Existence and uniqueness of viscosity solutions for QVI associated with impulse control of jump-diffusions

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Abstract

General theorems for existence and uniqueness of viscosity solutions for Hamilton–Jacobi–Bellman quasi-variational inequalities (HJBQVI) with integral term are established. Such nonlinear partial integro-differential equations (PIDE) arise in the study of combined impulse and stochastic control for jump-diffusion processes. The HJBQVI consists of an HJB part (for stochastic control) combined with a nonlocal impulse intervention term.

Existence results are proved via stochastic means, whereas our uniqueness (comparison) results adapt techniques from viscosity solution theory. This paper, to our knowledge is the first treating rigorously impulse control for jump-diffusion processes in a general viscosity solution framework; the jump part may have infinite activity. In the proofs, no prior continuity of the value function is assumed, quadratic costs are allowed, and elliptic and parabolic results are presented for solutions possibly unbounded at infinity.

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1. Introduction

Consider the combined stochastic and impulse control problem of the following SDE:

$$dX_t = \mu(t, X_{t-}, \beta_{t-}) \, dt + \sigma(t, X_{t-}, \beta_{t-}) \, dW_t + \int \ell(t, X_{t-}, \beta_{t-}, z) \, N(dz, dt),$$

(1)
for a standard Brownian motion \( W \) and a compensated Poisson random measure \( \widetilde{N}(dz, dt) = N(dz, dt) - 1_{|z|<1}v(dz)dt \) with possibly unbounded intensity measure \( v \) (the jumps of a Lévy process), and the stochastic control process \( \beta \) (with values in some compact set \( B \)). The impulses occur at stopping times \( (\tau_i)_{i \geq 1} \), and have the effect

\[
X_{\tau_i} = \Gamma(t, \tilde{X}_{\tau_i}, \zeta_i),
\]

after which the process continues to evolve according to the controlled SDE until the next impulse. (Detailed notation and definitions are introduced in Section 2.) We denote by \( \gamma = (\tau_i, \zeta_i)_{i \geq 1} \) the impulse control strategy, and by \( \alpha = (\beta, \gamma) \) a combined control consisting of a stochastic control \( \beta \) and an impulse control \( \gamma \). The aim is to maximise a certain functional, dependent on the impulse controlled process \( X^\alpha \) until the exit time \( \tau \) (e.g., for a finite time horizon \( T > 0, \tau := \tau_S \wedge T \), where \( \tau_S \) is the exit time of \( X^\alpha \) from a possibly unbounded set \( S \)):

\[
\nu(t, x) := \max_{\alpha} \mathbb{E}^{(t,x)}\left[ \int_t^T f(s, X_s^\alpha, \beta_s)ds + g(\tau, X_T^\alpha)1_{\tau<\infty} + \sum_{\tau_j \leq \tau} K(\tau_j, \tilde{X}_{\tau_j}, \zeta_j) \right]. \tag{2}
\]

Here the negative function \( K \) incorporates the impulse transaction costs, and the functions \( f \) and \( g \) are profit functions.

### 1.1. Quasi-variational inequality

The main purpose of this paper is to prove that the value function \( \nu \) of (2) is the unique viscosity solution of the following partial integro-differential equation (PIDE), a so-called Hamilton–Jacobi–Bellman quasi-variational inequality (HJBQVI):

\[
\min(- \sup_{\beta \in B} \{ u_t + L^\beta u + f^\beta \}, u - \mathcal{M} u) = 0 \quad \text{in } [0, T) \times S, \tag{3}
\]

for \( L^\beta \) the infinitesimal generator of the SDE (1) (where \( y = (t, x) \)),

\[
L^\beta u(y) = \frac{1}{2} tr \left( \sigma(y, \beta)\sigma^T(y, \beta)D_x^2 u(y) \right) + \langle \mu(y, \beta), \nabla_x u(y) \rangle
\]

\[+ \int u(t, x + \ell(y, \beta, z)) - u(y) - \langle \nabla_x u(y), \ell(y, \beta, z) \rangle 1_{|z|<1}v(dz), \]

and \( \mathcal{M} \) the intervention operator selecting the momentarily best impulse,

\[
\mathcal{M} u(t, x) = \sup_{\zeta} \{ u(t, \Gamma(t, x, \zeta)) + K(t, x, \zeta) \}.
\]

(3) is formally a nonlinear, nonlocal, possibly degenerate, second order parabolic PIDE. We point out that the investigated stochastic process is allowed to have jumps (jump-diffusion process), including so-called “infinite-activity processes” where the jump measure \( v \) may be singular at the origin. (It can be argued that infinite-activity processes are a good model for stock prices, see, e.g., [13,19].)

For a general introduction to viscosity solutions and their advantages, we recommend [6], the “User’s Guide” [14], and in the context of (stochastic) control [5,20]. For other solution approaches to (3) and impulse control problems, we refer to [8,16,28] and [31].
1.2. Impulse control & applications

The setting of our problem can be interesting for a number of applications, particularly in finance. Because impulse control problems typically involve fixed transaction costs — as opposed to singular control (only proportional costs), or stochastic control (no interventions) — they lend themselves readily to financial models in incomplete markets.

Clearly, the standard reference for applications as well as for theory is [8]; as a more recent overview for jump-diffusions, [31] can be helpful. For further applications in finance see the overview in [25], or specific examples concerning option pricing with transaction costs ([36,15,10]), optimal portfolios ([34,32,30,24]), options in long-term insurance contracts [12], or control of an exchange rate by the Central Bank ([29,11]).

This last application is a good example for combined control: there are two different means of intervention, namely interest rates (stochastic control) and FX market interventions. The stochastic control affects the process continuously (we neglect transaction costs here), and the impulses have fixed transaction costs, but have an immediate effect and thus can better react to jumps in the stochastic process.

Our goal in writing this paper was to establish a framework that can be readily used (and extended) in applications, without too many technical conditions.

1.3. Overview of the paper

Main contribution of this article (and of the working paper version [35]) is to rigorously treat viscosity solution existence and uniqueness of the HJBQVI (3) and of its elliptic counterpart, i.e., of the exit time problem for combined impulse and stochastic control of the jump-diffusion (1), whose jump part may have infinite activity.

Such a result is very well known in the diffusion case (in a general setting, see [22,37]; for specific applications, see [27,30,1]), and was established for piecewise deterministic processes (no exit time, and jumps with finite activity) in [26] without stochastic control. To our best knowledge, there is no such result for jump-diffusion processes yet (even in the finite activity case and without stochastic control). A singular integral term complicates the problem considerably; we cater for this using techniques and results from [7].

Our general setting is probably closest to the one in the book [31] (where a sketch of proof for existence in the jump-diffusion case is offered, under the assumption that the value function is continuous); see also [33] for optimal stopping and control until a finite time \( T \). In the diffusion case, [37] prove existence and uniqueness of continuous viscosity solutions for a finite time horizon (no exit time) including stochastic control and optimal switching, but under rather restrictive assumptions. Besides, their approach requires continuity of the value function (which is proved on 11 pages). We note also that our problem (no exit time, and without stochastic control) was already treated in [28] by non-viscosity solution techniques.

Let us now give a short overview on the methods we employ in this paper to prove existence and uniqueness. We prove that the value function is a discontinuous viscosity solution of (3), as done in the recent paper [27] in a portfolio optimisation context for a diffusion process. For the exit time problem, we need some continuity assumptions on the boundary \( \partial S \)—apart from that, the continuity will be a consequence of viscosity solution uniqueness. Because the jumps

1 Optimal switching can be considered as a special case of impulse control with higher-dimensional state space.
could lead outside \( S \) (and impulses could bring us back), we have to investigate the QVI on the whole space \( \mathbb{R}^d \) with appropriate boundary conditions (the “boundary” is in general not a null set of \( \mathbb{R}^d \)).

For the uniqueness proof, we use a perturbation technique with strict viscosity sub- and supersolutions (as in [22]); this also takes care of the unboundedness of the domain. Our solutions can be unbounded at infinity with arbitrary polynomial growth (provided appropriate conditions on the functions involved are satisfied), and superlinear transaction costs (e.g., quadratic) are allowed.

Because the Lévy measure is allowed to have a singularity of second order at 0, we cannot use the standard approach to uniqueness of viscosity solutions of PDE as used in [33] for optimal stopping, and in [9] for singular control. For a more detailed discussion, we refer to [23] (and the references therein), who were the first to propose a way to circumvent the problem for an HJB PIDE; see also the remark in the uniqueness section. For our proof of uniqueness, we will use and extend the framework as presented in the more recent paper [7] (the formulation in [23] does not permit an easy impulse control extension). The reader might also find helpful [3].

The paper consists of 4 main sections. Section 2 presents the detailed problem formulation, the assumptions and a summary of the main result; (substitutes for) the dynamic programming principle are derived in Section 3. The following Section 4 is concerned with existence of a QVI viscosity solution. After introducing the setting of our impulse control problem and several helpful results, we prove in Theorem 4.2 that the value function is a (discontinuous) QVI viscosity solution. The existence result for the elliptic QVI is deduced from the corresponding parabolic one. The last main section (Section 5) then starts with a reformulation of the QVI and several equivalent definitions for viscosity solutions. A maximum principle for impulse control is then derived, and used in a comparison result, which yields uniqueness and continuity of the QVI viscosity solution. The paper is complemented by a synthesis and summary at the end.

1.4. Notation

\( \mathbb{R}^d \) for \( d \geq 1 \) is the Euclidean space equipped with the usual norm and the scalar product denoted by \( \langle \cdot, \cdot \rangle \). For sets \( A, B \subset \mathbb{R}^d \), the notation \( A \subset B \) (compactly embedded) means that \( \overline{A} \subset B \), and \( A^c = \mathbb{R}^d \setminus A \) is the complement of \( A \). We denote the space of symmetric matrices \( \subset \mathbb{R}^{d \times d} \) by \( \mathbb{S}^d \), \( \geq \) is the usual ordering in \( \mathbb{R}^{d \times d} \), i.e. \( X \geq Y \Leftrightarrow X - Y \) positive semidefinite. \( |\cdot| \) on \( \mathbb{S}^d \) is the usual eigenvalue norm. \( C^2(\mathbb{R}^d) \) is the space of all functions two times continuously differentiable with values in \( \mathbb{R} \), and as usual, \( u_t \) denotes the time derivative of \( u \). \( L^2(\mathbb{P}; \mathbb{R}^d) \) is the Hilbert space of all \( \mathbb{P} \)-square-integrable measurable random variables with values in \( \mathbb{R}^d \), the measure \( \mathbb{P}^X = \mathbb{P} \circ X^{-1} \) is sometimes used to lighten notation.

2. Model and main result

Let a filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \) satisfying the usual assumptions be given. Consider an adapted \( m \)-dimensional Brownian motion \( W \), and an adapted independent \( k \)-dimensional pure-jump Lévy process represented by the compensated Poisson random measure \( \overline{N}(dz, dt) = N(dz, dt) - 1_{|z| < 1} \nu(dz)dt \), where as always \( \int (|z|^2 \land 1) \nu(dz) < \infty \) for the Lévy measure \( \nu \). We assume as usual that all processes are right-continuous. Assume the
$d$-dimensional state process $X$ follows the stochastic differential equation with impulses

$$
\begin{align*}
    dX_t &= \mu(t, X_t, \beta_t) \, dt + \sigma(t, X_t, \beta_t) \, dW_t + \int_{\mathbb{R}^k} \ell(t, X_t, \beta_t, z) \, \mathcal{N}(dz, dt), \\
    \tau_i &< t < \tau_{i+1}, \\
    X_{\tau_{i+1}} &= \Gamma(t, \tilde{X}_{\tau_{i+1}}, \zeta_{i+1}) \quad i \in \mathbb{N}_0
\end{align*}
$$

(4)

for $\Gamma : \mathbb{R}_0^+ \times \mathbb{R}^{2d} \to \mathbb{R}^d$ measurable, and $\mu, \ell : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}^d$, $\sigma : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}^{d \times m}$ satisfying the necessary conditions such that existence and uniqueness of the SDE is guaranteed. $\beta$ is a càdlàg adapted stochastic control (where $\beta(t, \omega) \in B$, $B$ compact non-empty metric space), and $\gamma = (\tau_1, \tau_2, \ldots, \zeta_1, \zeta_2, \ldots)$ is the impulse control strategy, where $\tau_i$ are stopping times with $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots$, and $\zeta_i$ are adapted impulses. The measurable transaction set $Z(t, x) \subset \mathbb{R}^d$ denotes the allowed impulses when at time $t$ in state $x$. We denote by $\alpha = (\beta, \gamma)$ the so-called combined stochastic control, where $\alpha \in A = A(t, x)$, the admissible region for the combined stochastic control. Admissible means here in particular that existence and uniqueness of the SDE be guaranteed, and that we only consider Markov controls (i.e., controls only dependent on current state and time).

The term $\tilde{X}^\alpha_{\tau_{j-1}}$ denotes the value of the controlled $X^\alpha$ in $\tau_j$ with a possible jump of the stochastic process, but without the impulse, i.e., $\tilde{X}^\alpha_{\tau_{j-1}} = X^\alpha_{\tau_{j-1}} + \Delta X^\alpha_{\tau_j}$, where $\Delta$ denotes the jump of the stochastic process. So for the first impulse, this would be the process $\tilde{X}^\alpha_{\tau_1} = X^\beta_{\tau_1}$ only controlled by the continuous control. If two or more impulses happen to be at the same time (e.g., $\tau_{j+1} = \tau_j$), then (4) is to be understood as concatenation, e.g., $\Gamma(t, \Gamma(t, \tilde{X}_{\tau_{j-1}}, \zeta_i), \zeta_{i+1})$. (The notation used here is borrowed from [31].)

