ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 245 (2008) 1413–1432

Journal of Differential Equations

www.elsevier.com/locate/jde

Evolution systems for paraxial wave equations of Schrödinger-type with non-smooth coefficients ☆

Maarten de Hoop [a], Günther Hörmann [b,*], Michael Oberguggenberger [c]

[a] *Center for Computational and Applied Mathematics, Purdue University, West Lafayette, USA*

[b] *Fakultät für Mathematik, Universität Wien, Austria*

[c] *Institut für Grundlagen der Bauingenieurwissenschaften, Universität Innsbruck, Austria*

Received 16 June 2006; revised 7 April 2008

Available online 15 July 2008

Abstract

We prove existence of strongly continuous evolution systems in L^2 for Schrödinger-type equations with non-Lipschitz coefficients in the principal part. The underlying operator structure is motivated from models of paraxial approximations of wave propagation in geophysics. Thus, the evolution direction is a spatial coordinate (depth) with additional pseudodifferential terms in time and low regularity in the lateral space variables. We formulate and analyze the Cauchy problem in distribution spaces with mixed regularity. The key point in the evolution system construction is an elliptic regularity result, which enables us to precisely determine the common domain of the generators. The construction of a solution with low regularity in the coefficients is the basis for an inverse analysis which allows to infer the lack of lateral regularity in the medium from measured data.

1. Introduction

The paraxial equations in models of wave propagation are based on parabolic symbol approximations in theories of wave operators. They have been extensively applied in integrated optics, underwater acoustic tomography as well as reflection seismic imaging (cf. [3,24]). They have also entered the analysis of time-reversal mirror experiments with waves taking into account

☆ Work supported by FWF grants P16820-N04 and Y237-N13.

* Corresponding author.

E-mail addresses: mdhoop@math.purdue.edu (M. de Hoop), guenther.hoermann@univie.ac.at (G. Hörmann), michael.oberguggenberger@uibk.ac.at (M. Oberguggenberger).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.06.022

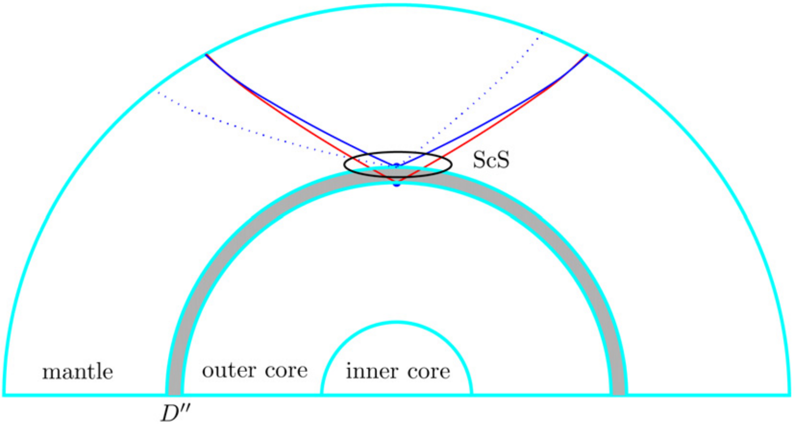

Fig. 1. The so-called D'' layer (in gray) above the core-mantle boundary (at approximately 2800 km depth) with a core-reflected (ScS) wave and two precursors. The precursors in the neighborhood of the top of D'' are locally modeled by our equation (indicated by ellipsoidal region), x coinciding locally with the radial direction and z coinciding with the tangential direction.

stochastic variations in the wave speed (cf. [2]). The paraxial equations can be derived from the reduced wave or Helmholtz equation and, since they split the wave fields according to a prescribed principal direction of propagation, are also called one-way wave equations. In particular, the leading-order parabolic symbol approximation, also referred to as the narrow-angle or beam-propagation approximation, leads to model equations of Schrödinger-type. The well-posedness of the one-way wave Cauchy problems has been discussed by [6,22]. The methodologies developed to date, however, have assumed smoothness of the wave speed function (i.e., the coefficients in the wave operators).

In the analysis presented here, we depart from this smoothness assumption by allowing the coefficients to be of any Hölder regularity between zero and one, but typically non-Lipschitz. The existence of distributional solutions to second-order strictly hyperbolic equations in general may fail below Log-Lipschitz regularity of the coefficients (cf. [4]). In case of Hölder regularity 2 or higher a constructive approach for hyperbolic evolution equations has been developed in [1], thereby extending results on propagation of singularities. The particular equation considered here is derived from such a second-order equation, but the existence of its solution is not restricted by the same conditions. Indeed, we exploit the framework of Sobolev space techniques, in particular, multiplication of distributions in scales of Sobolev spaces, to construct a strongly continuous evolution system. The novelty of the paper lies in the method of construction which not only provides a solution concept for the paraxial wave equation with low coefficient regularity, but also allows us to investigate how the coefficient regularity influences the solution.

The class of coefficients of Hölder regularity between zero and one arises in a variety of geophysical applications. Perhaps the most fundamental one concerns the study of thermo-chemical boundary layers and phase transitions in Earth's lowermost mantle—the so-called D'' layer overlaying the core-mantle boundary (see Fig. 1; cf. [25] for recent images of the phase transformation in D''). Such phase transitions can only be probed by earthquake generated seismic waves through scattering off these. The relevant scattered wave constituents appear as precursors to, for example, the core-reflected compressional PcP phase and the horizontally polarized shear ScS phase. The scattering is most prominent at large opening angles (towards grazing incidence).

It is to this situation that the paraxial approximation and coefficient dependence considered here applies. In this context the principal direction, z, of propagation is perpendicular to the direction of backscattering, x. The central question is whether an imprint of the coefficient regularity on the (regularity of the) scattered wave occurs. A positive answer to this question, as provided in this paper, allows to infer a lack of lateral regularity in the medium from a lack of regularity of the measured data.

Seismic inverse scattering has been formulated mathematically in terms of evolution equations with respect to the depth variable in [17–19] (with smooth symbols in the single scattering approximation). At the basis of these models are one-way wave equations, which are typically of the form

$$\big(\partial_z \pm i B(z, x, D_t, D_x)\big)u = f,$$

where B is (microlocally) a pseudodifferential operator with principal part

$$b(z, x, \tau, \xi) = \tau c(z, x)^{-1}\sqrt{1 - \tau^{-2} c(z, x)^2 |\xi|^2}.$$

Here, $t \in \mathbb{R}$ is time, $z \geqslant 0$ denotes depth, $x \in \mathbb{R}^d$ are lateral spatial directions, and $D = i^{-1}\partial$. Approximation of the square root to leading order results in the standard Schrödinger-type paraxial equation

$$\left(\partial_z + \frac{i}{c(z, x)} D_t\right) w + \frac{1}{2i} D_t^{-1} \sum_{j=1}^{d} D_{x_j}\big(c(z, x) D_{x_j} w\big) = 0,$$

where $c(z, x)$ is the local speed of propagation and z plays the role of the evolution parameter. In the frequency (τ-)domain the above equation is transformed with a so-called co-moving frame of reference according to $\hat{w}(z, x, \tau) = \hat{u}(z, x, \tau)\exp(i\tau T(z, x))$, where $T(z, x) = \int_0^z dz'/c(z', x)$. Then the paraxial equation attains the form

$$(\partial_z - i\bar{A})\hat{u} = 0,$$

in which $\bar{A}$ is given by

$$\bar{A}\hat{u} = \frac{1}{2} \sum_{j=1}^{d} e^{i\tau T(z,x)} D_{x_j}\big(c(z, x)\tau^{-1} D_{x_j}\big(e^{-i\tau T(z,x)}\hat{u}\big)\big).$$

The second-order differential operator $\bar{A}$ can be written as the sum of the self-adjoint operator

$$A(\tau; z, x, D_x) := \frac{1}{2} \sum_{j=1}^{d} D_{x_j} c(z, x)\tau^{-1} D_{x_j}$$

and a symmetric perturbation. We observe that if $\|D_x T(z, .)\|_{L^\infty(\mathbb{R}^d)} < K$, where K is an appropriate constant, then it is guaranteed that this perturbation is A-bounded with relative bound less than 1 (cf. [14, Section X.2]). From the viewpoint of generators of strongly continuous contraction semigroups on $L^2(\mathbb{R}^d)$ the simplification of $\bar{A}$ by A is of no consequence.

Remark 1.1 *(Directional decomposition and one-way wave equations).* Directional decomposition leads to the introduction of one-way wave equations [21]. One-way wave equations approximate solutions to the wave equation microlocally, relative to a principal direction of propagation. (This principal direction does not need to be defined globally; one can introduce curvilinear coordinates and an associated Riemannian metric to generate such directions locally.) The validity of one-way wave propagation breaks down when singularities tend to propagate in a direction perpendicular to the principal direction (that is, in a transverse direction). Hence, to make the statement concerning approximation above, precise, one needs to introduce a microlocal attenuation [16]. The mentioned procedures and results require smooth coefficients and symbols, and can be proven by making use of the calculus of pseudodifferential operators and Fourier integral operators with complex phase.

