{P}(\Lambda_0) \otimes \mathscr{P}(\Lambda_0)$ is

$$\begin{aligned} \mathrm{wt}(\eta \otimes p) &= \mathrm{wt}(\eta) + \mathrm{wt}(p) \\ &= \Lambda_0 - \Lambda_0 + \Lambda_k - k\delta \\ &= \Lambda_k - k\delta. \end{aligned} \tag{6.6}$$

Also, using the signature of the path $p = (\ldots b_0 b_0 b_1 b_2 \ldots b_k)$ we observe that

$$\begin{cases} \tilde{e}_i(p) = 0, & \text{for } i \neq 0 \\ \tilde{e}_0(\dots b_0 b_0 b_1 b_2 \dots b_k) = (\dots b_0 b_0 b_0 b_2 \dots b_k), \\ \tilde{e}_0^2(\dots b_0 b_0 b_1 b_2 \dots b_k) = 0. \end{cases} \tag{6.7}$$

Thus, $\tilde{e}_i^{\Lambda_0(h_i)+1}(p) = 0$ for $i = 0, 1, 2, \dots, n$. Hence, by Proposition 2.11 and (6.6), the connected component of $\eta \otimes p \in \mathscr{P}(\Lambda_0) \otimes \mathscr{P}(\Lambda_0)$ is the crystal graph of $L(\Lambda_k - k\delta)$, and $L(\Lambda_k - k\delta)$, $k = 1, 2, \dots, n-1$, occurs as a direct summand in $L(\Lambda_0) \otimes L(\Lambda_0)$. ∎

Now from (1.1), (1.5), and the above theorem we have the following.

THEOREM 6.8. *Any integrable highest weight $D_{n+1}^{(2)}$-module $L(\Lambda)$ is contained in the tensor product of $\Lambda(c) = k$ – fold tensor product of the level one fundamental module $L(\Lambda_0)$ up to degree shift.*

7. THE AFFINE LIE ALGEBRA $E_6^{(2)}$ AND $D_4^{(3)}$

For the affine Lie algebra $E_6^{(2)}$ the canonical central element and the null root are respectively $c = h_0 + 2h_1 + 3h_2 + 4h_3 + 2h_4$ and $\delta = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$. Hence $L(\Lambda_0)$ is the only level one fundamental module in this case. The other fundamental modules $L(\Lambda_1)$, $L(\Lambda_2)$, $L(\Lambda_3)$, and $L(\Lambda_4)$, are of levels 2, 3, 4, and 2, respectively. To show that these fundamental modules are contained in the tensor product of an appropriate number of copies of $L(\Lambda_0)$ we use the computational technique developed in [13].

THEOREM 7.1. *We have the following relations between the fundamental $E_6^{(2)}$-modules $L(\Lambda_k)$, $k = 1, 2, 3, 4$, and the tensor products of the level one fundamental module $L(\Lambda_0)$.*

(1) $L(\Lambda_1 - \delta) \subset L(\Lambda_0) \otimes L(\Lambda_0)$,

(2) $L(\Lambda_4 - 2\delta) \subset L(\Lambda_0) \otimes L(\Lambda_0)$,

(3) $L(\Lambda_2 - \delta) \subset L(\Lambda_0) \otimes L(\Lambda_1)$, *and*

(4) $L(\Lambda_3 - \delta) \subset L(\Lambda_0) \otimes L(\Lambda_2)$.

Proof. Let v_0 denote the highest weight vector in $L(\Lambda_0)$ with highest weight Λ_0. Consider the vector $v_1 = f_0 v_0 \otimes v_0 - v_0 \otimes f_0 v_0 \in L(\Lambda_0) \otimes L(\Lambda_0)$. Then clearly

$$e_i v_1 = 0 \qquad \text{for } i = 1, 2, 3, 4,$$

and

$$\begin{aligned} e_0 v_1 &= e_0(f_0 v_0 \otimes v_0 - v_0 \otimes f_0 v_0) \\ &= e_0 f_0 v_0 \otimes v_0 + f_0 v_0 \otimes e_0 v_0 - e_0 v_0 \otimes f_0 v_0 - v_0 \otimes e_0 f_0 v_0 \\ &= h_0 v_0 \otimes v_0 - v_0 \otimes h_0 v_0 = 0. \end{aligned}$$

Furthermore,

$$\mathrm{wt}(v_1) = 2\Lambda_0 - \alpha_0 = \Lambda_1 - \delta \in \mathfrak{h}^*,$$

where $\mathfrak{h}$ is the CSA of $E_6^{(2)}$. Hence (1) holds. To prove (2), consider

$$\begin{aligned} v_4 = {} & f_0 f_1 f_2 f_3 f_2 f_1 f_0 v_0 \otimes v_0 - f_1 f_2 f_3 f_2 f_1 f_0 v_0 \otimes f_0 v_0 \\ & + f_2 f_3 f_2 f_1 f_0 v_0 \otimes f_1 f_0 v_0 - f_3 f_2 f_1 f_0 v_0 \otimes f_2 f_1 f_0 v_0 \\ & + f_2 f_1 f_0 v_0 \otimes f_3 f_2 f_1 f_0 v_0 - f_1 f_0 v_0 \otimes f_2 f_3 f_2 f_1 f_0 v_0 \\ & + f_0 v_0 \otimes f_1 f_2 f_3 f_2 f_1 f_0 v_0 - v_0 \otimes f_0 f_1 f_2 f_3 f_2 f_1 f_0 v_0 \in L(\Lambda_0) \otimes L(\Lambda_0). \end{aligned}$$