The general (combined) impulse control problem is: find $\alpha = (\beta, \gamma) \in A$ that maximises the payoff starting in $t$ with $x$

$$
J^{(\alpha)}(t, x) = \mathbb{E}^{(t, x)} \left[ \int_0^\tau f(s, X^\alpha_s, \beta_s) \, ds + g(\tau, X^\alpha_\tau) \mathbf{1}_{\tau<\infty} + \sum_{\tau_j \leq \tau} K(\tau_j, \tilde{X}^\alpha_{\tau_{j-1}}, \zeta_j) \right],
$$

(5)

where $f : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}$, $g : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}$, $K : \mathbb{R}_0^+ \times \mathbb{R}^{2d} \to \mathbb{R}$ are measurable, and $\tau = \tau_S = \inf\{s \geq t : X^\alpha_s \notin S\}$ is the exit time from some open set $S \subseteq \mathbb{R}^d$ (possibly infinite horizon), or $\tau = \tau_S \wedge T$ for some $T > 0$ (finite horizon). Note that $X^\alpha_s$ is the value at $s$ after all impulses in $s$ have been applied; so “intermediate values” are not taken into account by this stopping time. The value function $v$ is defined by

$$
v(t, x) = \sup_{\alpha \in A(t, x)} J^{(\alpha)}(t, x).
$$

(6)

We have to assume further conditions on the admissibility set $A(t, x)$ to ensure well-definedness: First, we assume that existence and uniqueness of (4) hold with constant stochastic control (e.g., by Lipschitz conditions such as in [21]), which guarantees that $A(t, x)$ is non-empty. (Note that the typical property that a solution process has finite second moments is preserved after an impulse if for $X \in L^2_{\mathbb{P}}(\mathbb{R}; \mathbb{R}^d)$, also $\Gamma(t, X, \xi(t, X)) \in L^2(\mathbb{P}; \mathbb{R}^d)$. This is certainly the case if the impulses $\zeta_j$ are in a compact and $\Gamma$ continuous, which we will later assume.)
We further require the integrability condition on the negative parts of \( f, g, K \)
\[
\mathbb{E}^{(t,x)} \left[ \int_t^T f^-(s, X_s^\alpha, \beta_s)ds + g^-(\tau, X_\tau^\alpha)1_{\tau<\infty} + \sum_{\tau_j \leq \tau} K^-(\tau_j, \tilde{X}_{\tau_j^-}^\alpha, \zeta_j) \right] < \infty \tag{7}
\]
for all \( \alpha \in A(t, x) \).

### 2.1. Parabolic HJBQVI

For a fixed finite horizon \( T > 0 \), we define \( S_T := [0, T) \times S \) and its parabolic nonlocal “boundary” \( \partial^+ S_T := (\{0\} \times (\mathbb{R}^d \setminus S)) \cup ([T] \times \mathbb{R}^d) \). Further denote \( \partial^* S_T := ([0, T) \times \partial S) \cup ([T] \times \overline{S}) \). Let the impulse intervention operator \( \mathcal{M} = \mathcal{M}^{(t,x)} \) be defined by
\[
\mathcal{M}u(t, x) = \sup\{u(t, \Gamma(t, x, \zeta)) + K(t, x, \zeta) : \zeta \in Z(t, x)\} \tag{8}
\]
(define \( \mathcal{M}u(t, x) = -\infty \) if \( Z(t, x) = \emptyset \) – we will exclude this case later on). The hope is to find the value function by investigating the following parabolic Hamilton–Jacobi–Bellman QVI:
\[
\min_{\beta \in \mathcal{B}} (-\sup_{\beta} \{u(t, +\mathcal{L}^\beta u + f^\beta), u - \mathcal{M}u\} = 0 \quad \text{in } S_T \tag{9}
\]
\[
\min(u - g, u - \mathcal{M}u) = 0 \quad \text{in } \partial^+ S_T,
\]
for \( \mathcal{L}^\beta \) the generator of \( X \) in the SDE (4) for constant stochastic control \( \beta \), and \( f^\beta(\cdot) := f(\cdot, \beta) \). The generator \( \mathcal{L}^\beta \) has the form \( (y = (t, x)) \)
\[
\mathcal{L}^\beta u(y) = \frac{1}{2} \text{tr} \left( \sigma(y, \beta)\sigma^T(y, \beta) D^2_x u(y) \right) + \langle \mu(y, \beta), \nabla_x u(y) \rangle
\]
\[
+ \int u(t, x + \ell(y, \beta, z)) - u(y) - \langle \nabla_x u(y), \ell(y, \beta, z) \rangle 1_{|z| < 1} v(dz). \tag{10}
\]
While the equation for \( S \) in (9) can be motivated by Dynkin’s formula and the fact that \( v \geq \mathcal{M}v \) by the optimality of \( v \), we have to argue why we consider the value function \( v \) on \( [0, T) \times \mathbb{R}^d \) instead of the interesting set \( [0, T) \times S \): This is due to the jump term of the underlying stochastic process. While it is not possible to stay a positive time outside \( S \) (we stop in \( \tau_S \)), it is well possible in our setting that the stochastic process jumps outside, but we return to \( S \) by an impulse before the stopping time \( \tau_S \) takes notice. Thus we must define \( v \) outside \( S \), to be able to decide whether a jump back to \( S \) is worthwhile. The boundary condition has its origin in the following necessary condition for the value function:
\[
\min(v - g, v - \mathcal{M}v) = 0 \quad \text{in } [0, T) \times (\mathbb{R}^d \setminus S). \tag{11}
\]
This formalises that the controller can either do nothing (i.e., at the end of the day, the stopping time \( \tau_S \) has passed, and the game is over), or can jump back into \( S \), and the game continues. A similar condition holds at time \( T < \infty \), with the difference that the controller is not allowed to jump back in time (as the device permitting this is not yet available to the public). So the necessary terminal condition can be put explicitly as\(^2\)
\[
v = \sup(g, \mathcal{M}g, \mathcal{M}^2 g, \ldots) \quad \text{on } [T] \times \mathbb{R}^d. \tag{12}
\]
\(^2\text{If } g \text{ is lower semicontinuous, if } \mathcal{M} \text{ preserves this property and if the sup is finite, then it is well known that } v(T, \cdot) \text{ is lower semicontinuous. Even if } g \text{ is continuous, } v(T, \cdot) \text{ need not be continuous.} \)
Example 2.1. The impulse back from \( \mathbb{R}^d \setminus S \) to \( S \) could correspond to a capital injection into profitable business to avoid untimely default due to a sudden event.

2.2. Well-definedness of QVI terms

We need to establish conditions under which the terms \( L^\beta u \) and \( Mu \) in the QVI (9) are well-defined.

The integral operator in (10) is at the same time a differential operator of up to second order (if \( v \) singular). This can be seen by Taylor expansion for \( u \in C^{1,2}([0, T] \times \mathbb{R}^d) \) for some \( 0 < \delta < 1 \):

\[
\int_{|z| < \delta} |u(t, x + \ell(x, \beta, z)) - u(t, x) - \langle \nabla u(t, x), \ell(x, \beta, z) \rangle 1_{|z| < 1}| v(dz) \\
\leq \int_{|z| < \delta} |\ell(t, x, \beta, z)|^2 |D^2 u(t, \tilde{x})| v(dz)
\]

for an \( \tilde{x} \in B(x, \sup_{|z| < \delta} \ell(t, x, \beta, z)) \).

It depends on \( v \) and \( \ell \) for which \( u \) the term \( \sup_\beta L^\beta u \) is well-defined. We assume that \( \ell \) satisfies some growth constraint (in 0 and in \( \infty \)) uniformly in \( \beta \), locally in \( t, x \) of the type:

\[
|\ell(t, x, \beta, z)| \leq C_{t,x}(1_{|z| \geq 1} + |z|^p)
\]

for \( p \geq 0 \). If, e.g., \( |\ell(t, x, \beta, z)| \leq C_{t,x}(|z|) \), then \( \sup_\beta L^\beta u \) is well-defined for \( u \in C^{1,2}([0, T] \times \mathbb{R}^d) \) with certain growth conditions at infinity. Let the polynomial \( R : \mathbb{R}^d \to \mathbb{R} \) be such a function for which \( \sup_\beta L^\beta R \) is well-defined, and fix it throughout this paper. Then we define as in [7]:

**Definition 2.1 (Space of Polynomially Bounded Functions).** \( \mathcal{PB} = \mathcal{PB}([0, T] \times \mathbb{R}^d) \) is the space of all measurable functions \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) such that

\[
|u(t, x)| \leq C_u(1 + R(x))
\]

for a time-independent constant \( C_u > 0 \).

As pointed out in [7], this function space \( \mathcal{PB} \) is stable under lower and upper semicontinuous envelopes, and functions in \( \mathcal{PB} \) are locally bounded. Furthermore, it is stable under (pointwise) limit operations, and the conditions for Lebesgue’s Dominated Convergence Theorem are in general satisfied.

For \( Mu \) and the other remaining terms, well-definedness immediately follows from the assumptions below (compare also the discussion at the end of the section).

2.3. Assumptions and main result

Let us now formalise the conditions necessary for both the existence and the uniqueness proof in the following assumption (see also the discussion at the end of the section):

**Assumption 2.1.** (V1) \( \Gamma \) and \( K \) are continuous.

(V2) The transaction set \( Z(t, x) \) is non-empty and compact for each \( (t, x) \in [0, T] \times \mathbb{R}^d \). For a converging sequence \( (t_n, x_n) \to (t, x) \) in \( [0, T] \times \mathbb{R}^d \) (with \( Z(t_n, x_n) \) non-empty), \( Z(t_n, x_n) \) converges to \( Z(t, x) \) in the Hausdorff metric.

(V3) \( \mu, \sigma, \ell, f \) are continuous in \( (t, x, \beta) \) on \( [0, T] \times \mathbb{R}^d \times B \).

(V4) \( \ell \) satisfies a (growth) constraint as specified above, and \( \mathcal{PB} \) is fixed accordingly.
Apart from Assumption 2.1, we will need the following assumptions for the proof of existence:

**Assumption 2.2.** (E1) The value function \( v \) is in \( \mathcal{PB}([0, T] \times \mathbb{R}^d) \).

(E2) \( g \) is continuous.

(E3) The value function \( v \) satisfies for every \((t, x) \in \partial^+ S_T\), and all sequences \((t_n, x_n)_n \subset [0, T) \times S\) converging to \((t, x)\):

\[
\liminf_{n \to \infty} v(t_n, x_n) \geq g(t, x).
\]

\[
\text{If} \quad v(t_n, x_n) > \mathcal{M}_v(t_n, x_n) \forall n : \quad \limsup_{n \to \infty} v(t_n, x_n) \leq g(t, x).
\]

(E4) For all \( \rho > 0 \), \((t, x) \in [0, T) \times S\), and sequences \([0, T) \times S \ni (t_n, x_n) \to (t, x)\), there is a constant \( \hat{\beta} \) (not necessarily in \( B \)) and an \( N \in \mathbb{N} \) such that for all \( 0 < \epsilon < 1/N \) and \( n \geq N \),

\[
\mathbb{P}(\sup_{t_n \leq s \leq t_n + \epsilon} |X_{s}^{\hat{\beta}, t_n, x_n} - x_n| < \rho) \leq \mathbb{P}(\sup_{t_n \leq s \leq t_n + \epsilon} |X_{s}^{n, t_n, x_n} - x_n| < \rho),
\]

where \( X_{s}^{n, t_n, x_n} \) is the process according to SDE (4), started in \((t_n, x_n)\) and controlled by \( \hat{\beta} \).

The following assumptions are needed for the uniqueness proof \((\delta > 0)\):

**Assumption 2.3.** (U1) If \( v(\mathbb{R}^k) = \infty \): For all \((t, x) \in [0, T) \times S\), \(U_{\delta}(t, x) := \{ \ell(x, \beta, z) : |z| < \delta \} \) does not depend on \( \beta \).

(U2) If \( v(\mathbb{R}^k) = \infty \): For all \((t, x) \in [0, T) \times S\), \(\text{dist}(x, \partial U_{\delta}(t, x))\) is strictly positive for all \( \delta > 0 \) (or \( \ell \equiv 0 \)).

**Assumption 2.4.** (B1) \( \int_{\mathbb{R}^d} |\ell(t, x, \beta, z) - \ell(t, y, \beta, z)|^2 v(\mathrm{d}z) < C|x - y|^2 \), \( \int_{|z| \geq 1} |\ell(t, x, \beta, z) - \ell(t, y, \beta, z)| v(\mathrm{d}z) < C|x - y|\), and all estimates hold locally in \( t \in [0, T) \), \( x, y \), uniformly in \( \beta \).

(B2) Let \( \sigma(\cdot, \beta), \mu(\cdot, \beta), f(\cdot, \beta) \) be locally Lipschitz continuous, i.e. for each point \((t_0, x_0) \in [0, T) \times S\) there is a neighbourhood \( U \ni (t_0, x_0) \) (\( U \) open in \([0, T) \times \mathbb{R}^d\)), and a constant \( C \) (independent of \( \beta \)) such that \( |\sigma(t, x, \beta) - \sigma(t, y, \beta)| \leq C|x - y|\forall(t, x), (t, y) \in U \), and likewise for \( \mu \) and \( f \).

The space \( \mathcal{PB}_p = \mathcal{PB}_p([0, T] \times \mathbb{R}^d) \) consists of all functions \( u \in \mathcal{PB} \), for which there is a constant \( C \) such that \( |u(t, x)| \leq C(1 + |x|^p) \) for all \((t, x) \in [0, T] \times \mathbb{R}^d\).

Under the above assumptions, we can now formulate our main result in the parabolic case, whose proof is split up in Theorems 4.2 and 5.11 (the precise definition of viscosity solution is introduced in Section 4):

**Theorem 2.2** (Existence and Uniqueness of a Viscosity Solution in the Parabolic Case). Let Assumptions 2.1–2.4 be satisfied. Assume further that \( v \in \mathcal{PB}_p([0, T] \times \mathbb{R}^d) \), and that there is a nonnegative \( w \in \mathcal{PB} \cap C^2([0, T] \times \mathbb{R}^d) \) with \( w(t, x)/|x|^p \to \infty \) for \( |x| \to \infty \) (uniformly in \( t \)) and a constant \( \kappa > 0 \) such that

\[
\min(- \sup_{\beta \in B} \{w_t + \mathcal{L}^\beta w + f^\beta\}, w - \mathcal{M}w) \geq \kappa \quad \text{in } S_T
\]

\[
\min(w - g, w - \mathcal{M}w) \geq \kappa \quad \text{in } \partial^+ S_T.
\]

Then the value function \( v \) is the unique viscosity solution of the parabolic QVI (9), and it is continuous on \([0, T] \times \mathbb{R}^d\).
2.4. Elliptic HJBQVI

For finite time horizon $T$, (9) is investigated on $[0, T] \times \mathbb{R}^d$ (parabolic problem). For infinite horizon, typically a discounting factor $e^{-\rho(t+s)}$ for $\rho > 0$ applied to all functions takes care of the well-definedness of the value function, e.g., $f(t, x, \beta) = e^{-\rho(t+s)} \tilde{f}(x, \beta)$. In this time-independent case, a transformation $u(t, x) = e^{-\rho(t+s)} w(x)$ gives us the elliptic HJBQVI to investigate

$$\min(-\sup_{\beta \in B} \{ \mathcal{L}^\beta u + f^\beta \}, u - \mathcal{M}u) = 0 \quad \text{in } S$$

$$\min(u - g, u - \mathcal{M}u) = 0 \quad \text{in } \mathbb{R}^d \setminus S,$$

where the functions and variables have been appropriately renamed, and

$$\mathcal{L}^\beta u(x) = \frac{1}{2} \text{tr} \left( \sigma(x, \beta) \sigma^T(x, \beta) D^2 u(x) \right) + \langle \mu(x, \beta), \nabla u(x) \rangle - \rho \, u(x)$$

$$+ \int u(x + \ell(x, \beta, z)) - u(x) - \langle \nabla u(x), \ell(x, \beta, z) \rangle 1_{|z| < 1} \nu(dz),$$

$$\mathcal{M}u(x) = \sup\{ u(\Gamma(x, \zeta)) + K(x, \zeta) : \zeta \in Z(x) \}.$$ (16)

Under the time-independent version of the assumptions above, an essentially identical existence and uniqueness result holds for the elliptic QVI (14). We refrain from repeating it, and instead we refer to Sections 4.2 and 5.4 for a precise formulation.