It has been demonstrated that, in special cases, one can weaken the condition of smooth coefficients. For example, if the coefficients are independent of the coordinate along the principal direction, one can allow a step function (in a transverse coordinate) and still solve the associated scattering problem by methods of one-way wave equations. The approach to carry out such an evaluation can be found in [5]. Indeed, scattering in the transverse directions can, at least in special cases, be incorporated in the one-way wave equation. Moreover, in the case of wave propagation in random media, the (stochastic) paraxial equation naturally appears as well [12]. However, a general result concerning directional decomposition for—or recomposition to solutions of—the wave equation with non-smooth coefficients has not been obtained.

In this paper, we address the general problem of "transverse scattering" by a one-way wave equation, which constitutes one component in the development of a general theory referred to above.

Remark 1.2 *(Regularity and the second-order wave equation).* Both, mode decoupling of a second-order wave equation into one-way wave equations as well as the derivation of the narrow beam approximation outlined above, require higher-order differentiability of the coefficient $c(z,x)$ with respect to all variables to make sense (due to truncation of symbol expansions). However, the resulting paraxial wave equation displays precisely the same coefficient regularity as in the original second-order wave equation. Hence as a model equation it still reflects the correct medium properties on all scales. The (exact) solutions of the paraxial equation then serve as a narrow beam approximation to solutions of the original wave equation. In particular, the regularity properties are comparable on the same scales.

The fine tuned well-posedness theorem for wave equations by Colombini and Lerner in [4] assumes Log-Lipschitz regularity of the coefficients in the principal part. Moreover, their results are sharp in the sense that counterexamples to solvability exist when the coefficient regularity is below Log-Lipschitz (but still of any continuity type arbitrarily close to such). In order to indicate how their results relate to ours, we repeat the key energy estimate from [4]:

A function $a \in L^\infty(\mathbb{R}^d)$ is said to be a *Log-Lipschitz* function if

$$\|a\|_{LL} := \sup_{x\in\mathbb{R}^d} \left|a(x)\right| + \sup_{\substack{x\neq y\in\mathbb{R}^d \\ |x-y|\leqslant 1/2}} \frac{|a(x)-a(y)|}{|(x-y)\log(|x-y|)|} < \infty.$$

Colombini and Lerner consider second-order wave operators of the form

$$Pu := \partial_t^2 u - \sum_{1\leqslant j,k\leqslant n} \partial_{x_j}\bigl(a_{jk}(x,t)\partial_{x_k} u\bigr) + M(x,t,\partial_t,\partial_x)u,$$

where $(a_{jk})_{1\leqslant j,k\leqslant n}$ is real symmetric with Log-Lipschitz components and satisfies with some $1 \geqslant \delta_0 > 0$ the strong ellipticity condition

$$\sum_{1\leqslant j,k\leqslant n} a_{jk}(x,t)\xi_j\xi_k \geqslant \delta_0|\xi|^2 \quad \bigl(\xi \in \mathbb{R}^n,\ (x,t) \in \mathbb{R}^{n+1}\bigr),$$

and $M(x,t,\partial_t,\partial_x)$ is a first-order differential operator with Hölder-continuous coefficients.

Let P be as above and $\theta \in]0, 1/4]$. There exist $\beta > 0$ and $C > 0$ such that for $u \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$ and $t \in [0, 1/\beta]$ the *energy estimate*

$$\sup_{0\leqslant s\leqslant t} \bigl\|\partial_t u(.,s)\bigr\|_{H^{-\theta-\beta s}} + \sup_{0\leqslant s\leqslant t} \bigl\|u(.,s)\bigr\|_{H^{1-\theta-\beta s}} \leqslant C\left(\int_0^t \bigl\|Pu(.,s)\bigr\|_{H^{-\theta-\beta s}}\,ds + \bigl\|\partial_t u(.,0)\bigr\|_{H^{-\theta}} + \bigl\|u(.,0)\bigr\|_{H^{1-\theta}}\right)$$

holds (cf. [4, Eq. (2.6)]), where β depends only on δ_0, on the Log-Lipschitz norm of the a_{jk}, and on the Hölder norms of the coefficients in $M(x,t,\partial_t,\partial_x)$.

We show in the sequel that for the paraxial wave equation the condition on the coefficient regularity can be relaxed. For example, if $\varepsilon > 0$ any function in $H^{1+\varepsilon}(\mathbb{R}^2)$ of local behavior like $x \mapsto |x|^{1/2+\varepsilon}$ is not Log-Lipschitz continuous but satisfies the assumptions of our main results below.

The low coefficient regularity in our model conditions has its price in terms of a few technical aspects of the current paper: Additional care is needed in identifying the appropriate distribution and function spaces that allow for the description of mixed regularity properties and for the rigorous formulation of a solution concept. The existence proof then consists in showing a series of functional analytic properties to establish an evolution system of operators; among these the basic self-adjointness property—in the disguise of an elliptic regularity lemma—is derived by employing rather delicate regularity properties of multiplication in certain subspaces of the space of distributions.

The plan of the paper is as follows. In Section 2 we present the precise form of the operator and specify our (low) regularity assumptions on the coefficients. The solution will be sought as a continuous map of depth into the space of temperate L^2-valued distributions. Section 3 is devoted to the construction of the evolution system in the frequency domain. First, we prove that $A(\tau; z, x, D_x)$ generates a unitary group at fixed z and τ. The determination of its domain requires delicate use of the duality product of distributions as well as a bootstrap argument involving multiplication in scales of Sobolev spaces. This leads to the construction of an evolution system at fixed τ. Finally, strongly continuous dependence on the frequency parameter τ is established, based on a difference approximation. Again a subtle interplay between regularity arguments and distributional products is at the heart of our arguments. The strong continuity enables us, in Section 4, to construct a solution of the evolution system in frequency domain with distributional data. Finally, existence, uniqueness, and regularity of solutions to the original Cauchy problem is obtained. As an application to inverse regularity analysis, we obtain that a lack of H^2-regularity in the observed solution implies the existence of a region in which the lateral regularity of the medium is at most H^1 on the Sobolev scale.

2. The Cauchy problem: Function spaces and coefficient regularity

We recall the definition of temperate distributions on $\mathbb{R}$ with values in a Banach space E (cf. [23, Section 39.3]; but note that we use a different topology here): let $\mathscr{S}'(\mathbb{R}; E)$ be the space of continuous linear maps $\mathscr{S}(\mathbb{R}) \to E$, equipped with the topology of pointwise convergence; the Fourier transform $\mathcal{F}$ on $\mathscr{S}(\mathbb{R})$ is extended to $\mathscr{S}'(\mathbb{R}; E)$ by setting $(\mathcal{F}G)(\phi) = G(\mathcal{F}\phi)$ ($G \in \mathscr{S}'(\mathbb{R}; E)$, $\phi \in \mathscr{S}(\mathbb{R})$), which is easily seen to be an isomorphism (for the locally convex structure).

We denote the time variable by $t \in \mathbb{R}$ and introduce coordinates $z \in [0, \infty)$ for depth (the evolution direction in our context) and $x \in \mathbb{R}^d$ for the lateral variation, where $d \leqslant 2$. As basic space of the *wave components* we consider

$$\mathcal{V} := \mathcal{C}\big([0, \infty), \mathscr{S}'\big(\mathbb{R}; L^2\big(\mathbb{R}^d\big)\big)\big). \tag{2.1}$$

Its elements are continuous maps $z \mapsto u(z)$ of the depth variable z into temperate distributions of time t valued in L^2-functions of the lateral variables x. When we need to keep track of precise regularity information in the lateral variation of the waves, we may employ the Sobolev-scale $H^s(\mathbb{R}^d)$ ($s \in \mathbb{R}$) and define

$$\mathcal{V}^s := \mathcal{C}\big([0, \infty), \mathscr{S}'\big(\mathbb{R}; H^s\big(\mathbb{R}^d\big)\big)\big). \tag{2.2}$$

Let $\mathcal{F}_t : \mathscr{S}'(\mathbb{R}; H^s(\mathbb{R}^d)) \to \mathscr{S}'(\mathbb{R}; H^s(\mathbb{R}^d))$ denote (partial) Fourier transform with respect to the time variable. We extend $\mathcal{F}_t$ to an isomorphism $\widetilde{\mathcal{F}}_t$ of (the locally convex structure of) $\mathcal{V}^s$ by

$$(\widetilde{\mathcal{F}}_t u)(z) := \mathcal{F}_t\big(u(z, .)\big) \quad \forall z \geqslant 0.$$

We consider the following Cauchy problem for a prospective solution $u \in \mathcal{V}$ (div and grad with respect to $x \in \mathbb{R}^d$):

$$Pu := \partial_z u - i \operatorname{div}\big(C(z, x, D_t) \cdot \operatorname{grad} u\big) = f \in \mathcal{V}, \tag{2.3}$$

$$u|_{z=0} = u_0 \in \mathscr{S}'\big(\mathbb{R}; L^2\big(\mathbb{R}^d\big)\big). \tag{2.4}$$

Here, C is a pseudodifferential operator in t with parameters z and x. While in the classical paraxial wave equation it is of order 1, here we may assume it is of some order $m \in \mathbb{R}$. The precise conditions are collected in the following

Assumption 1. The symbol of $C(z, x, D_t)$ is of the form

$$C(z, x, \tau) = c(z, x, \tau) \cdot I_d = \left(c_0 + \sum_{l=1}^{N} c_l(z, x) h_l(\tau)\right) \cdot I_d, \tag{2.5}$$

where $N \in \mathbb{N}$, I_d is the $d \times d$ identity matrix and the following hold:

(i) For $l = 1, \dots, N$: h_l is a real-valued smooth symbol (of order m) on $\mathbb{R}$, i.e., for all $k \in \mathbb{N}_0$ an estimate $|\partial_\tau^k h_l(\tau)| = O(|\tau|^{m-k})$ holds when $|\tau|$ is large; in addition, we assume that $|h_l(\tau)| \geqslant \eta_0$ near $\tau = 0$ ($l = 1, \dots, N$) with some constant $\eta_0 > 0$ (this can be achieved by adding a cut-off function without changing the relevant frequency range).