As before, by direct computation it can be shown that

$$e_i v_4 = 0 \qquad \text{for } i = 0, 1, 2, 3, 4,$$

and

$$\begin{aligned} \mathrm{wt}(v_4) &= 2\Lambda_0 - 2\alpha_0 - 2\alpha_1 - 2\alpha_2 - \alpha_3 \\ &= \Lambda_4 - 2\delta \in \mathfrak{h}^*. \end{aligned}$$

Hence (2) holds. To prove (3) consider

$$v_2 = f_1 f_0 v_0 \otimes v_1 - f_0 v_0 \otimes f_1 v_1 + v_0 \otimes f_0 f_1 v_1 \in L(\Lambda_0) \otimes L(\Lambda_1).$$

It is easy to check that

$$e_i(v_2) = 0, \qquad \text{for } i = 0, 1, 2, 3, 4,$$

and

$$\mathrm{wt}(v_2) = \Lambda_0 + \Lambda_1 - \alpha_0 - \alpha_1 = \Lambda_2 - \delta \in \mathfrak{h}^*.$$

Hence, (3) holds. Finally, to prove (4), consider the vector

$$\begin{aligned} v_3 = {} & f_2 f_1 f_0 v_0 \otimes v_2 - f_1 f_0 v_0 \otimes f_2 v_2 \\ & + f_0 v_0 \otimes f_1 f_2 v_2 - v_0 \otimes f_0 f_1 f_2 v_2 \in L(\Lambda_0) \otimes L(\Lambda_2). \end{aligned}$$

Again, by direct calculations it is easy to check that

$$e_i(v_3) = 0 \qquad \text{for } i = 0, 1, 2, 3, 4,$$

and

$$\begin{aligned}\mathrm{wt}(v_3) &= \Lambda_0 + \Lambda_2 - \alpha_0 - \alpha_1 - \alpha_2 \\ &= \Lambda_3 - \delta \in \mathfrak{h}^*.\end{aligned}$$

Hence, (4) holds. ∎

For the affine Lie algebra $D_4^{(3)}$, the canonical central element and null root are respectively $c = h_0 + 2h_1 + 3h_3$ and $\delta = \alpha_0 + 2\alpha_1 + \alpha_2$. Hence in this case, $L(\Lambda_0)$ is the only level one fundamental module. The other fundamental modules $L(\Lambda_1)$ and $L(\Lambda_2)$ are of levels 2 and 3 respectively.

THEOREM 7.2. *We have the following relations between the fundamental $D_4^{(3)}$-modules $L(\Lambda_k)$, $k = 1, 2$, and the tensor products of the level one fundamental module $L(\Lambda_0)$.*

(1) $L(\Lambda_1 - \delta) \subset L(\Lambda_0) \otimes L(\Lambda_0)$,

and

(2) $L(\Lambda_2 - \delta) \subset L(\Lambda_0) \otimes L(\Lambda_1)$.

Proof. Let v_0 denote the highest weight vector in $L(\Lambda_0)$ with highest weight Λ_0. Consider the vector $v_1 = f_0 v_0 \otimes v_0 - v_0 \otimes f_0 v_0 \in L(\Lambda_0) \otimes L(\Lambda_0)$. Then it is easy to see that

$$e_i(v_1) = 0 \qquad \text{for } i = 0, 1, 2$$

and

$$\mathrm{wt}(v_1) = 2\Lambda_0 - \alpha_0 = \Lambda_1 - \delta \in \mathfrak{h}^*,$$

where $\mathfrak{h}$ is the CSA of $D_4^{(3)}$. Hence (1) holds. Now consider the vector $v_2 = f_1 f_0 v_0 \otimes v_1 - f_0 v_0 \otimes f_1 v_1 + v_0 \otimes f_0 f_1 v_1 \in L(\Lambda_0) \otimes L(\Lambda_1)$. Then it is easy to check that

$$e_i(v_2) = 0 \qquad \text{for } i = 0, 1, 2,$$

and

$$\mathrm{wt}(v_2) = \Lambda_0 + \Lambda_1 - \alpha_0 - \alpha_1 = \Lambda_2 - \delta \in \mathfrak{h}^*.$$

Hence, (2) follows. ∎

The following theorem now follows from (1.1), (1.5), and Theorems 7.1 and 7.2.

THEOREM 7.3. *For the affine Lie algebra $\mathfrak{g} = E_6^{(2)}$ or $D_4^{(3)}$, any integrable highest weight module $L(\Lambda)$ is contained in the tensor product of $\Lambda(c)$ = k-fold tensor product of the level one fundamental module $L(\Lambda_0)$ up to degree shift.*

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