2.5. Discussion of the assumptions

Of all assumptions, it is quite clear why we need the continuity assumptions, and they are easy to check.

By (V3), (V4) and the compactness of the control set $B$, the Hamiltonian $\sup_{\beta \in B} u(t, x) + \mathcal{L}^\beta u(t, x) + f^\beta(t, x)$ is well-defined and continuous in $(t, x) \in [0, T] \times S$ for $u \in \mathcal{P}B \cap C^{1,2}([0, T] \times \mathbb{R}^d)$. (This follows by sup manipulations, the (locally) uniform continuity and the DCT for the integral part.) Instead of (V3), assuming the continuity of the Hamiltonian is sufficient for the existence proof. For the stochastic process $X_t$, condition (V4) essentially ensures the existence of moments.

By (V1), (V2), we obtain that $\mathcal{M}u$ is locally bounded if $u$ locally bounded in $[0, T] \times \mathbb{R}^d$ (e.g., if $u \in \mathcal{P}B([0, T] \times \mathbb{R}^d)$). $\mathcal{M}u$ is even continuous if $u$ is continuous (so impulses preserve continuity properties), see Lemma 4.3.

In condition (V2), $Z(t, x) \neq \emptyset$ is necessary for the Hausdorff metric of sets to be well-defined, and to obtain general results on continuity of the value function (it is easy to construct examples of discontinuous value functions otherwise). The assumption is however no severe restriction, because we can set $Z(t, x) = \emptyset$ in the no-intervention region $\{ v > \mathcal{M}v \}$ without affecting the value function. The compactness of $Z(t, x)$ is not essential and can be relaxed in special cases—this restriction is however of no practical importance.

Condition (E3) connects the combined control problem with the continuity of the stochastic control problem at the boundary. In this respect, Theorem 2.2 roughly states that the value function is continuous except if there is a discontinuity on the boundary $\partial S$. (E3) is typically
satisfied if the stochastic process is regular at $\partial S$, as shown at the end of Section 4.1; see also [20], Theorem V.2.1, and the analytic approaches in [3,4]. In particular, this condition excludes problems with true or de facto state constraints, although the framework can be extended to cover state constraints.

(E4) can be expected to hold because the control set $B$ is compact and the functions $\mu, \sigma, \ell$ are continuous in $(t, x, \beta)$ (V3). The condition is very easy to check for a concrete problem—it would be a lot more cumbersome to state a general result, especially for the jump part. ((E4) is needed for lemmas in the working paper version of this article [35].)

Example 2.2. If $dX_t = \beta_t dt + dW_t$, and $\beta_t \in B = [-1, 1]$, then $\hat{\beta} := 2$ is a possible choice for (E4) to hold.

Assumption 2.3 collects some minor prerequisites that only need to be satisfied for small $\delta > 0$ (see also the remark in the beginning of Section 5.1), and the formulation can easily be adapted to a specific problem. (The assumption is needed for equivalence of viscosity solution definitions, skipped in this paper; for details see the working paper version [35].)

The local Lipschitz continuity in (B1) and (B2) is a standard condition; (B1) is satisfied if, e.g., the jump size of the stochastic process does not depend on $x$, or typical conditions for existence and uniqueness of the SDE are satisfied for a constant $\beta$. Condition (B1) can be relaxed if, e.g., $X$ has a state-dependent (finite) jump intensity—the uniqueness proof adapts readily to this case.

Certainly the most intriguing point is how to find a suitable function $w$ meeting all the requirements detailed in Theorem 2.2. (This requirement essentially means that we have a strict supersolution.) We first consider the elliptic case of QVI (14). Here such a function $w$ for a $\kappa > 0$ can normally be constructed by $w(x) = w_1|x|^q + w_2$ for suitable $w_i$ and $q > p$ (but still $w \in PB$!). Main prerequisites are then:

(L1) Positive interest rates: $\rho > \tilde{\kappa}$ for a suitably chosen constant $\tilde{\kappa} > 0$

(L2) Fixed transaction costs: e.g., $K(x, \zeta) = -k_0 < 0$.

If additionally we allow only impulses towards 0, then $w - Mw > \kappa$ is easily achieved, as well as $w - g > \kappa$ (if we require that $g$ have a lower polynomial order than $w$). For a given bounded set, choosing $w_2$ large enough makes sure that $-\sup_{\beta \in B} \{\mathcal{L}_\beta w + f_\beta\} > \kappa$ on this set (due to the continuity of the Hamiltonian and translation invariance in the integral). For $|x| \to \infty$, we need to impose conditions on $\tilde{\kappa}$—these depend heavily on the problem at hand, but can require the discounting factor to be rather large (e.g., for a geometric Brownian motion).

In the parabolic case, the same discussion holds accordingly, except that it is significantly easier to find a $w \in PB((0, T) \times \mathbb{R}^d)$ satisfying assumption (L1): By setting $w(t, x) = \exp(-\tilde{\kappa}t)$ $(w_1|x|^q + w_2)$, we have $w_t = -\tilde{\kappa}w$ for arbitrarily large $\tilde{\kappa}$.

3. Dynamic programming principle

In this section, we derive two inequalities, which we will need for the existence proof (instead of the dynamic programming principle (DPP)).

First of all, for Dynkin’s formula and several transformations, we need to establish the Markov property of the Markov-controlled process $X^\alpha$. In fact, one can prove the strong Markov property of $Y^\alpha_t := (s + t, X^\alpha_{s+t})$ for some $s \geq 0$. Given that the solution of SDE (4) is Markov, this comes as no surprise; we refer to [35] for a proof.
By the strong Markov property of the controlled process, we have for a stopping time $\tilde{\tau} \leq \tau$ ($\tau = \tau_\delta$ or $\tau = \tau_\delta \wedge T$):

$$J^{(\alpha)}(t, x) = \mathbb{E}^{(t, x)} \left\{ \int_t^{\tilde{\tau}} f(s, X_s^\alpha, \beta_s) ds + \sum_{\tau_j < \tilde{\tau}} K(\tau_j, \tilde{X}_{\tau_j -}^\alpha, \zeta_j) \right\}$$

$$+ \mathbb{E}^{(\tilde{\tau}, \tilde{X}_{\tilde{\tau} -})} \left\{ \int_{\tilde{\tau}}^\tau f(s, X_s^\alpha, \beta_s) ds + g(\tau, X_\tau^\alpha) 1_{\tau < \infty} + \sum_{\tilde{\tau} \leq \tau_j \leq \tau} K(\tau_j, \tilde{X}_{\tau_j -}^\alpha, \zeta_j) \right\}$$

$$= \mathbb{E}^{(t, x)} \left\{ \int_t^{\tilde{\tau}} f(s, X_s^\alpha, \beta_s) ds + \sum_{\tau_j < \tilde{\tau}} K(\tau_j, \tilde{X}_{\tau_j -}^\alpha, \zeta_j) + J^{(\alpha)}(\tilde{\tau}, \tilde{X}_{\tilde{\tau} -}) \right\}$$

(17)

$$\leq \mathbb{E}^{(t, x)} \left\{ \int_t^{\tilde{\tau}} f(s, X_s^\alpha, \beta_s) ds + \sum_{\tau_j < \tilde{\tau}} K(\tau_j, \tilde{X}_{\tau_j -}^\alpha, \zeta_j) + v(\tilde{\tau}, \tilde{X}_{\tilde{\tau} -}) \right\}. \quad (18)$$

Note especially that the second $J$ in (17) “starts” from $\tilde{X}_{\tilde{\tau} -}$, i.e. from $X$ before applying the possible impulses in $\tilde{\tau}$. This is to avoid counting a jump twice. $X_{\tilde{\tau}}$ instead of $\tilde{X}_{\tilde{\tau} -}$ in (17) would be incorrect (even if we replace the $=$ by a $\leq$). However, $J^{(\alpha)}(\tilde{\tau}, \tilde{X}_{\tilde{\tau} -}) \leq v(\tilde{\tau}, X_{\tilde{\tau}}) + K(\tilde{\tau}, \tilde{X}_{\tilde{\tau} -}, \zeta) 1_{\text{impulse in } \tilde{\tau}}$ holds because a (possibly non-optimal) decision to give an impulse $\zeta$ in $\tilde{\tau}$ influences $J$ and $v$ in the same way. So we have the modified inequality

$$J^{(\alpha)}(t, x) \leq \mathbb{E}^{(t, x)} \left\{ \int_t^{\tilde{\tau}} f(s, X_s^\alpha, \beta_s) ds + \sum_{\tau_j \leq \tilde{\tau}} K(\tau_j, \tilde{X}_{\tau_j -}^\alpha, \zeta_j) + v(\tilde{\tau}, X_{\tilde{\tau}}^\alpha) \right\}. \quad (19)$$

We will use both inequalities in the proof of Theorem 4.2 (viscosity existence).

By taking the sup, the above considerations can be formalised in the well-known dynamic programming principle (DPP) (if the admissibility set $A(t, x)$ satisfies certain natural criteria, see also [37]): For all $\tilde{\tau} \leq \tau$,

$$v(t, x) = \sup_{\alpha \in A(t, x)} \mathbb{E}^{(t, x)} \left\{ \int_t^{\tilde{\tau}} f(s, X_s^\alpha, \beta_s) ds + \sum_{\tau_j \leq \tilde{\tau}} K(\tau_j, \tilde{X}_{\tau_j -}^\alpha, \zeta_j) + v(\tilde{\tau}, X_{\tilde{\tau}}^\alpha) \right\}. \quad (20)$$

4. Existence

In this section, we are going to prove the existence of a QVI viscosity solution in the elliptic and parabolic case. Because a typical impulse control formulation will include the time, we will first prove the existence for the parabolic form, then transforming the problem including time component into a time-independent elliptic one (the problem formulation permitting).

4.1. Parabolic case

We consider in this section the parabolic QVI (9). Let us first define what exactly we mean by a viscosity solution of (9). Let $LSC(\Omega)$ (resp., $USC(\Omega)$) denote the set of measurable functions on the set $\Omega$ that are lower semicontinuous (resp., upper semicontinuous). Let $T > 0$, and let $u^*$
(\(u_+\)) define the upper (lower) semicontinuous envelope of a function \(u\) on \([0, T] \times \mathbb{R}^d\), i.e. the limit superior (limit inferior) is taken only from within this set. Let us also recall the definition of \(\mathcal{PB}\) encapsulating the growth condition from Section 2).

Recall the definition of \(S_T := [0, T) \times S\) and its parabolic “boundary” \(\partial^+ S_T := ([0, T) \times (\mathbb{R}^d \setminus S)) \cup \{T\} \times \mathbb{R}^d\).

**Definition 4.1 (Viscosity Solution).** A function \(u \in \mathcal{PB}([0, T] \times \mathbb{R}^d)\) is a (viscosity) subsolution of (9) if for all \((t_0, x_0) \in [0, T] \times \mathbb{R}^d\) and \(\varphi \in \mathcal{PB} \cap C^{1,2}([0, T) \times \mathbb{R}^d)\) with \(\varphi(t_0, x_0) = u^*(t_0, x_0)\), \(\varphi \geq u^*\) on \([0, T) \times \mathbb{R}^d\),

\[
\min \left( -\sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + L^\beta \varphi + f^\beta \right\}, u^* - M u^* \right) \leq 0 \quad \text{in} \quad (t_0, x_0) \in S_T
\]

\[
\min (u^* - g, u^* - M u^*) \leq 0 \quad \text{in} \quad (t_0, x_0) \in \partial^+ S_T.
\]

A function \(u \in \mathcal{PB}([0, T] \times \mathbb{R}^d)\) is a (viscosity) supersolution of (9) if for all \((t_0, x_0) \in [0, T) \times \mathbb{R}^d\) and \(\varphi \in \mathcal{PB} \cap C^{1,2}([0, T) \times \mathbb{R}^d)\) with \(\varphi(t_0, x_0) = u^*(t_0, x_0)\), \(\varphi \leq u^*\) on \([0, T) \times \mathbb{R}^d\),

\[
\min \left( -\sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + L^\beta \varphi + f^\beta \right\}, u^* - M u^* \right) \geq 0 \quad \text{in} \quad (t_0, x_0) \in S_T
\]

\[
\min (u^* - g, u^* - M u^*) \geq 0 \quad \text{in} \quad (t_0, x_0) \in \partial^+ S_T.
\]

A function \(u\) is a viscosity solution if it is sub- and supersolution.

The conditions on the parabolic boundary are included inside the viscosity solution definition (sometimes called “strong viscosity solution”, see, e.g., [22]) because of the implicit form of this “boundary condition”. In \(T\), we chose the implicit form too, because otherwise the comparison result would not hold. The time derivative in \(t = 0\) is of course to be understood as a one-sided derivative.

Now we can state the main result of the section, the existence theorem:

**Theorem 4.2 (Viscosity Solution: Existence).** Let Assumptions 2.1 and 2.2 be satisfied. Then the value function \(v\) is a viscosity solution of (9) as defined above.

For the proof of Theorem 4.2, we rely mainly on the proof given by [27], extending it to a general setting with jumps (compare also the sketch of proof in [31]). We need a sequence of lemmas beforehand. The following lemma states first and foremost that the operator \(\mathcal{M}\) preserves continuity. In a slightly different setting, the first two assertions can be found, e.g., in Lemma 5.5 of [27]. The complete proof is given in [35, Lemma 4.3].