(ii) c_0 is a positive constant.
(iii) There is an $r \in (0,1)$ such that for $1 \leqslant l \leqslant N$: c_l is in $\mathcal{C}^1([0,\infty), H^{r+1}(\mathbb{R}^d))$ and real-valued.
(iv) For all $(z,x,\tau) \in [0,\infty) \times \mathbb{R}^d \times \mathbb{R}$: $c(z,x,\tau) \geqslant c_0$.

Remark 2.1. (i) The operator action corresponding to a typical term $c_l(z,x)h_l(D_t)$ in the sum decomposition (2.5) on any element w of $\mathcal{V}$ is given as follows: for all $\phi \in \mathscr{S}(\mathbb{R})$

$$\big(h_l(D_t)w(z)\big)(\phi) = w(z)\big(\mathcal{F}\big(h_l\mathcal{F}^{-1}\phi\big)\big) \in L^2\big(\mathbb{R}^d\big),$$

which is then multiplied by the function $c_l(z,.)$.

(ii) Note that parts (i)–(iii) of Assumption 1 imply, for any z and τ fixed, the following (Zygmund–)Hölder-continuity:

$$c(z,.,\tau) - c_0 \in H^{r+1}\big(\mathbb{R}^d\big) \subseteq C_*^{r+1-\frac{d}{2}}\big(\mathbb{R}^d\big)$$

(cf. [9, Proposition 8.6.10]). Thus the coefficients have lateral Hölder regularity of order $r+1-d/2$. In the most relevant case from geophysics, $d=2$, this yields coefficients in $C_*^r(\mathbb{R}^2)$, but not necessarily Lipschitz continuous. (This includes the situation of a boundary layer, as discussed in the introduction, where the coefficient is smooth in one of the two variables.) Part (iv) implies uniform ellipticity of the lateral differential operator.

(iii) Note that in dimension $d \geqslant 3$ Sobolev regularity of order $r+1$ would not imply continuity of the coefficients (in case $1 < r+1 < d/2$).

Example 2.2. We consider a model with two-dimensional lateral variation ($d=2$) in the medium properties of low Hölder regularity depending on depth. In (2.5) we put $c_l = 0$ when $l \geqslant 2$ and let c_1 be of the form

$$c_1(z,x) = \chi(z,x)|x|^{\alpha(z)}$$

with the following specifications: $\chi \in \mathcal{C}^1([0,\infty) \times \mathbb{R}^d)$ such that $\chi(z,x) = \chi_0(z)$ when $|x| \leqslant R_1$ and $\chi(z,x) = 0$ when $|x| \geqslant R_2$ for certain radii $0 < R_1 < R_2$ and some $\chi_0 \in \mathcal{C}^1([0,\infty))$; $\alpha \in \mathcal{C}^1([0,\infty))$ with some uniform positive lower bound α_0, i.e., $\alpha(z) \geqslant \alpha_0 > 0$ for all z. Then we may choose any r such that $0 < r < \alpha_0$ and obtain the following regularity properties at arbitrary fixed values of z and τ:

$$c(z,.,\tau) - c_0 \in H^{r+1}\big(\mathbb{R}^2\big) \cap C_*^{\alpha(z)}\big(\mathbb{R}^2\big) \subseteq H^{r+1}\big(\mathbb{R}^2\big) \cap C_*^{\alpha_0}\big(\mathbb{R}^2\big).$$

(Observe that locally in two dimensions, for any $0 < s < 1$, the function $|x|^s$ belongs to C_*^s and to $H^{s+1-\varepsilon}$ for every $\varepsilon > 0$ but not to H^{s+1}.)

Applying $\widetilde{\mathcal{F}}_t$ to (2.3)–(2.4) we obtain an equivalent formulation of the Cauchy problem in the frequency domain:

$$\widetilde{P}v = \partial_z v - i\,\mathrm{div}\big(c(z,x,\tau)\,\mathrm{grad}\,v\big) = g \in \mathcal{V}, \tag{2.6}$$

$$v|_{z=0} = v_0 \in \mathscr{S}'\big(\mathbb{R}; L^2\big(\mathbb{R}^d\big)\big). \tag{2.7}$$

Eq. (2.6) is an evolution equation for depth z with the second-order operator

$$A(\tau; z, x, D_x)v := \operatorname{div}\bigl(c(z, x, \tau)\operatorname{grad} v\bigr) \tag{2.8}$$

acting in the lateral x-domain and smoothly depending on the "external" parameter τ. Note that $A(\tau; z, x, D_x)$ is uniformly elliptic by Assumption 1(iv).

Remark 2.3. Note that P in (2.3) commutes with convolution in the time variable. Therefore, damping (or cut-off) of high frequencies in the data of (2.6)–(2.7) corresponds to time-smoothing the data in the original problem (2.3)–(2.4): more precisely, if a frequency filter $\hat{\chi} \in \mathscr{S}(\mathbb{R})$ is applied by $g := \widetilde{\mathcal{F}}_t(f) \cdot \hat{\chi}(\tau)$, $v_0 := \mathcal{F}_t(u_0) \cdot \hat{\chi}(\tau)$ and v is a solution to (2.6)–(2.7) then $u := \widetilde{\mathcal{F}}_t^{-1}(v)$ solves (2.3)–(2.4) with the data changed to $f \underset{(t)}{*} \chi$ and $u_0 \underset{(t)}{*} \chi$.

3. Evolution system

In this section, we will show that, up to any finite depth $Z > 0$, the family of unbounded operators $iA(\tau; z, x, D_x)$ ($\tau \in \mathbb{R}$, $z \geqslant 0$) generates a strongly continuous evolution system (or fundamental solution) $\{U(\tau; z_1, z_2)\colon 0 \leqslant z_1 \leqslant z_2 \leqslant Z\}$ on $L^2(\mathbb{R}^2)$ in the sense of [20, Chapter 4] which, in addition, is strongly continuous in the frequency variable $\tau \in \mathbb{R}$. In a first step we freeze both parameters, τ as well as z, and construct a strongly continuous (semi)group of operators on $L^2(\mathbb{R}^d)$.

3.1. *Unitary group at frozen values of τ and z*

Notational simplifications. By abuse of notation we will employ the short-hand symbols $A := A(\tau; z, x, D_x)$, $c(x) := c(z, x, \tau)$, and $c_1(x)$ now denoting $\sum_{l \geqslant 1} c_l(z, x)h_l(\tau)$. To summarize, using the above conventions and Assumption 1, we have

$$Av = \operatorname{div}\bigl(c(x)\operatorname{grad} v\bigr) \tag{3.9}$$

as unbounded, formally self-adjoint operator on $L^2(\mathbb{R}^d)$ with coefficient

$$c(x) = c_0 + c_1(x) \quad \text{with } c_0 > 0 \text{ and } 0 \leqslant c_1 \in H^{r+1}\bigl(\mathbb{R}^d\bigr). \tag{3.10}$$

We will show that A is a self-adjoint operator with domain $D(A) = H^2(\mathbb{R}^d)$.

Remark 3.1. Note that self-adjointness of A with domain H^2 would be immediate from uniform ellipticity in case the coefficient were smooth. On the other hand, self-adjointness on some domain could be obtained in an abstract fashion via quadratic forms under mere L^∞-assumptions [15, Section VIII.6]. However, in accordance with our focus on the interplay of the coefficient regularity class with qualitative solution properties, we will give an explicit domain description in terms of Sobolev spaces, which in addition is uniform with respect to τ and z.

Observe that, due to the low coefficient regularity, we also have to establish that A is well defined on all of $H^2(\mathbb{R}^d)$ as an operator into $L^2(\mathbb{R}^d)$. This is included in the following lemma as the special case $s = 0$.

Lemma 3.2. *Let* $0 \leqslant s < r < 1$ *and* $v \in H^{s+2}(\mathbb{R}^d)$. *Then* $Av \in H^s(\mathbb{R}^d)$.