**Lemma 4.3.** Let (V1), (V2) be satisfied for all parts except (v). Let \(u\) be a locally bounded function on \([0, T] \times \mathbb{R}^d\). Then

(i) \(M u_+ \in LSC([0, T] \times \mathbb{R}^d)\) and \(M u_+ \leq (M u)_+\).

(ii) \(M u_- \in USC([0, T] \times \mathbb{R}^d)\) and \((M u)_- \leq M u_-\).

(iii) If \(u \leq M u\) on \([0, T] \times \mathbb{R}^d\), then \(u_- \leq M u_-\) on \([0, T] \times \mathbb{R}^d\).

(iv) For an approximating sequence \((t_n, x_n) \to (t, x)\), \((t_n, x_n) \subset [0, T) \times \mathbb{R}^d\) with \(u(t_n, x_n) \to u^*(t, x)\): If \(u^*(t, x) > M u^*(t, x)\), then there exists \(N \in \mathbb{N}\) such that \(u(t_n, x_n) > M u(t_n, x_n)\forall n \geq N\).
Lemma 4.3 \[\text{continuity holds true uniformly in the control (this is of course a consequence of (E4)).}\]

Continuous processes on a compact time interval are uniformly stochastically continuous. This continuous Markov processes (see \[\text{outside probability processes are stochastically continuous, which means that at least the probability of } X_\tau \text{ being outside } \overline{B}(x, \rho_1) \text{ converges to } 1, \text{if } \rho_2 \to 0. \text{ Stochastic continuity as well holds for normal right-continuous Markov processes (see [18], Lemma 3.2), and thus for our SDE solutions.}\]

The lemma destined to overcome this problem essentially states the fact that stochastically continuous processes on a compact time interval are uniformly stochastically continuous. This lemma can be found in the working paper version of this article [35, Lemma 7.2]. A further lemma (Lemma 7.1 in [35]) shows that for a continuously controlled process, stochastic continuity holds true uniformly in the control (this is of course a consequence of (E4)).

Now we are ready for the proof of the existence theorem. Recall the necessary condition \((11)\) for the value function on the parabolic boundary \(\partial^+ S_T\).

**Proof of Theorem 4.2.** \(v\) is supersolution: First, for any \((t_0, x_0) \in [0, T] \times \mathbb{R}^d\), the inequality \(v(t_0, x_0) \geq \mathcal{M}v(t_0, x_0)\) holds, because otherwise an immediate jump would increase the value function. By Lemma \(4.3\) (i), \(\mathcal{M}v_\ast(t_0, x_0) \leq (\mathcal{M}v)_\ast(t_0, x_0) \leq v_\ast(t_0, x_0)\).

We then verify the condition on the parabolic boundary: Since we can decide to stop immediately, \(v \geq g\) on \(\partial^+ S_T\), so \(v_\ast \geq g\) follows by the continuity of \(g\) (outside \(\overline{S}\)) and requirement (E3) (if \(x_0 \in \partial S\) or \(t_0 = T\)).

So it remains to show the other part of the inequality

\[
- \sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}_\beta \varphi + f(\varphi) \right\} \geq 0
\]

in a fixed \((t_0, x_0) \in [0, T] \times S\), for \(\varphi \in \mathcal{P}B \cap C^{1,2}([0, T] \times \mathbb{R}^d), \varphi(t_0, x_0) = v_\ast(t_0, x_0), \varphi \leq v_\ast\) on \([0, T] \times \mathbb{R}^d\). From the definition of \(v_\ast\), there exists a sequence \((t_n, x_n) \in [0, T] \times S\) such that \((t_n, x_n) \to (t_0, x_0), v(t_n, x_n) \to v_\ast(t_0, x_0)\) for \(n \to \infty\). By continuity of \(v, \delta_n := v(t_n, x_n) - v(t_0, x_0)\) converges from above to \(0\) as \(n\) goes to infinity. Because \((t_0, x_0) \in [0, T] \times S\), there exists \(\rho > 0\) such that for \(n\) large enough, \(t_n < T\) and \(B(x_n, \rho) \subset B(x_0, 2\rho) = \{ |y - x_0| < 2\rho \} \subset S\).

Let us now consider the combined control with no impulses and a constant stochastic control \(\beta \in B\), and the corresponding controlled stochastic process \(X^{\beta, t_n, x_n}\) starting in \((t_n, x_n)\). Choose a strictly positive sequence \((h_n)\) such that \(h_n \to 0\) and \(\delta_n / h_n \to 0\) as \(n \to \infty\). For

\[
\bar{t}_n := \inf\{s \geq t_n : |X^{\beta, t_n, x_n} - x_n| \geq \rho\} \wedge (t_n + h_n) \wedge T,
\]

we get by the strong Markov property and Dynkin’s formula for \(\rho\) sufficiently small \((\mathbb{E}^n = \mathbb{E}^{(t_n, x_n)})\) denotes the expectation when \(X\) starts in \(t_n\) with \(x_n\):
\[ v(t_n, x_n) \geq \mathbb{E}^n \left[ \int_{t_n}^{\bar{\tau}_n} f(s, X_s^\beta, \beta) ds + v\left(\bar{\tau}_n, \hat{X}_{\bar{\tau}_n}^\beta\right) \right] \]
\[ \geq \mathbb{E}^n \left[ \int_{t_n}^{\bar{\tau}_n} f(s, X_s^\beta, \beta) ds + \varphi\left(\bar{\tau}_n, \hat{X}_{\bar{\tau}_n}^\beta\right) \right] \]
\[ = \varphi(t_n, x_n) + \mathbb{E}^n \left[ \int_{t_n}^{\bar{\tau}_n} f(s, X_s^\beta, \beta) + \frac{\partial \varphi}{\partial t}(s, X_s^\beta) + \mathcal{L}^\beta \varphi(s, X_s^\beta) \right] ds. \]

Here, our assumptions on the SDE coefficients of \( X \) were sufficient to apply Dynkin’s formula because of the localizing stopping time \( \bar{\tau}_n \). Using the definition of \( \delta_n \), we obtain

\[ \frac{\delta_n}{h_n} \geq \mathbb{E}^n \left[ \frac{1}{h_n} \int_{t_n}^{\bar{\tau}_n} f(s, X_s^\beta, \beta) + \frac{\partial \varphi}{\partial t}(s, X_s^\beta) + \mathcal{L}^\beta \varphi(s, X_s^\beta) \right] ds. \]  \tag{22} \]

Now, we want to let converge \( n \to \infty \) in (22), but it is not possible to apply the mean value theorem because \( s \mapsto f(s, X_s^\beta, \beta) \) (for fixed \( \omega \)) is in general not continuous. Select \( \varepsilon \in (0, \rho) \).

By Lemma 7.2 in [35], \( \mathbb{P}(\sup_{t_n \leq s \leq \bar{\tau}} |X_s^\beta| - x_n| > \varepsilon) \to 0 \) for \( r \downarrow t_n \). Define now

\[ A_{n,\varepsilon} = \{ \omega : \sup_{t_n \leq s \leq t_n + h_n} |X_s^\beta - x_n| \leq \varepsilon \}, \]

and split the integral in (22) into two parts

\[ \left( \int_{t_n}^{t_n + h_n} \right) 1_{A_{n,\varepsilon}} + \left( \int_{t_n}^{\bar{\tau}_n} \right) 1_{A_{n,\varepsilon}^c}. \]

On \( A_{n,\varepsilon} \), for the integrand \( G \) of the right hand side in (22),

\[ \left| G(t_0, x_0, \beta) - \frac{1}{h_n} \int_{t_n}^{t_n + h_n} G(s, X_s^\beta, \beta) \right| ds \]
\[ \leq \frac{1}{h_n} \int_{t_n}^{t_n + h_n} |G(t_0, x_0, \beta) - G(s, X_s^\beta, \beta)| ds \]
\[ \leq |G(t_0, x_0, \beta) - G(\hat{\tau}_{n,\varepsilon}, \hat{X}_{\hat{\tau}_{n,\varepsilon}}^\beta, \beta)|, \]  \tag{23}

the latter because \( G \) is continuous by (V3) and assumption on \( \varphi \), and the maximum distance of \( |G(t_0, x_0, \beta) - G(\cdot, \cdot, \beta)| \) is assumed in a \((\tilde{t}_{n,\varepsilon}, \hat{X}_{\hat{\tau}_{n,\varepsilon}}^\beta) \in [t_n, t_n + h_n] \times B(x_n, \varepsilon) \).

On the complement of \( A_{n,\varepsilon} \),

\[ \frac{1}{h_n} \left( \int_{t_n}^{\bar{\tau}_n} \right) 1_{A_{n,\varepsilon}^c} \leq \text{ess sup}_{t_n \leq s \leq \bar{\tau}_n} \left| f(s, X_s^\beta, \beta) + \frac{\partial \varphi}{\partial t}(s, X_s^\beta) + \mathcal{L}^\beta \varphi(s, X_s^\beta) \right| 1_{A_{n,\varepsilon}^c}, \]

which is bounded by the same arguments as above and because a jump in \( \bar{\tau}_n \) does not affect the essential supremum.

Because \( h_n \to 0 \) and \((t_n, x_n) \to (t_0, x_0)\) for \( n \to \infty \) and by stochastic continuity, \( \mathbb{P}(A_{n,\varepsilon}) \to 1 \), for all \( \varepsilon > 0 \) or, equivalently, \( 1_{A_{n,\varepsilon}} \to 1 \) almost surely. So by \( n \to \infty \) and then \( \varepsilon \to 0 \), we can conclude by the dominated convergence theorem that \( f(t_0, x_0, \beta) + \frac{\partial \varphi}{\partial t}(t_0, x_0) + \mathcal{L}^\beta \varphi(t_0, x_0) \leq 0 \forall \beta \in B \), and thus (21) holds. \( \square \)

\( v \) is subsolution: Let \((t_0, x_0) \in [0, T] \times \mathbb{R}^d \) and \( \varphi \in \mathcal{PB} \cap C^{1,2}([0, T] \times \mathbb{R}^d) \) such that \( v_+(t_0, x_0) = \varphi(t_0, x_0) \) and \( \varphi \geq v^\ast \) on \([0, T] \times \mathbb{R}^d \). If \( v_+(t_0, x_0) \leq \mathcal{M}v^\ast(t_0, x_0) \), then the subsolution inequality holds trivially. So consider from now on the case \( v^\ast(t_0, x_0) > \mathcal{M}v^\ast(t_0, x_0) \).
Consider \((t_0, x_0) \in \partial^+ S_T\). For an approximating sequence \((t_n, x_n) \to (t_0, x_0)\) in \([0, T] \times \mathbb{R}^d\) with \(v(t_n, x_n) \to v^*(t_0, x_0)\), the relation \(v(t_n, x_n) > \mathcal{M}v(t_n, x_n)\) holds by Lemma 4.3(iv). Thus by the continuity of \(g\) (outside \(\overline{S}\)) and requirement (E3) (if \(x_0 \in \partial S\) or \(t_0 = T\)),

\[
g(t_0, x_0) = \lim_{n \to \infty} g(t_n, x_n) = \lim_{n \to \infty} v(t_n, x_n) = v^*(t_0, x_0).
\]

Now let us show that, for \(v^*(t_0, x_0) > \mathcal{M}v^*(t_0, x_0)\),

\[
- \sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}^\beta \varphi + f^\beta \right\} \leq 0
\]

in \((t_0, x_0) \in [0, T) \times S\). We argue by contradiction and assume that there is an \(\eta > 0\) such that

\[
\sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}^\beta \varphi + f^\beta \right\} < -\eta < 0.
\]

Because \(\varphi \in \mathcal{PB} \cap C^{1,2}([0, T) \times \mathbb{R}^d)\) and the Hamiltonian is continuous in \((t, x)\) by (V3), there is an open set \(\mathcal{O}\) surrounding \((t_0, x_0)\) in \(S_T\) where \(\sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}^\beta \varphi + f^\beta \right\} < -\eta/2\).

From the definition of \(v^*\), there exists a sequence \((t_n, x_n) \in S_T \cap \mathcal{O}\) such that \((t_n, x_n) \to (t_0, x_0), v(t_n, x_n) \to v^*(t_0, x_0)\) for \(n \to \infty\). By continuity of \(\varphi\), again \(\delta_n := v(t_n, x_n) - \varphi(t_n, x_n)\) converges to 0 as \(n\) goes to infinity.

By definition of the value function, there exists for all \(\alpha \in \mathbb{R}\), \(\rho > 0\), \(\psi(\cdot)\), \(\beta \geq 0\), \(\beta_0 \geq 0\), \(\alpha_n = \alpha_n(\varphi) = (\beta^n, \gamma^n), \gamma^n = (\tau^n_i, \xi^n_i)_{i \geq 1}\) such that

\[
v(t_n, x_n) \leq J(\alpha^n)(t_n, x_n) + \varepsilon_n.
\]

For a \(\rho > 0\) chosen suitably small (i.e. \(B(\alpha_i, \rho) \subset B(x_0, 2 \rho) \subset S\), \(t_n + \rho < t_0 + 2\rho < T\) for large \(n\)), we define the stopping time \(\bar{\tau}_n := \bar{\tau}_n^\rho \wedge \tau^n_1\), where

\[
\bar{\tau}_n^\rho := \inf\{s \geq t_n : |X_s^{\alpha^n, t_n, x_n} - x_n| \geq \rho\} \wedge (t_n + \rho).
\]

We want to show that \(\bar{\tau}_n \to t_0\) in probability. From (26) combined with the Markov property (18), it immediately follows that (again \(E^n = E^{(t_n, x_n)}\), and \(\beta = \beta^n, \alpha = \alpha^n\))

\[
v(t_n, x_n) \leq E^n \left[ \int_{t_n}^{\bar{\tau}_n} f(s, X_s^{\beta^n, \beta_n})ds + \varphi(\bar{\tau}_n, \bar{X}_{\bar{\tau}_n}^{\beta^n}) \right] + \varepsilon_n
\]

\[
\leq E^n \left[ \int_{t_n}^{\bar{\tau}_n} f(s, X_s^{\beta^n, \beta_n})ds + \varphi(\bar{\tau}_n, \bar{X}_{\bar{\tau}_n}^{\beta^n}) + \bar{\varphi}(\bar{\tau}_n, \bar{X}_{\bar{\tau}_n}^{\beta^n}) \right] + \varepsilon_n.
\]

Thus by Dynkin’s formula on \(\varphi(\bar{\tau}_n, \bar{X}_{\bar{\tau}_n}^{\beta^n})\) and using \(v(t_n, x_n) = \varphi(t_n, x_n) + \delta_n\),

\[
\delta_n \leq E^n \left[ \int_{t_n}^{\bar{\tau}_n} f(s, X_s^{\beta^n, \beta_n}) + \frac{\partial \varphi}{\partial s}(s, X_s^{\beta^n, \beta_n}) + \mathcal{L}^\beta \varphi(s, X_s^{\beta^n})ds \right] + \varepsilon_n
\]

\[
\leq \frac{\eta}{2} E^n [\bar{\tau}_n - t_n] + \varepsilon_n,
\]

\(^3\) In the following, we will switch between \(\alpha\) and \(\beta\) in our notation, where the usage of \(\beta\) indicates that there is no impulse to take into account.
where for $\rho$ small enough, we have applied (25). This implies that $\lim_{n \to \infty} \mathbb{E}[\tilde{\tau}_n] = t_0$, which is equivalent to $\tilde{\tau}_n \to t_0$ in probability (as one can easily check with Chebyshev’s inequality; $\tilde{\tau}_n$ is bounded).