Proof. Each component of $\operatorname{grad} v$ is in $H^{s+1}(\mathbb{R}^d)$. Hence multiplying with the H^{r+1}-coefficient c_1, as well as with the constant c_0, is well defined within $H^{s+1}(\mathbb{R}^d)$ since this is an algebra. □

We set $D(A) := H^2(\mathbb{R}^d)$ and note that an integration by parts immediately yields that A is symmetric, i.e., $D(A) \subseteq D(A^*)$ and $A^*|_{D(A)} = A$. We proceed to show that also $D(A^*) \subseteq D(A)$ by which self-adjointness will be established.

By definition, the adjoint operator has domain

$$D(A^*) = \{ v \in L^2 \mid \text{for some } w \in L^2\text{: } \langle \psi | w \rangle = \langle A\psi | v \rangle \text{ for all } \psi \in H^2 \},$$

where $\langle\ |\ \rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$. Let $v \in D(A^*)$ and choose a sequence $(v_k)_{k\in\mathbb{N}}$ in $H^2(\mathbb{R}^d)$ which converges to v in $L^2(\mathbb{R}^d)$. On the one hand, there exists $w \in L^2(\mathbb{R}^d)$ such that we have for all test functions $\psi \in \mathcal{D}(\mathbb{R}^d)$

$$\langle A^* v_k | \psi \rangle = \langle v_k | A\psi \rangle \to \langle v | A\psi \rangle = \langle w | \psi \rangle \quad \text{when } k \to \infty.$$

Thus, $(A^* v_k)_{k\in\mathbb{N}}$ has the distributional limit $w \in L^2(\mathbb{R}^d)$. On the other hand, since $c_1 \in H^1(\mathbb{R}^d)$ and $\operatorname{grad} v_k \to \operatorname{grad} v$ in $H^{-1}(\mathbb{R}^d)$ (as $k \to \infty$) we may employ the continuous duality product of distributions (cf. [11, Proposition 5.2]) $H^1 \times H^{-1} \to W^{-1,1}_{\text{loc}}$ and deduce that $A^* v_k = A v_k \to \operatorname{div}(c(x) \operatorname{grad} v) = Av$ in $W^{-2,1}_{\text{loc}}$, hence in the sense of distributions. By uniqueness of distributional limits, we deduce that $Av = w \in L^2(\mathbb{R}^d)$. We obtain

$$D(A^*) = \{ v \in L^2 \mid Av \in L^2 \}.$$

The assertion $D(A^*) \subseteq D(A) = H^2(\mathbb{R}^d)$ follows now from the following result. For later reference, it is stated in slightly more general terms. (We first consider the important case $d = 2$ and leave the case of one-dimensional lateral variation for a remark below.)

Lemma 3.3 *(Elliptic regularity). Let* $0 \leqslant s < r < 1$ *and* $v \in H^s(\mathbb{R}^2)$ *such that* $Av \in H^s(\mathbb{R}^2)$. *Then* v *belongs to* $H^{s+2}(\mathbb{R}^2)$.

The proof will be based on repeated use of the following three facts, which we collect in a preparatory list of “ingredients”:

Fact A: Let $w_j \in H^{s_j}(\mathbb{R}^2)$ $(j = 1, 2)$ such that $s_1 + s_2 \geqslant 0$. Then

$$w_1 \cdot w_2 \in H^{s_0}(\mathbb{R}^2),$$

where

$$s_0 = \begin{cases} \min(s_1, s_2, s_1 + s_2 - 1) & \text{if } s_1 \neq \pm 1,\ s_2 \neq \pm 1, \text{ and } s_1 + s_2 \neq 0, \\ \min(s_1, s_2, s_1 + s_2 - 1 - \varepsilon) & \text{with } \varepsilon > 0 \text{ arbitrary, otherwise.} \end{cases}$$

This is included in the statement of [9, Theorem 8.3.1]. As can be seen from the proof therein, one also obtains continuity of the multiplication $H^{s_1} \times H^{s_2} \to H^{s_0}$ (with respect to the corresponding Sobolev norms).

Fact B: We can find a function $F \in \mathcal{C}^\infty(\mathbb{R})$, $F(0) = 0$, such that

$$\frac{1}{c(x)} = \frac{1}{c_0} + F\big(c_1(x)\big).$$

In particular, we obtain $1/c - 1/c_0 = F(c_1) \in H^{r+1}(\mathbb{R}^2)$. Since $r > 0$, this follows from [9, Theorem 8.5.1] once F is given. To find F, we simply set $F(y) := -y/(c_0(c_0 + y))$ when $y \geqslant c_0/2$ and extend it in a smooth way to $\mathbb{R}$ such that $F(0) = 0$.

Fact C: If $v \in L^2(\mathbb{R}^2)$ the equation

$$Av = \operatorname{div}(c \operatorname{grad} v) = \Delta(cv) - \operatorname{div}(v \operatorname{grad} c) \tag{C1}$$

holds in $\mathcal{D}'(\mathbb{R}^2)$, where the occurring products are defined as follows: using $\operatorname{grad} v \in H^{-1}$ we get $c \operatorname{grad} v \in W^{-1,1}_{\text{loc}}$ by the duality product [11, Proposition 5.2]; $cv \in L^2$ since $c \in L^\infty$; and $v \operatorname{grad} c \in L^1$ because $\partial_j c \in H^r \subseteq L^2$.

Under the stronger assumption $v \in H^{r+1}(\mathbb{R}^2)$ we have, in addition, that

$$Av = \operatorname{grad} c \cdot \operatorname{grad} v + c\Delta v \tag{C2}$$

in $\mathcal{D}'(\mathbb{R}^2)$, with the meaning of the products on the right-hand side as follows: since $\Delta v \in H^{r-1}$, Fact A applies and yields $c\Delta v \in H^{r-1}$; furthermore, $\operatorname{grad} c$ and $\operatorname{grad} v$ both lie in H^r, so that another application of Fact A shows that their (Euclidean inner) product belongs to $H^{\min(r,2r-1)}$.

Remark 3.4. Note that formula (C2) represents A as an operator with a (Hölder-) continuous coefficient in its principal part and Sobolev regularity in the lower orders. We observe that in such situation, [8, Theorem 17.1.1] gives local solvability in H^2 for right-hand sides in L^2. However, the latter does not imply H^2-regularity of any L^2-solution. For the pure regularity question, it also seems that methods based on perturbations of constant coefficient operators do not apply either, since A is not necessarily of constant strength (cf. [7, Chapter XIII]).

Proof of Lemma 3.3. To begin with, we only know that v as well as Av belong to $H^s(\mathbb{R}^2)$. The proof proceeds in three steps, successively revealing higher regularity.

Claim 1. $v \in H^{s+r}(\mathbb{R}^2)$.

We have $\operatorname{grad} c \in H^r$, so that application of Fact A, noting that $r - 1 < 0$, gives $v \operatorname{grad} c \in H^{s+r-1}$, hence $\operatorname{div}(v \operatorname{grad} c) \in H^{s+r-2}$. Since $Av \in H^s$ we deduce from Eq. (C1) that $\Delta(cv) \in H^{s+r-2}$, which in turn implies that $cv \in H^{s+r}$. Now invoke the decomposition from Fact B and write

$$v = \frac{1}{c_0} cv + F(c_1)cv.$$

The first part clearly is in H^{s+r}, and for the second summand the same is true by Fact A. (Note that $\min(s + r, s + 2r - \varepsilon) = s + r$ if $0 < \varepsilon < r$.)

Claim 2. $v \in H^{r+1}(\mathbb{R}^2)$.

We may start from $v \in H^{r+s}$ by Claim 1 and proceed inductively to show that

$$v \in H^{r+\min(1,s+jr/2)}, \quad j \geqslant 0. \tag{$\star$}$$

Claim 2 then follows upon choosing j sufficiently large (i.e., $j \geqslant 2(1-s)/r$ steps will be required).

The case $j = 0$ is just Claim 1, so we assume that the assertion holds for some $j \geqslant 0$. If $s + jr/2 \geqslant 1$ it is trivially satisfied for larger values of j, therefore we assume $t_j := s + jr/2 < 1$ and that

$$v \in H^{r+t_j}.$$

Fact A gives $v \operatorname{grad} c \in H^{r+t_j} \cdot H^r \subseteq H^{\min(r,t_j+2r-1-\varepsilon)} \subseteq H^{\min(r,t_j+3r/2-1)}$ upon choosing $\varepsilon < r/2$. Thus, using the short-hand notation $r_j := \min(r+1, t_j + 3r/2) \leqslant r+1$ we may infer that $\operatorname{div}(v \operatorname{grad} c) \in H^{r_j-2}$. Since $r_j - 2 \leqslant r - 1 < 0 < s$ we also have $Av \in H^s \subset H^{r_j-2}$. Eq. (C1) now implies that $\Delta(cv) \in H^{r_j-2}$, hence $cv \in H^{r_j}$. Again by Fact B, combined with Fact A, we obtain

$$v = \frac{1}{c_0} cv + F(c_1) cv \in H^{r+1} \cdot H^{r_j} \subseteq H^{r_j} = H^{r+\min(1,t_j+r/2)},$$

which means ($\star$) for $j+1$ in place of j.