Now let us continue with our proof. In the following, we make again use of the stochastic continuity of $X_{\beta_t}^{I_n,t_n,x_n}$ (up until the first impulse). We define

$$A_n(\rho) = \{\omega: \sup_{t_n \leq s \leq \tilde{\tau}_n} |X_{\beta_t}^{I_n,t_n,x_n} - x_n| \leq \rho\}.$$ 

(26) combined with the Markov property (19) gives us

$$v(t_n, x_n) \leq \mathbb{E}^n \left[ \int_{t_n}^{\tilde{\tau}_n} f(s, X^\beta_{\xi(s)}, \beta_s)ds + K(\tau_1^0, \dot{X}^\beta_{\tau_1^0}, \xi_1^m)1_{\tilde{\tau}_n^0 \geq \tau_1^0} + v(\tilde{\tau}_n, X^\alpha_{\tilde{\tau}_n}) \right] + \varepsilon_n. \quad (28)$$

To find upper estimates for $v(t_n, x_n)$, we use indicator functions for three separate cases:

1. $\{\tilde{\tau}_n^0 < \tau_1^0\}$
2. $\{\tilde{\tau}_n^0 \geq \tau_1^0\} \cap A_n(\rho)^c$
3. $\{\tilde{\tau}_n^0 \geq \tau_1^0\} \cap A_n(\rho)$

(III) is the predominant set: For any sequence $(\hat{\tau}_n)$, by basic probability, $\mathbb{P}(\tilde{\tau}_n^0 \geq \tau_1^0) \geq 1 - \mathbb{P}(\tilde{\tau}_n^0 < \hat{\tau}_n) \geq \mathbb{P}(\tau_1^0 \geq \hat{\tau}_n) \to 0$. Choose $\hat{\tau}_n \downarrow t_0$ such that $\mathbb{P}(\tau_1^0 \geq \hat{\tau}_n) \to 0$. By Lemma 7.2(iii) in [35] (wlog, we need only consider the setting without impulses), also $\limsup_{n \to \infty} \mathbb{P}(\tilde{\tau}_n^0 < \hat{\tau}_n) = 0$. In total, we have $\mathbb{P}(\text{III}) \to 1$ or $1_{\text{III}} \to 1$ a.s. for $n \to \infty$.

Thus, using that if there is an impulse in $\tilde{\tau}_n$ (i.e. $\tilde{\tau}_n^0 \geq \tau_1^0$), then $v(\tilde{\tau}_n, X^\alpha_{\tilde{\tau}_n}) + K(\tilde{\tau}_n, \dot{X}^\beta_{\tilde{\tau}_n}, \xi_1^m) \leq \mathcal{M}v(\tilde{\tau}_n, \dot{X}^\beta_{\tilde{\tau}_n})$, $4$

$$v(t_n, x_n) \leq \sup_{|t' - t_0| < \rho, |y' - x_0| < \rho} f(t', y', \beta_{t'}) \mathbb{E}[\tilde{\tau}_n - t_n]$$

$$+ \mathbb{E}[|v(\tilde{\tau}_n, X^\beta_{\tilde{\tau}_n})|1_{(I)}] + \mathbb{E}[|\mathcal{M}v(\tilde{\tau}_n, \dot{X}^\beta_{\tilde{\tau}_n}, \xi_1^m)|1_{(II)}]$$

$$+ \sup_{|t' - t_0| < \rho, |y' - x_0| < \rho} \mathcal{M}v(t', y'). \quad \text{(III)}.$$ 

To prove the boundedness in term (I) (uniform in $n$), we can assume wlog that $v(\mathbb{R}^k) = 1$, and consider only jumps bounded away from 0 for $x_0 > 0$. Then by (E1), $\mathbb{E} \sup_n |v(\tilde{\tau}_n, X^\beta_{\tilde{\tau}_n})| \leq C \mathbb{E}[1 + R(x_0 + 2\rho + Y)]$ for a jump $Y$ with distribution $v^{\ell(t_0 + 2\rho, x_0 + 2\rho, \beta, \cdot)}$, which is finite by the definition of $\mathcal{PB}$. The same is true for $\mathcal{M}v(\tilde{\tau}_n, \dot{X}^\beta_{\tilde{\tau}_n})$ (for $\mathcal{M}v \geq 0$ because $\mathcal{M}v \leq v$, for negative $\mathcal{M}v$ this follows from the definition).

Sending $n \to \infty$ (lim sup $n \to \infty$), the $f$-term, and term (I) converge to 0 by the dominated convergence theorem. For term (II), a general version of the DCT shows that it is bounded by

$$\mathbb{E}[\limsup_{n \to \infty} |\mathcal{M}v(\tilde{\tau}_n, \dot{X}^\beta_{\tilde{\tau}_n}, \xi_1^m)|1_{A_n(\rho)^c}].$$

---

4 Note: More than one impulse could occur in $\tilde{\tau}_n$ if the transaction cost structure allows for it (e.g., $K$ quadratic in $\xi$). In this case however, the result follows by monotonicity of $\mathcal{M}$ (Lemma 4.3, (v)).
and term (III) becomes $\sup M v(t', y')$. Now we let $\rho \to 0$ and obtain:

$$v^*(t_0, x_0) \leq \lim_{\rho \downarrow 0} \sup_{|y' - y_0| < \rho} M v(t', y') = (M v)^*(t_0, x_0) \leq M v^*(t_0, x_0)$$

by Lemma 4.3(ii), a contradiction. Thus (24) is true. □

Let us elaborate on some details of the proof:

- In the proof, we have only used that all constant controls with values in $B$ are admissible for the SDE (4). So actually, we are quite free how to choose the set of admissible controls—the value function always turns out to be a viscosity solution.
- Another approach for the subsolution part would be tempting, although we do not see how this can work: In the subsolution proof, we assumed $v^*(t_0, x_0) - M v^*(t_0, x_0) > 0$. This implies, using Lemma 4.3(iv) and the $0\cd 1$ law, that for $n$ large enough, $\tau^{n}_1 > t_n$ a.s. On the other hand, from $\tau_n \to t_0$, it follows by Lemma 7.1 in [35] that $\tau^{n}_1 \to t_0$ in probability (it is sufficient to consider the setting of Lemma 7.1 in [35] without impulse, since otherwise the first impulse would anyhow converge to 0 in probability). So the convergence of $\tau^{n}_1$ points already to a contradiction.

We promised to come back to the "regularity at $\partial S$" issue, and present here conditions sufficient for condition (E3). A proof can be found in [35], Prop 4.4.

(E1*) For any point $(t, x) \in [0, T) \times \partial S$, any sequence $(t_n, x_n) \subset [0, T) \times S$, $(t_n, x_n) \to (t, x)$, and for each small $\varepsilon > 0$, there is an admissible combined control $\alpha_n = (\beta_n, \gamma_n)$ such that

$$v(t_n, x_n) \leq J^{(\alpha_n)}(t_n, x_n) + \varepsilon, \quad (29)$$

and such that for all $\delta > 0$, $\mathbb{P}(\tilde{\tau}^{n}_S < t_n + \delta) \to 1$ for $n \to \infty$ (where $\tilde{\tau}^{n}_S = \inf\{s \geq t_n : X^{b_n, \alpha_n}(s) \notin S\}$).

(E2*) For any point $(t, x) \in \partial^* S_T$, if there is a sequence $(t_n, x_n) \subset [0, T) \times S$ converging to $(t, x)$ with $v(t_n, x_n) > M v(t_n, x_n)$, then there is a neighbourhood of $(t, x)$ (open in $[0, T) \times \mathbb{R}^d$) where $v > M v$.

Example 4.1. Let $X$ be a one-dimensional Brownian motion with $\sigma > 0$, and assume it is never optimal to give an impulse near the boundary. Then (E1*) and (E2*) are satisfied.

Remark 4.1. (E3) (resp, (E1*), (E2*)) excludes in particular problems with de facto state constraints, where it is optimal to stay inside $S$. We note however that the framework presented here allows for an adaptation to (true and de facto) state constraints, which can be pretty straightforward for easy constraints. Apart from the stochastic proof that we can restrain ourselves to controls keeping the process inside $S$, the adaptation involves changing the function $w$ used in the uniqueness part, such that only values in $S$ need to be considered in the comparison proof. For an example in the diffusion case, see [27]; jumps outside $S$ however may be difficult to handle.

4.2. Elliptic case

The existence result for the elliptic QVI (14) now follows from the parabolic result by an exponential time transformation (see the working paper version of this article [35] for more details). The definition of viscosity solution is completely analogous to the parabolic case (see also Section 5.2).
Here the functions \( f, g, \) and \( K \) to be used in the parabolic result are all multiplied by \( e^{-\rho(s+t)} \) (for \( s \geq 0, \) and the rate \( \rho > 0 \)). \( \Gamma, \mu, \sigma, \ell \) and the transaction set \( Z \) have to be time-independent. If the value function of the corresponding parabolic problem is denoted by \( \tilde{v} \), then we define
\[
v(x) := e^{\alpha(x+t)} \tilde{v}(t,x) .
\]

The existence of the elliptic QVI then follows by an easy time transformation:

**Corollary 4.4.** Let Assumptions 2.1 and 2.2 be satisfied. Then the value function \( v \) as defined above is a viscosity solution of (14).

5. Uniqueness

The purpose of this section is to prove uniqueness results both for the elliptic and the parabolic QVI by analytic means. Using such a uniqueness result, together with the existence results of Section 4, we can conclude that the viscosity solution of the QVI is equal to the value function.

We were inspired mainly by the papers [22] (for the impulse part) and [7] (for the PIDE part). As general reference for viscosity solutions, the “User’s Guide” [14] was used and will be frequently cited. Some ideas have come from [30,1,23,2].

In this section, \( v \) does not denote the value function any longer, and some other variables may serve new purposes as well.

First, we will investigate uniqueness of QVI viscosity solutions for the elliptic case of Eq. (14); the parabolic case will follow at the end.

5.1. Preliminaries

Whereas in the last section, we did not care about the specific form of the generator (as long as Dynkin’s formula was valid), we now need to investigate the operator \( \mathcal{L}^\beta \) more in detail:
\[
\mathcal{L}^\beta u(x) = \frac{1}{2} \text{tr} \left( \sigma(x, \beta)\sigma^T(x, \beta)D^2 u(x) \right) + \langle \mu(x, \beta), \nabla u(x) \rangle - c(x)u(x)
+ \int u(x + \ell(x, \beta, z)) - u(x) - \langle \nabla u(x), \ell(x, \beta, z) \rangle 1_{|z|<1} v(dz),
\]
where \( c \) is some positive function related to the discounting in the original model.\(^5\)

We recall the definition of the function space \( \mathcal{PB} = \mathcal{PB}(\mathbb{R}^d) \) from Section 2, such that the differential operator \( \mathcal{L}^\beta \) is well-defined for \( \phi \in \mathcal{PB} \cap C^2(\mathbb{R}^d) \). Denoting for \( 0 < \delta < 1, y, p \in \mathbb{R}^d, X \in \mathbb{R}^{d \times d}, r \in \mathbb{R} \) and \( (l_\beta)_{\beta \in B} \subset \mathbb{R}:
\]

\[
F(x, r, p, X, (l_\beta)) = - \sup_{\beta \in B} \left\{ \frac{1}{2} \text{tr} \left( \sigma(x, \beta)\sigma^T(x, \beta)X \right) + \langle \mu(x, \beta), p \rangle - c(x)r + f(x, \beta) + l_\beta \right\}
\]

\[
\mathcal{I}^1_{\beta, \delta} [x, \phi] = \int_{|z|<\delta} \phi(x + \ell(x, \beta, z)) - \phi(x) - \langle \nabla \phi(x), \ell(x, \beta, z) \rangle v(dz)
\]

\[
\mathcal{I}^2_{\beta, \delta} [x, p, \phi] = \int_{|z|\geq\delta} \phi(x + \ell(x, \beta, z)) - \phi(x) - \langle p, \ell(x, \beta, z) \rangle 1_{|z|<1} v(dz)
\]

\[
\mathcal{I}_\beta [x, \phi] = \mathcal{I}^1_{\beta, \delta} [x, \phi] + \mathcal{I}^2_{\beta, \delta} [x, \nabla \phi(x), \phi],
\]

\(^5\) Note that the normal definition of a Lévy integral is with the indicator function \( 1_{|z| \leq 1} \); this is however equivalent.
we have to analyse the problem
\[
\min(F(x, u(x), \nabla u(x), D^2 u(x), \mathcal{I}_\beta[x, u(\cdot)]), u(x) - \mathcal{M} u(x)) = 0,
\]
where the notation \( u(\cdot) \) in the integral indicates that nonlocal terms are used on \( u \), not only from \( x \). As well, \( \mathcal{I}_\beta[x, u(\cdot)] \) within \( F \) always stands for a family \( (\beta \in B) \) of integrals. Denote by \( F^\beta \) the function \( F \) without the sup, i.e. for a concrete \( \beta \).

**Remark 5.1.** The following properties hold for our problem:

(P1) Ellipticity of \( F \): \( F(x, r, p, X^1, (l^1_\beta)) \leq F(x, r, p, X^2, (l^2_\beta)) \) if \( X^1 \geq X^2, l^1_\beta \geq l^2_\beta \ \forall \beta \in B \)

(P2) Translation invariance: \( u - \mathcal{M} u = (u + l) - \mathcal{M} (u + l), \mathcal{I}_{[y_0, \phi]} = \mathcal{I}_{[y_0, \phi + l]} \) for constants \( l \in \mathbb{R} \)

(P3) \( (l_\beta) \beta \mapsto \mathcal{F}(x, r, p, X, (l_\beta)) \) is continuous in the sense that
\[
|\mathcal{F}(x, r, p, X, (l^1_\beta)) - \mathcal{F}(x, r, p, X, (l^2_\beta))| \leq \sup_\beta |l^1_\beta - l^2_\beta|.
\]

The last statement – proved by easy sup manipulations – is just for the sake of completeness; we will not use it explicitly because uniform convergence needs continuous functions, which in general we do not have.

For \( x \in S \) and \( \delta > 0 \), recall the definition of \( \bar{U}_\delta = U_\delta(x) = \{\ell(x, \beta, z) : |z| < \delta\} \). \( U_\delta \) facilitates splitting the integral \( \mathcal{I}_\beta[x, \phi] \): changing \( \phi \) on \( U_\delta \) only affects \( \mathcal{I}^{1,\delta}_\beta[x, \phi] \), and reversely.

Henceforth let Assumptions 2.1 and 2.3 be satisfied (the latter needed mainly for the equivalence of different viscosity solution definitions). Further assume

(U1*) \( c \) is continuous.

**Remark 5.2.** It is sufficient for the comparison theorem if Assumption 2.3 holds only for small \( \delta > 0 \): “For any \( x_0 \), there is a small environment and a \( \bar{\delta} > 0 \), where the assumption holds for \( 0 < \delta < \bar{\delta} \ldots \).” This is why we will carry out all proofs for \( \ell \) depending on \( \beta \) in the following.

Immediately from (V3) and (U1*), it follows that \( \sup_{\beta \in B} \sigma(x, \beta) < \infty \), and by sup manipulations that \( (x, r, p, X) \mapsto \mathcal{F}(x, r, p, X, l_\beta) \) is continuous; but even more can be deduced:

**Proposition 5.1.** Let \( (\beta_k) \subset B \) with \( \beta_k \to \beta \), and \( (x_k), (p_k) \subset \mathbb{R}^d \) with \( x_k \to x \in S, p_k \to p \).

(i) If \( u \in \mathcal{PB} \cap \text{USC}(\mathbb{R}^d) \) and \( v \in \mathcal{PB} \cap \text{LSC}(\mathbb{R}^d) \) with \( u(x_k) \to u(x), v(x_k) \to v(x) \), then
\[
\limsup_{k \to \infty} \mathcal{I}^{2,\delta}_{\beta_k}[x_k, p_k, u(\cdot)] \leq \mathcal{I}^{2,\delta}_\beta[x, p, u(\cdot)],
\]
\[
\liminf_{k \to \infty} \mathcal{I}^{2,\delta}_{\beta_k}[x_k, p_k, v(\cdot)] \geq \mathcal{I}^{2,\delta}_\beta[x, p, v(\cdot)].
\]

Moreover, for \( (\varphi_k), (\psi_k) \subset \mathcal{PB} \cap C^2(\mathbb{R}^d) \) with \( \varphi_k \to u \) and \( \psi_k \to v \) monotonously, \( \varphi_k(x_k) \to u(x), \psi_k(x_k) \to v(x) \),
\[
\limsup_{k \to \infty} \mathcal{I}^{2,\delta}_{\beta_k}[x_k, p_k, \varphi_k(\cdot)] \leq \mathcal{I}^{2,\delta}_\beta[x, p, u(\cdot)],
\]
\[
\liminf_{k \to \infty} \mathcal{I}^{2,\delta}_{\beta_k}[x_k, p_k, \psi_k(\cdot)] \geq \mathcal{I}^{2,\delta}_\beta[x, p, v(\cdot)].
\]
(ii) If $\varphi \in C^2(\mathbb{R}^d)$, then $(x, \beta) \mapsto T^1_{\beta} [x, \varphi(\cdot)]$ is continuous. Moreover, for $(\varphi_k) \subset C^2(\mathbb{R}^d)$ with $\varphi_k \to \varphi$ monotonously, $\varphi_k = \varphi$ in an environment of $x$,

$$\lim_{k \to \infty} T^1_{\beta_k} [x_k, \varphi_k(\cdot)] = T^1_{\beta} [x, \varphi(\cdot)].$$

(iii) If $u \in \mathcal{PB} \cap U SC(\mathbb{R}^d)$ and $v \in \mathcal{PB} \cap L SC(\mathbb{R}^d)$, $(\varphi_k), (\psi_k) \subset \mathcal{PB} \cap C^2(\mathbb{R}^d)$ with $\varphi_k \to u$ and $\psi_k \to v$ monotonously, $\varphi_k = u \in C^2$ and $\psi_k = v \in C^2$ in an environment of $x$, then

$$\liminf_{k \to \infty} F^\beta_k (x, r, p, X, I_{\beta_k} [x_k, \varphi_k(\cdot)]) \geq F^\beta (x, r, p, X, I_{\beta} [x, u(\cdot)])$$

$$\limsup_{k \to \infty} F^\beta_k (x, r, p, X, I_{\beta_k} [x_k, \psi_k(\cdot)]) \leq F^\beta (x, r, p, X, I_{\beta} [x, v(\cdot)]).$$

(iv) If $u \in \mathcal{PB} \cap U SC(\mathbb{R}^d)$, $\varphi \in C^2(\mathbb{R}^d)$, then $\beta \mapsto -F^\beta (x, r, p, X, I_{\beta}^{1, \delta} [x, \varphi(\cdot)]) + I_{\beta}^{2, \delta} [x, p, u(\cdot)]$ is in $U SC(\mathbb{R}^d)$. In particular, the supremum in $\beta \in B$ is assumed.

**Proof.** (i): We prove only the first statement for $u \in U SC$ (the lsc proof being analogous). By a general version of the DCT and the definition of $\mathcal{PB}$,

$$\limsup_{k \to \infty} \int_{|z| \leq \delta} u(x_k + \ell(x_k, \beta_k, z)) - u(x_k) - \langle p_k, \ell(x_k, \beta_k, z) \rangle 1_{|z| < 1} \nu(\mathrm{d}z)$$

$$\leq \int_{|z| \leq \delta} \limsup_{k \to \infty} u(x_k + \ell(x_k, \beta_k, z)) - u(x_k) - \langle p_k, \ell(x_k, \beta_k, z) \rangle 1_{|z| \leq 1} \nu(\mathrm{d}z)$$

$$\leq \int_{|z| \leq \delta} u(x + \ell(x, \beta, z)) - u(x) - \langle p, \ell(x, \beta, z) \rangle 1_{|z| < 1} \nu(\mathrm{d}z),$$

(34)

where we have used the continuity of $\ell$ (V3). The fact for $\varphi_k$ follows because by monotonicity of $(\varphi_k)$, and a general version of Dini’s Theorem (cf. [17], Th. 7.3), $\limsup_{k \to \infty} \varphi_k (x_k + \ell(x_k, \beta_k, z)) \leq u^*(x + \ell(x, \beta, z))$.

(ii): Outside of the singularity of $\nu$, the result follows from (i). In the environment of $x$, the Taylor expansion (13) gives the upper bound for the application of the DCT (where the local boundedness of $U_S(x)$ holds by (V4)).

(iii) follows immediately from (i), (ii). Finally, (iv) holds by the continuity conditions (V3), (U1*) and (i), (ii). □

5.2. Viscosity solutions: Different definitions

In this subsection, we introduce (as in [7]) two equivalent definitions that are necessary to handle the (possibly) singular integral part. The reader will certainly be grateful that we have omitted a further definition which is only needed to prove the equivalence. It can be found in the working paper version of this article [35], or in [7] without (impulse) control.

First let us state in the new notation the definition of viscosity solution in the elliptic case (equivalent thanks to the translation invariance property):

**Definition 5.2 (Viscosity Solution 1).** A function $u \in \mathcal{PB}$ is a (viscosity) subsolution of (14) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ such that $u^* - \varphi$ has a global maximum in $x_0$,

$$\min \left( F(x_0, u^*, \nabla \varphi, D^2 \varphi, I_{\beta}[x_0, \varphi(\cdot)]), u^* - \mathcal{M} u^* \right) \leq 0 \text{ in } x_0 \in S$$

$$\min (u^* - g, u^* - \mathcal{M} u^*) \leq 0 \text{ in } x_0 \in (\mathbb{R}^d \setminus S).$$
A function $u \in \mathcal{PB}$ is a (viscosity) supersolution of (14) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ such that $u_* - \varphi$ has a global minimum in $x_0$,

$$\min \left( F(x_0, u_*, \nabla \varphi, D^2 \varphi, I_\beta[x_0, \varphi(\cdot)]), u_* - \mathcal{M}u_* \right) \geq 0 \quad \text{in } x_0 \in S$$

$$\min (u_* - g, u_* - \mathcal{M}u_*) \geq 0 \quad \text{in } x_0 \in (\mathbb{R}^d \setminus S).$$

A function $u$ is a viscosity solution if it is sub- and supersolution.

We recall the semijets needed for another equivalent definition (Definition 3 because we have omitted Definition 2). They are motivated by a classical property of differentiable functions. Let $u : \mathbb{R}^d \to \mathbb{R}$.

$$J^+u(x) = \{(p, X) \in \mathbb{R}^d \times \mathbb{S}^d : u(x + z) \leq u(x) + \langle p, z \rangle + \frac{1}{2} \langle Xz, z \rangle + o(|z|^2) \text{ as } z \to 0\}$$

$$J^-u(x) = \{(p, X) \in \mathbb{R}^d \times \mathbb{S}^d : u(x + z) \geq u(x) + \langle p, z \rangle + \frac{1}{2} \langle Xz, z \rangle + o(|z|^2) \text{ as } z \to 0\}.$$ If $u$ is twice differentiable at $x$, then $J^+u(x) \cap J^-u(x) = \{(\nabla u(x), D^2u(x))\}$. The limiting semijets are defined by, e.g.,

$$\overline{J}^+u(x) = \{(p, X) \in \mathbb{R}^d \times \mathbb{S}^d : \text{ there exist } (x_k, p_k, X_k) \to (x, p, X), (p_k, X_k) \in J^+u(x_k) \text{ such that } u(x_k) \to u(x)\}.$$}

**Definition 5.3** *(Viscosity Solution 3).* A function $u \in \mathcal{PB}$ is a (viscosity) subsolution of (14) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $u^* - \varphi$ has a maximum in $x_0$ on $U_\delta(x_0)$ and for $(p, X) \in J^+u(x_0)$ with $p = D\varphi(x_0)$ and $X \leq D^2\varphi(x_0)$,

$$\min \left( F(x_0, u^*, p, X, I_\beta^{1,\delta}[x_0, \varphi(\cdot)] + I_\beta^{2,\delta}[x_0, p, u^*(\cdot)]), u^* - \mathcal{M}u^* \right) \leq 0 \quad \text{in } x_0 \in S$$

$$\min (u^* - g, u^* - \mathcal{M}u^*) \leq 0 \quad \text{in } x_0 \in (\mathbb{R}^d \setminus S).$$

A function $u \in \mathcal{PB}$ is a (viscosity) supersolution of (14) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $u_* - \varphi$ has a minimum in $x_0$ on $U_\delta(x_0)$ and for $(q, Y) \in J^-u(x_0)$ with $q = D\varphi(x_0)$ and $Y \geq D^2\varphi(x_0)$,

$$\min \left( F(x_0, u_*, q, Y, I_\beta^{1,\delta}[x_0, \varphi(\cdot)] + I_\beta^{2,\delta}[x_0, q, u_*(\cdot)]), u_* - \mathcal{M}u_* \right) \geq 0 \quad \text{in } x_0 \in S$$

$$\min (u_* - g, u_* - \mathcal{M}u_*) \geq 0 \quad \text{in } x_0 \in (\mathbb{R}^d \setminus S).$$

A function $u$ is a viscosity solution if it is sub- and supersolution.

Note that the definition is of course still valid if $U_\delta(x_0) \cap (\mathbb{R}^d \setminus S) \neq \emptyset$. The conditions $p = D\varphi(x_0)$ and $X \leq D^2\varphi(x_0)$ etc. and the maximum condition seem to be superfluous at first view. However, they are needed to ensure consistency of $\varphi$ with the “local” derivatives $(p, X)$.

**Proposition 5.4.** Definition 1 and Definition 3 are equivalent.

**Proof.** See [35], and [7]. The proof first shows the equivalence of Definition 1 and Definition 2 (which basically replaces $\varphi$ in the integral by $u$, and instead considers the local maxi-
5.3. A maximum principle

Following [7] we give here a nonlocal theorem which should replace the “maximum principle”. Prior to this, we have to collect some properties of the intervention operator \( \mathcal{M} \) (compare also Lemma 4.3):

**Lemma 5.5.** (i) \( \mathcal{M} \) is convex, i.e. for \( \lambda \in [0, 1], \mathcal{M}(\lambda a + (1 - \lambda)b) \leq \lambda \mathcal{M}a + (1 - \lambda)\mathcal{M}b. \)
(ii) For \( \lambda > 0, \mathcal{M}(-\lambda a + (1 + \lambda)b) \geq -\lambda \mathcal{M}a + (1 + \lambda)\mathcal{M}b \) (assuming the latter is not \( \infty - \infty \)).

**Proof.** Follows easily from \( \sup_x (a(x) + b(x)) \leq \sup_x a(x) + \sup_x b(x) \) and \( \sup_x (a(x) + b(x)) \geq \sup_x a(x) + \inf_x b(x) \), respectively. \( \square \)

We need the following nonlocal Jensen–Ishii lemma that can be applied in the PIDE case (compare the discussion below):

**Lemma 5.6 (Lemma 1 in [7]).** Let \( u \in USC(\mathbb{R}^d) \) and \( v \in LSC(\mathbb{R}^d), v \in C^2(\mathbb{R}^{2d}). \) If \( (x_0, y_0) \in \mathbb{R}^{2d} \) is a zero global maximum point of \( u(x) - v(y) - \varphi(x, y) \) and if \( p = D_x \varphi(x_0, y_0), q = D_y \varphi(x_0, y_0), \) then for any \( K > 0, \) there exists \( \tilde{a}(K) > 0 \) such that, for any \( 0 < \alpha < \tilde{a}(K), \) we have: There exist sequences \( x_k \to x_0, y_k \to y_0, p_k \to p, q_k \to q, \) matrices \( X_k, Y_k \) and a sequence of functions \( (\varphi_k), \) converging to the function \( \varphi_\alpha(x, y) := R^\alpha[\varphi](x, y, (p, q)) \) uniformly in \( \mathbb{R}^{2d} \) and in \( C^2(B((x_0, y_0), K)), \) such that

\[
\begin{align*}
  u(x_k) &\to u(x_0), & v(y_k) &\to v(y_0) \\
  (x_k, y_k) &\text{ is a global maximum point of } u - v - \varphi_k \\
  (p_k, X_k) &\in J^+ u(x_k) \\
  (q_k, Y_k) &\in J^- v(y_k) \\
  -\frac{1}{\alpha} I &\leq \begin{bmatrix} X_k & 0 \\ 0 & -Y_k \end{bmatrix} \leq D^2 \varphi_k(x_k, y_k). 
\end{align*}
\]

Here \( p_k = \nabla_x \varphi_k(x_k, y_k), q_k = \nabla_y \varphi_k(x_k, y_k), \) and \( \varphi_\alpha(x_0, y_0) = \varphi(x_0, y_0), \) \( \nabla \varphi_\alpha(x_0, y_0) = \nabla \varphi(x_0, y_0). \)

**Remark 5.3.** The expression \( \varphi_\alpha(x, y) = R^\alpha[\varphi](x, y, (p, q)) \) is the “modified super-convolution” as used by [7]. For all compacts \( C, \varphi_\alpha \) converges uniformly to \( \varphi \) in \( C^2(C) \) as \( \alpha \to 0. \) This was already used in [7], and can be seen by classical arguments using the implicit function theorem.