Claim 3. $v \in H^{s+2}(\mathbb{R}^2)$.

We use a similar strategy as in the proof of Claim 2 and prove inductively that

$$v \in H^{\min(s+2,1+(j+1)r/2)}, \quad j \geqslant 1. \tag{$\star\star$}$$

Claim 3 then follows when j is chosen sufficiently large (i.e., $j \geqslant 2(1+s)/r - 1$ steps are required).

The basic case $j = 1$ corresponds to Claim 2, so we proceed with some $j \geqslant 1$, under the additional assumption $r \leqslant s_j := (j+1)r/2 < s+1$ to exclude trivial cases. In other words, we assume that

$$v \in H^{s_j+1}.$$

Therefore, again by Fact A and choosing a possibly occurring $\varepsilon < r/2$, we deduce

$$\operatorname{grad} c \cdot \operatorname{grad} v \in H^r \cdot H^{s_j} \subseteq H^{\min(r,s_j+r/2-1)} = H^{\min(r,q_j)},$$

where we have introduced $q_j := s_j + r/2 - 1$. Exploiting Eq. (C2) and noting that $Av \in H^s$, $s < r$, we extract the information that $c\Delta v \in H^{\min(s,q_j)}$. Once again we use the decomposition corresponding to Fact B and the statement of Fact A to deduce

$$\Delta v \in H^{\min(s,q_j)} + H^{r+1} \cdot H^{\min(s,q_j)} \subseteq H^{\min(s,q_j)}$$

and a fortiori that $\Delta v \in H^{\min(s+2,q_j+2)}$. But $q_j = (j+1)r/2 + r/2 - 1$ hence $q_j + 2 = 1 + (j+2)r/2$ and ($\star\star$) is proven with $j+1$ in place of j. □

Remark 3.5. The one-dimensional analogue of Lemma 3.3 is more elementary: $Av = (cv')' \in H^s(\mathbb{R})$ implies $cv' \in H^{s+1}(\mathbb{R})$ and, since Fact B is valid for $d = 1$ as well, we obtain $v' \in H^{s+1}(\mathbb{R})$; thus, $v \in H^{s+2}(\mathbb{R})$.

We summarize the intermediate conclusions from the discussion so far in a separate statement, where we appeal to Stone's theorem providing us with the exponential unitary group $T(z) = \exp(izA)$.

Theorem 3.6. *Let* $c \in \mathcal{C}(\mathbb{R}^d)$ *satisfy* (3.10) *and define* $Av = \operatorname{div}(c(x) \operatorname{grad} v)$ *with domain* $D(A) = H^2(\mathbb{R}^d)$ *in* $L^2(\mathbb{R}^d)$. *Then* A *is self-adjoint and* iA *generates a strongly continuous unitary group* $(T(z))_{z\in\mathbb{R}}$ *on* $L^2(\mathbb{R}^d)$. *Moreover, we have the following resolvent estimate, valid for* $\lambda \in \mathbb{R} \setminus \{0\}$:

$$\left\|(iA - \lambda)^{-1} v\right\|_{L^2} \leqslant \frac{\|v\|_{L^2}}{|\lambda|} \quad \textit{for all } v \in L^2\big(\mathbb{R}^d\big). \tag{3.11}$$

We briefly recall how $(T(z))_{z\in\mathbb{R}}$ can be used to construct solutions to the Cauchy problem on $\mathbb{R}^d \times \mathbb{R}$

$$\partial_z v - iAv = g, \qquad v(0) = v_0.$$

Let $v_0 \in L^2(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}, L^2(\mathbb{R}^d))$ then the *mild solution*

$$v(z) := T(z)v_0 + \int_0^z T(z-\rho)g(\rho)\, d\rho \tag{3.12}$$

is in $\mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^d))$.

If $v_0 \in H^2(\mathbb{R}^d)$ and $g \in \mathcal{C}(\mathbb{R}, H^2(\mathbb{R}^d))$ or $g \in \mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^d))$ then v belongs to $\mathcal{C}^1(\mathbb{R}, L^2(\mathbb{R}^d))$ and is the unique classical solution with pointwise values in $H^2(\mathbb{R}^d)$ (cf. [13, Section 4.2, Corollaries 2.5 and 2.6]).

Remark 3.7. The mild solution (3.12) defines a weak solution in the following sense: $v(0) = v_0$ and for all $\phi \in \mathcal{D}(\mathbb{R}^{d+1})$

$$-\int_{\mathbb{R}} \big(\big\langle v(z) | \partial_z \phi(z,.) - iA\phi(z,.)\big\rangle\big)\, dz = \int_{\mathbb{R}} \big\langle g(z) | \phi(z,.)\big\rangle\, dz,$$

where $\langle\ |\ \rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$. To see this, one approximates the mild solution by classical solutions $(v_k)_{k\in\mathbb{N}}$ to equations with regularized right-hand side and initial data [13, Section 4.2, Theorem 2.7]: L^2-convergence of $v_k(z) \to v(z)$ (as $k \to \infty$), uniformly when z varies in compact intervals, together with the convergence $g_k \to g$ in $L^1(\mathbb{R}, L^2(\mathbb{R}^d))$ implies convergence in the integral formula above.

3.2. Evolution system at fixed frequency τ

Let $\tau \in \mathbb{R}$ be fixed, but arbitrary. We consider the z-parameterized family of unbounded self-adjoint operators in (2.8) and put

$$A(\tau; z) := A(\tau; z, x, D_x) \quad (z \geqslant 0).$$

Let $Z > 0$ be arbitrary. We will check that, for every τ, $(iA(\tau; z))_{z\geqslant 0}$ defines an evolution system (or fundamental solution) $(U(\tau; z_1, z_2))_{Z\geqslant z_1\geqslant z_2\geqslant 0}$ on $L^2(\mathbb{R}^d)$ by applying [20, Section 4.4, Corollary to Theorem 4.4.2, p. 102] (cf. also [13, Sections 5.3–5.5]). We have to check that the corresponding hypotheses are satisfied.

First, observe that $D(A(\tau; z)) = H^2(\mathbb{R}^d)$ is independent of the evolution parameter z (and of τ), and every $iA(\tau; z)$ is the skew-adjoint generator of a strongly continuous (unitary) semigroup $(T(\tau, z; \zeta)_{\zeta\geqslant 0})$ on $L^2(\mathbb{R}^d)$. Furthermore, the resolvent estimates (3.11), valid for all z (and τ), immediately imply that $(A(\tau; z))_{z\geqslant 0}$ is a stable family of generators *with stability constants* 1 *and* 0 *for all* τ (cf. [20, Definition 4.3.1]).

Finally, we have to check that for all $v \in H^2(\mathbb{R}^d)$ the map

$$[0, \infty) \ni z \mapsto A(\tau; z)v \in L^2(\mathbb{R}^d)$$

is continuously differentiable. We may use Eq. (C2) from Fact C to write (with grad taken with respect to x only)

$$A(\tau; z)v = \operatorname{grad} c(z, x, \tau) \cdot \operatorname{grad} v + c(z, x, \tau)\Delta v.$$

By Assumption 1(ii)–(iii), we have

$$\operatorname{grad} c(.,.,\tau) \in \mathcal{C}^1\big([0,\infty), H^r(\mathbb{R}^d)\big), \qquad c(.,.,\tau) - c_0 h_0(\tau) \in \mathcal{C}^1\big([0,\infty), H^{r+1}(\mathbb{R}^d)\big).$$

Since $\operatorname{grad} v \in H^1(\mathbb{R}^d)$ and $\Delta v \in L^2(\mathbb{R}^d)$ the multiplication rules plus continuity properties in Fact A apply (where in case $H^r \cdot H^1$ we choose $\varepsilon < r$, if $d = 2$) and yield that $A(\tau; .)v \in \mathcal{C}^1([0, \infty), L^2(\mathbb{R}^d))$.

Thus, all hypotheses of [20, Section 4.4, Corollary to Theorem 4.4.2, p. 102] are fulfilled. Note that the evolution system is constructed as the strong operator limit of discretizations based on the unitary semigroups of each generator, hence is contractive. This implies the following intermediate result.

Proposition 3.8. *Let $Z > 0$. Then for all $\tau \in \mathbb{R}$ the family $(iA(\tau; z))_{z\geqslant 0}$ defines a unique evolution system $(U(\tau; z_1, z_2))_{Z\geqslant z_1\geqslant z_2\geqslant 0}$ on $L^2(\mathbb{R}^2)$ with the following properties: The map $(z_1, z_2) \mapsto U(\tau; z_1, z_2)$ is strongly continuous, $U(\tau; z, z) = I$, $U(\tau; z_1, z_2)$ is contractive, and*

$$U(\tau; z_1, z_2) \circ U(\tau; z_2, z_3) = U(\tau; z_1, z_3), \quad 0 \leqslant z_3 \leqslant z_2 \leqslant z_1 \leqslant Z; \tag{3.13}$$

moreover, $H^2(\mathbb{R}^d)$ is invariant under $U(\tau; z_1, z_2)$, for all $v \in H^2(\mathbb{R}^d)$ the map $(z_1, z_2) \mapsto U(\tau; z_1, z_2)v$ is continuously differentiable, separately in both variables, and the following equations hold:

$$\frac{\partial}{\partial z_1} U(\tau; z_1, z_2)v = A(\tau; z_1)U(\tau; z_1, z_2)v, \tag{3.14}$$

$$\frac{\partial}{\partial z_2} U(\tau; z_1, z_2)v = -U(\tau; z_1, z_2)A(\tau; z_2)v. \tag{3.15}$$

At this stage, we obtain solutions to a version of the Cauchy problem (2.6)–(2.7) at fixed frequency τ, i.e.,

$$\partial_z v - iA(\tau; z)v = g \in L^1\big([0, Z], L^2\big(\mathbb{R}^d\big)\big), \tag{3.16}$$

$$v|_{z=0} = v_0 \in L^2\big(\mathbb{R}^d\big). \tag{3.17}$$

The *mild solution* is defined by

$$v(z) := U(\tau; z, 0)v_0 + \int_0^z U(\tau; z, \rho)g(\rho)\,d\rho \tag{3.18}$$

and belongs to $\mathcal{C}([0, Z], L^2(\mathbb{R}^d))$ [13, Section 5.5, Definition 5.1].

Remark 3.9. (i) In the case of classical solutions, we have the following regularity property: If $v_0 \in H^2(\mathbb{R}^d)$ and $g \in \mathcal{C}([0, Z], H^2(\mathbb{R}^d))$ or $g \in \mathcal{C}^1([0, Z], L^2(\mathbb{R}^d))$ then $v \in \mathcal{C}^1([0, Z], L^2(\mathbb{R}^d))$ is the unique *H^2-valued solution* and satisfies the equation in the strong sense (cf. [13, Section 5.5, Theorems 5.2 and 5.3]).

(ii) Observe that, at frozen value of τ, one may apply [10, Chapter 3, Theorem 10.1 and Remark 10.2] directly by putting $H = L^2$, $V = H^1$, and

$$a(z; u, v) = \sum_{j=1}^{d} \big\langle c(z, .)\partial_j u | \partial_j v\big\rangle \quad \forall u, v \in V.$$

It suffices to assume $c \in \mathcal{C}^1([0, Z], L^\infty(\mathbb{R}^d))$, then for any initial value $v_0 \in H^1(\mathbb{R}^d)$ and right-hand side $g \in L^2([0, Z] \times \mathbb{R}^d)$ such that $\partial_z g \in L^2([0, Z], H^{-1})$ there is a unique solution $v \in \mathcal{C}([0, Z], H^1) \cap \mathcal{C}^1([0, Z], H^{-1})$ to the Cauchy problem (3.16)–(3.17). However, our approach allows for a precise investigation of the τ-dependence, which is needed to solve the full Cauchy problem (2.6)–(2.7) with distributional data as well as to transform back to the original problem (2.3)–(2.4) in Section 4. Furthermore, our results show that lateral H^2-regularity of the data is preserved in the solution.

We thus have established an evolution system in the L^2-setting. Note that by Lemma 3.3 we have, in fact, that $A(\tau; z, x, D_x)$ is an unbounded operator on H^s with domain H^{s+2} for any $0 \leqslant s < r$. If we were able to establish an evolution system on H^s then the regularity information encoded into A would be more directly preserved.

3.3. *Frequency dependence of the evolution system*

Throughout this subsection, let $Z > 0$ be arbitrary but fixed. So far, the frequency parameter τ was arbitrary, but fixed, throughout the construction of the evolution system

$(U(\tau; z_1, z_2))_{Z \geqslant z_1 \geqslant z_2 \geqslant 0}$. We will prove that the dependence on all parameters (τ, z_1, z_2) jointly is strongly continuous. In the sequel, let $L(E, F)$ (resp. $L(E)$) denote the set of bounded linear operators between the Banach spaces E and F (resp. on E).

We begin with an observation on the general level of semigroups and evolution systems.

Lemma 3.10. *Assume that*

$$(\tau, z) \mapsto A(\tau; z) \text{ is continuous } \mathbb{R} \times [0, \infty) \to L\big(H^2\big(\mathbb{R}^d\big), L^2\big(\mathbb{R}^d\big)\big) \tag{3.19}$$

(*with respect to the operator norm*) *and*

$$(\tau, z, \zeta) \mapsto T(\tau, z; \zeta) \text{ is strongly continuous } \mathbb{R} \times [0, \infty) \times [0, \infty) \to L\big(L^2\big(\mathbb{R}^d\big)\big), \tag{3.20}$$

where $(T(\tau, z; \zeta))_{\zeta \geqslant 0}$ denotes the semigroup generated by $A(\tau; z)$. Then the map $(\tau, z_1, z_2) \mapsto U(\tau; z_1, z_2)$ is strongly continuous from $B := \mathbb{R} \times \{(z_1, z_2): Z \geqslant z_1 \geqslant z_2 \geqslant 0\}$ into $L(L^2(\mathbb{R}^d))$.

Proof. We inspect the basic construction of the evolution system from the family of semigroups in the proof of [13, Section 5.3, Theorem 3.1] and keep track of the additional parameter $\tau \in \mathbb{R}$ in our case. For all $(\tau, z_1, z_2) \in B$ we obtain $U(\tau; z_1, z_2)$ as the strong limit of $U_n(\tau; z_1, z_2)$ (as $n \to \infty$), where $U_n(\tau; ., .)$ is the evolution system defined as follows: put $z_n^j = jZ/n$ ($j = 0, \dots, n$) then for $\tau \in \mathbb{R}$, $0 \leqslant y \leqslant z \leqslant Z$ let

$$U_n(\tau; z, y) := T\big(\tau, z_n^l; z - y\big) \quad \text{if } z_n^l \leqslant y \leqslant z \leqslant z_n^{l+1},$$

and

$$U_n(\tau; z, y) := T\big(\tau, z_n^k; z - z_n^k\big) \cdot \prod_{l+1 \leqslant j \leqslant k-1} T\big(\tau, z_n^j; Z/n\big) \cdot T\big(\tau, z_n^l; z_n^{l+1} - y\big)$$
$$\text{if } z_n^l \leqslant y \leqslant z_n^{l+1} \leqslant z_n^k \leqslant z \leqslant z_n^{k+1},\ k > l.$$

By (3.20) the right-hand side of each formula is strongly continuous with respect to (τ, z, y), and the boundary values, when $k = l + 1$ and $y = z_n^{l+1}$ or $z = z_n^{l+1}$, match. Hence U_n is strongly continuous on B and $\|U_n(\tau; z, y)\| = 1$.

As in [13, (3.13) on p. 136] we have the following integral representation for the action on any $v \in H^2$

$$U_n(\tau; z, y)v - U_m(\tau; z, y)v = \int_y^z U_n(\tau; z, \rho)\big(A_n(\tau; \rho) - A_m(\tau; \rho)\big)U_m(\tau; \rho, y)v\, d\rho,$$

where $A_n(\tau; \rho)$ is the piecewise constant approximation of $A(\tau; \rho)$ with $A_n(\tau; \rho) := A(\tau; z_n^k)$, when $z_n^k \leqslant \rho < z_n^{k+1}$, and $A_n(\tau; Z) = A(\tau; Z)$. By (3.19) we have $\|A_n(\tau; \rho) - A(\tau; \rho)\|_{L(H^2, L^2)} \to 0$ uniformly for (τ, ρ) in compact sets. Passing to the limit $m \to \infty$ in the integral representation above yields the estimate

$$\big\|U_n(\tau; z, y)v - U(\tau; z, y)v\big\|_{L^2} \leqslant \|v\|_{H^2} \int_y^z \big\|A_n(\tau; \rho) - A(\tau; \rho)\big\|_{L(H^2, L^2)}\, d\rho.$$

By the uniform convergence of $A_n(\tau;\rho)$ (as $n\to\infty$) we thus obtain (local) uniform convergence of $U_n(\tau;z,y)v$, which proves the asserted continuity of $(\tau,z,y)\mapsto U(\tau;z,y)v$. □

We have to establish conditions (3.19)–(3.20) in the specific context of the assumptions described in Section 2. In due course, we will make repeated use of (τ,z)-parameterized variants of Facts A–C, stated in Section 3.1. Note that, in particular, the function F used in Fact B does not depend on (τ,z).

Lemma 3.11. *If $A(\tau;z)$ ($\tau\in\mathbb{R}$, $z\geqslant 0$) is defined by* (2.8) *then Assumption* 1 *implies Lipschitz-continuity of the map in condition* (3.19).