We would obtain a variant of the local Jensen–Ishii lemma (also called maximum principle), if we were not interested in the sequence \( (\varphi_k) \) converging in \( C^2 \)—in this case the statement could be expressed in terms of the \textit{limiting semijets} (or “closures”) \( \mathcal{T}^+, \mathcal{T}^- \) (e.g., \( (p, X) \in \mathcal{T}^+ u(x_0) \)). However, the local Jensen–Ishii lemma is only useful (in the PDE case), because it can be directly used to deduce, e.g., \( F(x_0, u^*(x_0), p, X) \leq 0 \) by continuity of \( F. \) Compare also the more detailed explanation in [23].

This immediate consequence in the PDE case is a bit more tedious to show in our PIDE case (because the Lévy measure \( \upsilon \) is possibly singular at 0), and needs the \( C^2 \) convergence of the \( (\varphi_k) \).

The corollary for our impulse control purposes takes the following form:
Corollary 5.7. Assume (V1), (V2). Let $u$ be a viscosity subsolution and $v$ a viscosity supersolution of (14), and $\varphi \in C^2(\mathbb{R}^{2d})$. If $(x_0, y_0) \in \mathbb{R}^{2d}$ is a global maximum point of $u^*(x) - v_\ast(y) - \varphi(x, y)$, then, for any $\delta > 0$, there exists $\tilde{\alpha}$ such that for $0 < \alpha < \tilde{\alpha}$, there are $(p, X) \in \mathcal{J}^+ u(x_0)$ and $(q, Y) \in \mathcal{J}^- v(y_0)$ with
\[
\min \left( F(x_0, u^*(x_0), p, X, \mathcal{T}^{1,\delta}_\beta [x_0, \varphi_\alpha(\cdot, y_0)]) + \mathcal{T}^{2,\delta}_\beta [x_0, p, u^*(\cdot)], u^* - \mathcal{M} u^* \right) \leq 0
\]
\[
\min \left( F(y_0, v_\ast(y_0), q, Y, \mathcal{T}^{1,\delta}_\beta [y_0, -\varphi_\alpha(x_0, \cdot)] + \mathcal{T}^{2,\delta}_\beta [y_0, q, v_\ast(\cdot)], v_\ast - \mathcal{M} v_\ast \right) \geq 0
\]
if $x_0 \in S$ or $y_0 \in S$, respectively. Here $p = \nabla_x \varphi(x_0, y_0) = \nabla_x \varphi_\alpha(x_0, y_0)$, $q = -\nabla_y \varphi(x_0, y_0) = -\nabla_y \varphi_\alpha(x_0, y_0)$, and furthermore,
\[
- \frac{1}{\alpha} I \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq D^2 \varphi_\alpha(x_0, y_0) = D^2 \varphi(x_0, y_0) + o_\alpha(1).
\] (40)

If none of $x_0$, $y_0$ is in $S$, then we do not need this corollary, because the boundary conditions are not dependent on derivatives of functions $\varphi$.

Remark 5.4. Note that the fact $(p, X) \in \mathcal{J}^+ u(x_0)$ (and the corresponding for the supersolution) is not needed in the statement of the corollary, because the subsolution (supersolution) inequality directly holds by the approximation procedure in the proof. An abstract way of formulating Lemma 5.6 and Corollary 5.7 in the style of the local Jensen–Itô Lemma (only subsolution without impulses) would be to define a new “limiting superjet” containing the $(p, X, \varphi_\alpha)$ obtained as limit of the terms in Lemma 5.6. Then Corollary 5.7 could be stated as “For $(p, X, \varphi_\alpha)$ in the limiting superjet, $F(x_0, u^*(x_0), p, X, \varphi_\alpha(\cdot), u^*(\cdot)) \leq 0$” and would follow directly from Lemma 5.6, provided some “continuity” of $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times C^2 \times \mathcal{P} \mathcal{B} \rightarrow \mathbb{R}$ hold.

Proof of Corollary 5.7. Because of translation invariance, we can assume wlog that $u^*(x_0) - v_\ast(y_0) - \varphi(x_0, y_0) = 0$. Choose sequences according to Lemma 5.6 (applied for $u^*$ and $v_\ast$), and $K := \max \{ \text{dist}(x_0, U_\delta(x_0)), \text{dist}(y_0, U_\delta(y_0)) \} + 1$. Fix $\alpha \in (0, \tilde{\alpha}(K))$. We prove in the following only the subsolution case, as the supersolution case is proved in an analogous way.

If $x_0 \in S$, then $x_k \in S$ for $k$ large, and by Definition 3 and (36)–(39),
\[
\min \left( F(x_k, u^*(x_k), p_k, X_k, \mathcal{T}^{1,\delta}_\beta [x_k, \varphi_\alpha(\cdot, y_k)]) + \mathcal{T}^{2,\delta}_\beta [x_k, p_k, u^*(\cdot)], u^*(x_k) - \mathcal{M} u^*(x_k) \right) \leq 0.
\] (41)

First let us prove convergence of the PIDE part $F$. First, $x_k \rightarrow x_0$, $u^*(x_k) \rightarrow u^*(x_0)$, $p_k \rightarrow p$ by Lemma 5.6. $(X_k)$ is contained in a compact set in $\mathbb{R}^{d \times d}$ by (39), so it admits a convergent subsequence to an $X$ satisfying (40). For each $k$, by Proposition 5.1(iv), the supremum in $F$ is attained in a $\beta_k$. Choose another (sub-)subsequence converging to $\beta \in B$.

We now only need a reinforced version of (32) in Proposition 5.1 for $\mathcal{T}^{1,\delta}_\beta$. By Lebesgue’s Dominated Convergence Theorem, (V3) and uniform convergence,
\[
\lim_{k \rightarrow \infty} \int_{|z| < \delta} \varphi_\alpha(x_k + \ell(x_k, \beta_k, z)) - \varphi_\alpha(x_k) - \langle \nabla \varphi_\alpha(x_k), \ell(x_k, \beta_k, z) \rangle v(dz)
\]
\[
= \int_{|z| < \delta} \varphi_\alpha(x_0 + \ell(x_0, \beta, z)) - \varphi_\alpha(x_0) - \langle \nabla \varphi_\alpha(x_0), \ell(x_0, \beta, z) \rangle v(dz),
\]
where the ν-integrable upper estimate can be derived by the Taylor expansion and the estimates for k large
\[
\sup_{|z-x|<\kappa_1} |D^2\varphi_k(z)| \leq \sup_{|z-x|<\kappa_1} |D^2\varphi_k(z)| + \kappa_2
\]
for some \(\kappa_1, \kappa_2 > 0\) (recall that \(\int_C |z|^2\nu(dz) < \infty\) for all compacts C, and that \(U_\delta(y_0) \downarrow 0\) for singular \(\nu\)). For \(I_{2,\delta}^{\beta}\), we use Proposition 5.1 (i). For the impulse part, we know by Lemma 4.3 (ii) that \(M^u_\ast\) is usc, so
\[
\liminf_{k \to \infty} u_\ast(x_k) - M^u_\ast(x_k) = u_\ast(x_0) - \limsup_{k \to \infty} M^u_\ast(x_k) \geq u_\ast(x_0) - M^u_\ast(x_0).
\]
Now we have to combine the estimates derived so far. By iteratively taking subsequences and using (32), we have the desired result for \(k \to \infty\) in (41).

\[\square\]

**Remark 5.5.** By inspecting the proof of Corollary 5.7, we see that the statement also holds if \(u\) and \(v\) are subsolution and supersolution, respectively, of different QVIs, provided of course that the conditions are satisfied. This will be used in the proof of the comparison Theorem 5.9.

### 5.4. A comparison result

The final step separating us from the comparison result (inspired by [22]) is the following lemma:

**Lemma 5.8.** Assume (V1), (V2). Let \(u\) be a subsolution and \(v\) a supersolution of (14), further assume that there is a \(w \in \mathcal{PB} \cap C^2(\mathbb{R}^d)\) and a positive function \(\kappa : \mathbb{R}^d \to \mathbb{R}\) such that
\[
\min(-\sup_{\beta \in B} \{\mathcal{L}^\beta w + f^\beta\}, w - Mw) \geq \kappa \quad \text{in } S
\]
\[
\min(w - g, w - Mw) \geq \kappa \quad \text{in } \mathbb{R}^d \setminus S.
\]
Then \(v_m := (1 - \frac{1}{m})v + \frac{1}{m}w\) is a supersolution of
\[
\min(-\sup_{\beta \in B} \{\mathcal{L}^\beta u + f^\beta\}, u - Mu) - \kappa/m = 0 \quad \text{in } S
\]
\[
\min(u - g, u - Mu) - \kappa/m = 0 \quad \text{in } \mathbb{R}^d \setminus S,
\]
and \(u_m := (1 + \frac{1}{m})u - \frac{1}{m}w\) is a subsolution of (14), and of (42) with \(-\kappa\) replaced by \(+\kappa\).

**Proof.** Using the first viscosity solution definition, the \(\mathcal{L}^\beta \varphi\) part follows by sup manipulations, and the \(M^u\) part by using the (anti-)convexity from Lemma 5.5. For a detailed proof see [35].

\[\square\]

In the proof of the now following comparison result, we are going to use the above perturbations of sub- and supersolutions to make sure that the maximum of \(u_m - v_m\) is attained. So we want to find a \(w \geq 0\) growing faster than \(|u|\) and \(|v|\) as \(|x| \to \infty\) (how to find such a \(w\) is discussed in Section 2; the requirements lead to the function \(F\) being proper in the sense of [14]).

If \(\sigma(\cdot, \beta), \mu(\cdot, \beta), f(\cdot, \beta), c\) are Lipschitz continuous, then by classical results (see, e.g., Lemma V.7.1 in [20]), our function \(F\) has the property:
For any $R > 0$, there exists a modulus of continuity $\omega_R$, such that, for any $|x|, |y|, |v| \leq R$, $l \in \mathbb{R}$ and for any $X, Y \in S^d$ satisfying
\[
\begin{bmatrix}
  X & 0 \\
  0 & -Y
\end{bmatrix} \leq \frac{1}{\varepsilon} \begin{bmatrix}
  I & -l \\
  -I & I
\end{bmatrix} + o_\alpha(1)
\]
for some $\varepsilon > 0$ ($o_\alpha(1)$ does not depend on $\varepsilon$), then
\[
F(y, v, \varepsilon^{-1}(x - y), Y, l) - F(x, v, \varepsilon^{-1}(x - y), X, l) 
\leq \omega_R(|x - y| + \varepsilon^{-1}|x - y|^2) + o_\alpha(1),
\]
where $o_\alpha(1)$ again does not depend on $\varepsilon$, and the first term is independent of $\alpha$. In the proof of Theorem 5.9, $x_S, y_e$ converge to the same limit $x_0$, so the requirement can be relaxed to locally Lipschitz (as required in Assumption 2.4) and the property holds only for $x, y$ in a suitable neighbourhood of $x_0$.

Recall that $\mathcal{PB}_p$ is the space of functions in $\mathcal{PB}$ at most polynomially growing with exponent $p$. Assuming essentially that there is a strict supersolution of (14), the following comparison theorem holds:

**Theorem 5.9.** Let Assumptions 2.1, 2.3 and 2.4 be satisfied and $c$ be locally Lipschitz continuous. Assume further that there is a $w \geq 0$ as in Lemma 5.8 (for a constant $\kappa > 0$) with $w(x)/|x|^p \to \infty$ for $|x| \to \infty$. If $u \in \mathcal{PB}_p(\mathbb{R}^d)$ is a subsolution and $v \in \mathcal{PB}_p(\mathbb{R}^d)$ a supersolution of (14), then $u^* \leq v_*$. 

**Corollary 5.10** (Viscosity Solution: Uniqueness). Under the same assumptions, there is at most one viscosity solution of (14), and it is continuous.

The now following proof of Theorem 5.9 uses the strict sub-/supersolution technique (adapted from [22]). We first prove that a maximum cannot be attained outside $S$ (because then this would have been because of an impulse back to $S$). Then we use the classical doubling of variables technique, apply the nonlocal maximum principle, and by a case distinction reduce the problem to a PIDE without impulse part (then adapting techniques of [7]).

**Proof of Theorem 5.9.** Write $u$ instead of $u^*$ and $v$ instead of $v_*$ to make the notation more convenient. It is sufficient to prove that $u_m - v_m \leq 0$ for all $m$ large (where $u_m, v_m$ are as defined in Lemma 5.8). Let $m \in \mathbb{N}$ be fixed for the moment. To prove by contradiction, let us assume that $M := \sup_{x \in \mathbb{R}^d} u_m(x) - v_m(x) > 0$.

**Step 1.** We want to show that the supremum cannot be approximated from within $\mathbb{R}^d \setminus S$. Assume that for each $\varepsilon_1 > 0$, we can find an $\hat{x} = \hat{x}_{\varepsilon_1} \in \mathbb{R}^d \setminus S$ such that $u_m(\hat{x}) - v_m(\hat{x}) + \varepsilon_1 > M$ (and wlog $u_m(\hat{x}) - v_m(\hat{x}) > 0$). By the sub- and supersolution definition, we have
\[
\begin{align*}
\min(u_m(\hat{x}) - g(\hat{x}), u_m(\hat{x}) - Mu_m(\hat{x})) &\leq 0 \\
\min(v_m(\hat{x}) - g(\hat{x}), v_m(\hat{x}) - Mu_m(\hat{x})) &\geq \kappa/m.
\end{align*}
\]
If $u_m(\hat{x}) - g(\hat{x}) \leq 0$, then $\kappa/m + u_m(\hat{x}) - v_m(\hat{x}) \leq u_m(\hat{x}) - g(\hat{x}) \leq 0$ which is already a contradiction. If $u_m(\hat{x}) \leq Mu_m(\hat{x})$, then select for $\varepsilon_2 > 0$ a $\hat{\zeta} = \hat{\zeta}_{\varepsilon_1, \varepsilon_2}$ such that $u_m(\Gamma(\hat{x}, \hat{\zeta})) + K(\hat{x}, \hat{\zeta}) + \varepsilon_2 > Mu_m(\hat{x})$. Then,
\[
M - \varepsilon_1 < u_m(\hat{x}) - v_m(\hat{x}) 
\leq u_m(\Gamma(\hat{x}, \hat{\zeta})) + K(\hat{x}, \hat{\zeta}) + \varepsilon_2 - \kappa/m - K(\hat{x}, \hat{\zeta}) - v_m(\Gamma(\hat{x}, \hat{\zeta})) 
\leq \varepsilon_2 - \kappa/m + M,
\]
which is a contradiction for $\varepsilon_1, \varepsilon_2$ sufficiently small. This shows that the supremum $M$ cannot be attained in $\mathbb{R}^d \setminus S$, neither can it be approached from within $\mathbb{R}^d \setminus S$.

**Step 2.** Now that we are sure we do not have to take into account the boundary conditions, we employ the doubling of variables device as usual. We define for $\varepsilon > 0$ and $u_m, v_m$ chosen as in Lemma 5.8

$$M_\varepsilon = \sup_{x,y \in \mathbb{R}^d} \left( u_m(x) - v_m(y) - \frac{1}{2\varepsilon} |x-y|^2 \right).$$

In view of the definition of $w$ and $u_m, v_m$, the maximum is attained in a compact set $C$ (independent of small $\varepsilon$). Choose a point $(x_\varepsilon, y_\varepsilon) \in C$ where the maximum is attained. By applying Lemma 3.1 in [14], we obtain that $\frac{1}{2\varepsilon} |x_\varepsilon - y_\varepsilon|^2 \to 0$ as $\varepsilon \to 0$, and that $M_\varepsilon \to M = u_m(x_0) - v_m(x_0)$ for all limit points $x_0$ of $(x_\varepsilon)$. We assume from now on wlog that we have chosen a convergent subsequence of $(x_\varepsilon), (y_\varepsilon)$, converging to the same limit $x_0 \in C$. Let $\varepsilon$ be small enough such that $x_\varepsilon, y_\varepsilon \in S$ (by Step 1), and that all local estimates in (B1), (B2) hold.

Hence, we can apply Corollary 5.7 in $(x_\varepsilon, y_\varepsilon)$ for $\varphi(x, y) = \frac{1}{2\varepsilon} |x-y|^2$: For any $\delta > 0$, there is a range of $\alpha > 0$, for which there are matrices $X, Y$ satisfying (40), and $(p, -q) = \nabla \varphi(x_\varepsilon, y_\varepsilon)$ (so $p = q = \frac{1}{16}(x_\varepsilon - y_\varepsilon)$) such that

$$\min \left( F(x_\varepsilon, u_m(x_\varepsilon), p, X, T_\beta^{1,\delta} [x_\varepsilon, \varphi(\cdot, y_\varepsilon)] + T_\beta^{2,\delta} [x_\varepsilon, p, u_m(\cdot)], u_m(x_\varepsilon) - M u_m(x_\varepsilon) \right) \leq 0$$

$$\min \left( F(y_\varepsilon, v_m(y_\varepsilon), q, Y, T_\beta^{1,\delta} [y_\varepsilon, -\varphi(\cdot, x_\varepsilon)] + T_\beta^{2,\delta} [y_\varepsilon, q, v_m(\cdot)], v_m(y_\varepsilon) - M v_m(y_\varepsilon) \right) \geq \frac{\kappa}{m}.$$ 

**Case 2a** $(u_m(x_\varepsilon) - M u_m(x_\varepsilon) \leq 0)$: Using $v_m(y_\varepsilon) - M v_m(y_\varepsilon) \geq \frac{\kappa}{m}$, for $\varepsilon > 0$ small enough,

$$M = \limsup_{\varepsilon \to 0} (u_m(x_\varepsilon) - v_m(y_\varepsilon))$$

$$\leq \limsup_{\varepsilon \to 0} \sup M u_m(x_\varepsilon) - \liminf_{\varepsilon \to 0} M v_m(y_\varepsilon) - \frac{\kappa}{m} \leq M u_m(x_0) - M v_m(x_0) - \frac{\kappa}{m},$$

where we have used the upper and lower semicontinuity of $M u_m$ and $M v_m$, respectively (Lemma 5.5). The contradiction is obtained as in Step 1.

**Case 2b** $(u_m(x_\varepsilon) - M u_m(x_\varepsilon) > 0)$: It remains to treat the PIDE part

$$F(x_\varepsilon, u_m(x_\varepsilon), p, X, T_\beta^{1,\delta} [x_\varepsilon, \varphi(\cdot, y_\varepsilon)] + T_\beta^{2,\delta} [x_\varepsilon, p, u_m(\cdot)]) \leq 0 \quad (43)$$

$$F(y_\varepsilon, v_m(y_\varepsilon), q, Y, T_\beta^{1,\delta} [y_\varepsilon, -\varphi(\cdot, x_\varepsilon)] + T_\beta^{2,\delta} [y_\varepsilon, q, v_m(\cdot)]) \geq \frac{\kappa}{m}. \quad (44)$$

Before we can proceed, we have to compare the integral terms in both inequalities. First note that because $|x + \ell(x, \beta, z) - y|^2 = |x-y|^2 + 2(x-y, \ell(x, \beta, z)) + |\ell(x, \beta, z)|^2$, for all $\beta$

$$T_\beta^{1,\delta} [x_\varepsilon, \varphi(\cdot, y_\varepsilon)] = \frac{1}{2\varepsilon} \int_{|z|<\delta} |\ell(x_\varepsilon, \beta, z)|^2 v(dz) < \infty$$

$$T_\beta^{1,\delta} [y_\varepsilon, -\varphi(\cdot, x_\varepsilon)] = \frac{1}{2\varepsilon} \int_{|z|<\delta} -|\ell(y_\varepsilon, \beta, z)|^2 v(dz) < \infty,$$
(finite by (V4) and definition of $\mathcal{P}\mathcal{B}$) so trivially $\mathcal{I}_\beta^{-1,\delta}[x_\varepsilon, \varphi(\cdot, y_\varepsilon)] \leq \mathcal{I}_\beta^{1,\delta}[y_\varepsilon, -\varphi(x_\varepsilon, \cdot)] + \frac{1}{\varepsilon} o_\delta(1)$.

Because we know that $\varphi_\alpha$ converges to $\varphi$ uniformly in $C^2(C)$ for any compact $C$, we can see analogously to the proof of Corollary 5.7 that $\mathcal{I}_\beta^{1,\delta}[x_\varepsilon, \varphi_\alpha(\cdot, y_\varepsilon)] \leq \mathcal{I}_\beta^{1,\delta}[y_\varepsilon, -\varphi_\alpha(x_\varepsilon, \cdot)] + \frac{1}{\varepsilon} o_\delta(1) + o_\alpha(1)$, where $o_\alpha(1)$ may depend on $\varepsilon$, but is independent of small $\delta$.

Using that $(x_\varepsilon, y_\varepsilon)$ is a maximum point and again $|x + y|^2 = |x|^2 + 2\langle x, y \rangle + |y|^2$,

$$u_m(x_\varepsilon + d) - u_m(x_\varepsilon) - \frac{1}{\varepsilon}\langle x_\varepsilon - y_\varepsilon, d \rangle \leq v_m(y_\varepsilon + d') - v_m(y_\varepsilon)$$

$$- \frac{1}{\varepsilon}\langle x_\varepsilon - y_\varepsilon, d' \rangle + \frac{1}{2\varepsilon}|d - d'|^2,$$

(45)

where $d, d'$ are arbitrary vectors. We find by integrating (45) for all $\beta$ and $d = \ell(x_\varepsilon, \beta, z)$, $d' = \ell(y_\varepsilon, \beta, z)$ that

$$\mathcal{I}_\beta^{2,\delta}[x_\varepsilon, p, u_m(\cdot)] \leq \mathcal{I}_\beta^{2,\delta}[y_\varepsilon, q, v_m(\cdot)] + \frac{1}{2\varepsilon}\int_{|z| \geq \delta} |\ell(x_\varepsilon, \beta, z) - \ell(y_\varepsilon, \beta, z)|^2 v(dz)$$

$$+ \int_{|z| \geq 1} (p, \ell(x_\varepsilon, \beta, z) - \ell(y_\varepsilon, \beta, z)) v(dz).$$

We then have by (B1) for $\varepsilon > 0$ small enough, (denoting $I^1_\beta = \mathcal{I}_\beta^{1,\delta}[x_\varepsilon, \varphi(\cdot, y_\varepsilon)] + \mathcal{I}_\beta^{2,\delta}[x_\varepsilon, p, u_m(\cdot)]$ and $I^2_\beta = \mathcal{I}_\beta^{1,\delta}[y_\varepsilon, -\varphi(x_\varepsilon, \cdot)] + \mathcal{I}_\beta^{2,\delta}[y_\varepsilon, q, v_m(\cdot)]$ that

$$I^1_\beta \leq O\left(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2\right) + \frac{1}{\varepsilon} o_\delta(1) + o_\alpha(1),$$

where $O(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2)$, $o_\delta(1)$ and $o_\alpha(1)$ are independent of $\beta$ because of (B1). Likewise, $O(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2)$ is independent of $\delta$ and $\alpha$. Thus

$$\frac{\kappa}{m} \leq F(y_\varepsilon, v_m(y_\varepsilon), q, Y, I^2_\beta) - F(x_\varepsilon, u_m(x_\varepsilon), p, X, I^1_\beta)$$

by (43) and (44)

$$\leq F(y_\varepsilon, v_m(y_\varepsilon), q, Y, I^2_\beta) - F(x_\varepsilon, v_m(y_\varepsilon), p, X, I^1_\beta)$$

for small $\varepsilon$ because $c \geq 0$

$$\leq F(y_\varepsilon, v_m(y_\varepsilon), q, Y, I^1_\beta) - F(x_\varepsilon, v_m(y_\varepsilon), p, X, I^1_\beta)$$

$$+ O\left(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2\right) + \frac{1}{\varepsilon} o_\delta(1) + o_\alpha(1),$$

where we have used ellipticity (P1) and Lipschitz continuity (P3) in the last component for the $O$ and $o$ values independent of $\beta$. The matrix inequality (40) becomes

$$- \frac{1}{\alpha} I \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \frac{1}{\varepsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + o_\alpha(1).$$

By assumption (B2) for $R > 0$ large enough ($v_m$ is locally bounded) and $\varepsilon$ small enough,

$$\frac{\kappa}{m} \leq \omega_R(|x_\varepsilon - y_\varepsilon| + \varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2) + O\left(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2\right) + \frac{1}{\varepsilon} o_\delta(1) + o_\alpha(1).$$
Now let us subsequently converge $\delta \to 0$ (because of the special dependence of $\alpha$ – the smaller $\delta$, the larger $\alpha$ – this does not affect $\alpha$), and then $\alpha \to 0$. The contradiction is finally obtained by $\varepsilon \to 0$. □

5.5. Parabolic case

Now let us deduce the parabolic result from the preceding discussion. Apart from a few necessary adjustments (e.g., use the parabolic semijet instead of the elliptic one) the reasoning follows the above lines. More details can be found in [35].

Recall the definition of a viscosity solution of (9) from Section 4.1. The original motivation for introducing the different definitions of viscosity solutions was to cater for the singularity in the integral. Because this integral, started in $(t_0, x_0)$, only takes into account values at the time $t_0$, the different definitions in Section 5.2 are equivalent in the parabolic case, too (where the time derivative in $t = 0$ is only the one-sided derivative).

Define $\mathcal{P}_B_p = \mathcal{P}_B_p([0, T] \times \mathbb{R}^d)$ in the parabolic case by all functions $u \in \mathcal{P}_B$, for which there is a (time-independent!) constant $C$ such that $|u(t, x)| \leq C(1 + |x|^p)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$. The upper (lower) semicontinuous envelope $u^*$ ($v_*$) is again taken from within $[0, T] \times \mathbb{R}^d$.

**Theorem 5.11.** Let Assumptions 2.1, 2.3 and 2.4 be satisfied. Assume further that there is a $w \geq 0$ as in Lemma 5.8 (for a constant $\kappa > 0$) with $w(t, x)/|x|^p \to \infty$ for $|x| \to \infty$ (uniformly in $t$). If $u \in \mathcal{P}_B_p([0, T] \times \mathbb{R}^d)$ is a subsolution and $v \in \mathcal{P}_B_p([0, T] \times \mathbb{R}^d)$ a supersolution of (9), then $u^* \leq v_*$ on $[0, T] \times \mathbb{R}^d$.

**Corollary 5.12.** Under the same assumptions, there is at most one viscosity solution of (9), and it is continuous on $[0, T] \times \mathbb{R}^d$.

6. Conclusion

We have shown in the present paper existence and uniqueness of viscosity solutions for impulse control QVI. The results we have obtained are quite general, and the (minimal) assumptions required (basically (local Lipschitz) continuity, continuity of the value function at the boundary, and compactness and “continuity” of the transaction set) are sufficient to guarantee a continuous solution on $\mathbb{R}^d$. We note that the Lipschitz continuity assumptions are already needed to ensure existence and uniqueness of the underlying SDE.

The complications to be overcome were mainly:

- The discontinuous stochastic process and definition of the value function on $\mathbb{R}^d$
- The possibly singular integral term in the PIDE (arisen from the Lévy jumps)
- The additional stochastic control.

It is our hope that the parabolic and elliptic results presented can be used to great benefit in applications of impulse control without the need to go into details of viscosity solutions (as typically the value function in – at least financial – applications will be continuous). The comparison result can also be used to carry out a basic stability analysis for numerical calculations.

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6 By $\alpha \to 0$, we lose the first part of the inequality (46) (so we cannot be sure anymore that $X, Y$ are bounded because they are dependent on $\alpha$).
Admittedly, our results do not cover all special cases—but quite frequently, one should be able to extend the results of this paper easily; e.g., state constraints can be handled with a modified framework, where the continuity inside $S$ in general should still hold (see also [27]). This leaves some room for future research.

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References