Proof. Let $M(\tau,z)$ denote the operator of multiplication of pairs $(v_1,v_2)\in H^1\times H^1$ by the scalar function $c(\tau,z,x)-c(\tau_0,z_0,x)$. We write $A(\tau;z)-A(\tau_0;z_0)=\operatorname{div}\circ M(\tau,z)\circ\operatorname{grad}$ as a composition of operators and get the following norm inequality

$$\begin{aligned}&\big\|A(\tau;z)-A(\tau_0;z_0)\big\|_{L(H^2,L^2)}\\&\quad\leqslant\|\operatorname{div}\|_{L(H^1\times H^1,L^2)}\cdot\big\|M(\tau,z)\big\|_{L(H^1\times H^1)}\cdot\|\operatorname{grad}\|_{L(H^2,H^1\times H^1)}\\&\quad\leqslant\sqrt{2}\,\big\|M(\tau,z)\big\|_{L(H^1\times H^1)}.\end{aligned}$$

To estimate $\|M(\tau,z)(v_1,v_2)\|_{H^1\times H^1}$ it suffices to find an upper bound of $\|(c(z,.,\tau)-c(z_0,.,\tau_0))v\|_{H^1}$ for $v\in H^1$. We have

$$\begin{aligned}&\big(c(z,x,\tau)-c(z_0,x,\tau_0)\big)v(x)\\&\quad=\begin{pmatrix}z-z_0\\ \tau-\tau_0\end{pmatrix}\cdot\int_0^1\operatorname{grad}_{(z,\tau)}c\big(z_0+\sigma(z-z_0),x,\tau_0+\sigma(\tau-\tau_0)\big)\,d\sigma\,v(x),\end{aligned}$$

which, upon taking the H^1-norm with respect to x and assuming $\max(|z-z_0|,|\tau-\tau_0|)\leqslant 1$, yields

$$\begin{aligned}&\big\|\big(c(z,.,\tau)-c(z_0,.,\tau_0)\big)v\big\|_{H^1}\\&\quad\leqslant\max\big(|z-z_0|,|\tau-\tau_0|\big)\cdot\sup\big(\big\|\partial_z c(z',.,\tau')v\big\|_{H^1}+\big\|\partial_\tau c(z',.,\tau')\big\|_{H^1}\big),\end{aligned}$$

where the supremum is taken over $(z',\tau')\in[z_0-1,z_0+1]\times[\tau_0-1,\tau_0+1]$. Assumption 1 implies that $\partial_z c$, $\partial_\tau c$ both are continuous functions of (z,τ) valued in $H^{r+1}(\mathbb{R}^d)$, which combined with Fact A gives

$$\begin{aligned}&\big\|\partial_z c(z',.,\tau')v\big\|_{H^1}+\big\|\partial_\tau c(z',.,\tau')\big\|_{H^1}\\&\quad\leqslant C_1\|v\|_{H^1}\big(\big\|\partial_z c(z',.,\tau')\big\|_{H^{r+1}}+\big\|\partial_\tau c(z',.,\tau')\big\|_{H^{r+1}}\big)\leqslant C_2\|v\|_{H^1}\end{aligned}$$

with positive constants C_1,C_2 and for all $(z',\tau')\in[z_0-1,z_0+1]\times[\tau_0-1,\tau_0+1]$. Combining all estimates we deduce that there is $C_3>0$ such that $|z-z_0|+|\tau-\tau_0|\leqslant 1$ implies

$$\big\|M(\tau,z)(v_1,v_2)\big\|_{H^1\times H^1} \leqslant C_3 \max\big(|z-z_0|,|\tau-\tau_0|\big)\big\|(v_1,v_2)\big\|_{H^1\times H^1},$$

which proves the asserted Lipschitz-continuity. □

Lemma 3.12. *If* $A(\tau;z)$ ($\tau\in\mathbb{R}$, $z\geqslant 0$) *is defined by* (2.8) *then Assumption* 1 *implies the continuity condition* (3.20).

Proof. We apply the Kato–Trotter theorem on convergence of semigroups (cf. [26, Chapter IX, Section 12, Theorem 1]). According to this theorem, we obtain

$$T(\tau,z;\zeta)\to T(\tau_0,z_0;\zeta) \quad \text{strongly as } (\tau,z)\to(\tau_0,z_0),$$
$$\text{uniformly on any compact interval containing } \zeta,$$

thus (3.20) by uniformity, provided that we show strong continuity of the resolvent map $(\tau,z)\mapsto(\lambda-iA(\tau;z))^{-1}=:R(\lambda,iA(\tau;z))$ for some $\lambda>0$.

Fix $\lambda>0$ and let $f\in L^2(\mathbb{R}^d)$ be arbitrary. Define $u(\tau,z):=R(\lambda,iA(\tau;z))f\in H^2(\mathbb{R}^d)$, so that u solves

$$\lambda u(\tau,z)-iA(\tau;z)u(\tau,z)=f. \tag{3.21}$$

Adding the difference $iA(\tau;z)u(\tau,z)-iA(\tau_0;z_0)u(\tau,z)$ yields

$$\big(\lambda-iA(\tau_0;z_0)\big)u(\tau,z)=f+i\big(A(\tau;z)-A(\tau_0;z_0)\big)u(\tau,z)=:f+iw(\tau,z).$$

Hence, $u(\tau,z)=R(\lambda,iA(\tau_0;z_0))(f+iw(\tau,z))$ and it suffices to prove that $w(\tau,z)\to 0$ in $L^2(\mathbb{R}^d)$ as $(\tau,z)\to(\tau_0,z_0)$. Applying (C2) from Fact C in Section 3.1, we may write

$$w(\tau,z)=\operatorname{grad}\big(c(z,x,\tau)-c(z_0,x,\tau_0)\big)\cdot\operatorname{grad}u(\tau,z)$$
$$+\big(c(z,x,\tau)-c(z_0,x,\tau_0)\big)\Delta u(\tau,z).$$

By Assumption 1, the difference $c(z,.,\tau)-c(z_0,.,\tau_0)$ tends to 0 in $H^{r+1}(\mathbb{R}^d)$. In view of Fact A this implies $w(\tau,z)\to 0$, if $\operatorname{grad}u(\tau,z)$ as well as $\Delta u(\tau,z)$ stays bounded. To prove the latter, we take the L^2-inner product with u on both sides of Eq. (3.21) and obtain

$$\lambda\|u\|_{L^2}^2-i\sum_j\langle c\partial_{x_j}u|\partial_{x_j}u\rangle=\langle f|u\rangle.$$

Note that taking real parts here yields the estimate (3.11), which is $\|u\|_{L^2}\leqslant\|f\|_{L^2}/\lambda$. If we take absolute values of the imaginary parts, we may use the lower bound $c(z,x,\tau)\geqslant c_0$ and the resolvent estimate to deduce $\sum_j\|\partial_{x_j}u(\tau,z)\|_{L^2}^2\leqslant\|f\|_{L^2}^2/(\lambda c_0)$, uniformly in (τ,z).

Finally, the boundedness of $\|\Delta u(\tau,z)\|_{L^2}$ is revealed in several steps. First, note that (C2) from Fact C applied to (3.21) yields

$$c(z,.,\tau)\Delta u(\tau,z)=i\big(f-\lambda u(\tau,z)\big)-\operatorname{grad}c(z,.,\tau)\cdot\operatorname{grad}u(\tau,z).$$

The first term on the right-hand side is bounded in L^2, uniformly for all (τ,z), whereas the second term is uniformly bounded in H^{r-1} by Fact A. Hence $c\,\Delta u(\tau,z)$ is a bounded family in H^{r-1} and, combining Facts A and C, we find that $\Delta u(\tau, z)$ is uniformly bounded in H^{r-1} as well. Therefore, $u(\tau, z)$ has a uniform bound in H^{r+1}-norms. From here we may proceed as in Claim 3 from the proof of Proposition 3.3 (with $s = 0$). Indeed, the arguments used there preserve uniform boundedness properties throughout, since we have such in H^{r+1} already. Thus, $u(\tau, z)$ is uniformly bounded in H^2, in particular, $\Delta u(\tau, z)$ is bounded uniformly for all (τ, z), which completes the proof. □

We summarize the preceding results in the announced continuity statement for the evolution system.

Theorem 3.13. *Let $(U(\tau; z_1, z_2))_{Z \geqslant z_1 \geqslant z_2 \geqslant 0}$ be the evolution system generated by the family of operators $A(\tau; z)$ ($\tau \in \mathbb{R}$, $z \geqslant 0$), defined in* (2.8) *and satisfying Assumption* 1. *Then $(\tau, z_1, z_2) \mapsto U(\tau; z_1, z_2)$ is strongly continuous $\mathbb{R} \times \{(z_1, z_2)\colon Z \geqslant z_1 \geqslant z_2 \geqslant 0\} \to L(L^2(\mathbb{R}^d))$.*

4. Solution of the Cauchy problem

In this section we present our main results: existence and uniqueness of solutions to the Cauchy problem (2.6)–(2.7) in the frequency domain and to (2.3)–(2.4) in the time domain.

If E is a Banach space, let $\mathcal{C}_b(\mathbb{R}, E)$ denote the space of E-valued continuous bounded functions. Observe that for any $G \in \mathcal{C}_b(\mathbb{R}, L^2) \subset \mathscr{S}'(\mathbb{R}; L^2)$ the expression $U(\tau; z_1, z_2)G(\tau)$ is well defined pointwise for all (τ, z_1, z_2). Therefore, collecting the results obtained so far we arrive at the following assertion.

Proposition 4.1. *If $v_0 \in \mathcal{C}_b(\mathbb{R}, L^2(\mathbb{R}^d))$ and $g \in \mathcal{C}_b([0, Z] \times \mathbb{R}, L^2(\mathbb{R}^d))$ then the formula*

$$v(z, \tau) := U(\tau; z, 0)v_0(\tau) + \int_0^z U(\tau; z, \rho)g(\rho, \tau)\, d\rho \tag{4.22}$$

defines a mild solution $v \in \mathcal{C}^1([0, Z], \mathcal{C}_b(\mathbb{R}, L^2(\mathbb{R}^d))) \subset \mathcal{V}$ to (2.6)–(2.7). *Moreover, when v is a strong solution then $u := \mathcal{F}_t^{-1} v$ is a strong solution of* (2.3)–(2.4) *with initial data $u_0 = \mathcal{F}_t^{-1} v_0$ and right-hand side $f = \mathcal{F}_t^{-1} g$.*

For example, the hypotheses leading to strong solvability are satisfied if $v_0 \in \mathcal{C}_b(\mathbb{R}, H^2)$ and $g \in \mathcal{C}_b([0, Z] \times \mathbb{R}, H^2)$ or $g \in \mathcal{C}^1([0, Z], \mathcal{C}_b(\mathbb{R}, L^2))$. Of course, using functions that are bounded and continuous with respect to the frequency variable τ here is just one simple way to ensure that all constructions described above work with all involved objects staying temperate. More generally, it would suffice to consider elements in $\mathcal{V}$ whose distributional action (with respect to the frequency variable) is given by (weak) integration over a continuous function (times the test function).

To apply formula (4.22) to the original Cauchy problem (2.3)–(2.4) we only need to state conditions on the data u_0 and f that imply $\mathcal{F}_t u_0 \in \mathcal{C}_b(\mathbb{R}, H^2)$ and $\mathcal{F}_t f \in \mathcal{C}_b([0, Z] \times \mathbb{R}, H^2)$ or $\mathcal{F}_t f \in \mathcal{C}^1([0, Z], \mathcal{C}_b(\mathbb{R}, L^2))$. Note that, for example, in our physical application such conditions would be met if the source or force terms are active only for some finite time interval and vanish otherwise. Then by the uniqueness of H^2-valued solutions, as stated in Remark 3.9, we obtain the following result.

Theorem 4.2. *Assume that the right-hand side f in Eq.* (2.3) *satisfies either $f \in \mathcal{C}([0,Z], L^1(\mathbb{R}, H^2(\mathbb{R}^2)))$ or $f \in \mathcal{C}^1([0,Z], L^1(\mathbb{R}, L^2(\mathbb{R}^2)))$.*

Then for every $u_0 \in L^1(\mathbb{R}, H^2(\mathbb{R}^2))$ the Cauchy problem (2.3)–(2.4) *has a unique strong solution $u \in \mathcal{C}^1([0,Z], \mathscr{S}'(\mathbb{R}, L^2(\mathbb{R}^2)))$ which is H^2-valued in the following sense: for all $z \in [0,Z]$ and $\phi \in \mathscr{S}'(\mathbb{R})$ we have $\langle u(z), \phi\rangle \in H^2(\mathbb{R}^2)$.*

Moreover, u is obtained by inverse partial Fourier transform (with respect to τ) of $v \in \mathcal{C}^1([0,Z], \mathcal{C}_b(\mathbb{R}, L^2(\mathbb{R}^2)))$ as defined in Eq. (4.22)*, where $v_0 := \mathcal{F}_t u_0$, $g := \mathcal{F}_t f$, and U is the evolution system from Proposition* 3.8*. In addition, $v(z,\tau)$ belongs to $H^2(\mathbb{R}^2)$ for every (z,τ).*

Inverse analysis of medium regularity. We conclude with a brief indication of a potential application of Theorem 4.2 to an inverse analysis of medium regularity in wave propagation. Suppose that we are in a model situation where parts (i), (ii) and (iv) of Assumption 1 in Section 2 are satisfied and the regularity property (iii) of the medium is in question; assume that we have the a priori knowledge that $c_l \in \mathcal{C}^1([0,\infty), L^\infty(\mathbb{R}^2))$ for all l. Let the sources of a seismic experiment be calibrated to produce data in accordance with the hypotheses in Theorem 4.2. If then the measured wave solution u (or v) fails to display the asserted H^2-regularity then we may conclude that for some c_l (and near some depth z) there is no $r \in (0,1)$ such that the H^{r+1}-regularity holds; in other words, the lateral regularity of the medium there cannot be better than H^1 (on the Sobolev scale). If we were in the possession of analogous H^s-results ($s > r$) for the Cauchy problem it would enable us to draw sharper conclusions in such an "inverse regularity analysis." Note that the exact location of the most singular region need not be known. For the application hence, precise imaging of the singularities is not required prior to the regularity analysis.

References

[1] F. Andersson, M.V. de Hoop, H. Smith, G. Uhlmann, A multi-scale approach to hyperbolic evolution equations with limited smoothness, Comm. Partial Differential Equations 33 (6) (2008) 988–1017.
[2] G. Bal, G. Papanicolaou, L. Ryzhik, Self-averaging in time reversal for the parabolic wave equation, Stoch. Dyn. 2 (2002) 507–531.
[3] J.F. Claerbout, Imaging the Earth's Interior, Blackwell Scientific Publications, Palo Alto, 1985.
[4] F. Colombini, N. Lerner, Hyperbolic operators with non-Lipschitz coefficients, Duke Math. J. 77 (3) (1995) 657–698.
[5] L. Fishman, M.V. de Hoop, M. van Stralen, Exact constructions of square-root Helmholtz operator symbols: The focusing quadratic profile, J. Math. Phys. 41 (2000) 4881–4938.
[6] L. Halpern, L.N. Trefethen, Wide-angle one-way wave equations, J. Acoust. Soc. Am. 84 (1988) 1397–1404.
[7] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol. II, Springer-Verlag, 1983.
[8] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol. III, Springer-Verlag, 1985, second ed., 1994.
[9] L. Hörmander, Lectures on Nonlinear Hyperbolic Differential Equations, Springer-Verlag, Berlin, Heidelberg, 1997.
[10] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. 1, Springer-Verlag, Berlin, New York, 1972.
[11] M. Oberguggenberger, Multiplication of Distributions and Applications to Partial Differential Equations, Longman Scientific & Technical, 1992.
[12] G. Papanicolaou, L. Ryzhik, K. Solna, Statistical stability in time reversal, SIAM J. Appl. Math. 64 (2004) 1133–1155.
[13] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[14] M. Reed, B. Simon, Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness, Academic Press, New York, 1975.
[15] M. Reed, B. Simon, Methods of Modern Mathematical Physics, I: Functional Analysis, second ed., Academic Press, New York, 1980.

[16] C.C. Stolk, A pseudodifferential equation with damping for one-way wave propagation in inhomogeneous acoustic media, Wave Motion 40 (2004) 111–121.
[17] C.C. Stolk, M.V. de Hoop, Microlocal analysis of seismic inverse scattering in anisotropic, elastic media, Comm. Pure Appl. Math. 55 (2002) 261–301.
[18] C.C. Stolk, M.V. de Hoop, Modeling of seismic data in the downward continuation approach, SIAM J. Appl. Math. 65 (2005) 1388–1406.
[19] C.C. Stolk, M.V. de Hoop, Seismic inverse scattering in the downward continuation approach, Wave Motion 43 (2006) 579–598.
[20] H. Tanabe, Equations of Evolution, Pitman, London, 1979.
[21] M. Taylor, Reflection of singularities of solutions to systems of partial differential equations, Comm. Pure Appl. Math. 28 (1975) 457–478.
[22] L.N. Trefethen, L. Halpern, Well-posedness of one-way wave equations and absorbing boundary conditions, Math. Comp. 47 (1986) 421–435.
[23] F. Treves, Basic Linear Partial Differential Equations, Academic Press, New York, 1975.
[24] S.C. Wales, J.J. McCoy, A comparison of parabolic wave theories for linearly elastic solids, Wave Motion 5 (1983) 99–113.
[25] R.D. van der Hilst, M.V. de Hoop, P. Wang, S.-H. Shim, P. Ma, L. Tenorio, Seismo-stratigraphy and thermal structure of Earth's core-mantle boundary region, Science 315 (2007) 1813–1817.
[26] K. Yosida, Functional Analysis, sixth ed., Springer-Verlag, Berlin, 1980.