Invariant Approximations and \( R \)-Subweakly Commuting Maps

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The existence of invariant approximations for \( R \)-subweakly commuting mappings is proved. Our results extend most of the known results to a new class of noncommuting mappings.

Key Words: \( R \)-subweakly commuting map; common fixed point; best approximation; normed space.

1. INTRODUCTION AND PRELIMINARIES

Let \( S \) be a subset of a normed space \( X = (X, \| \cdot \|) \) and \( T, I \) self-mappings of \( X \). Then \( T \) is called (1) nonexpansive on \( S \) if \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in S \); (2) \( I \)-nonexpansive on \( S \) if \( \|Tx - Ty\| \leq \|Ix - Iy\| \) for all \( x, y \in S \); (3) an \( I \)-contraction on \( S \) if there exists \( k \in [0, 1) \) such that \( \|Tx - Ty\| \leq k \|Ix - Iy\| \) for all \( x, y \in S \). The set of fixed points of \( T \) (resp. \( I \)) is denoted by \( FT \) (resp. \( FI \)). The set \( S \) is called (4) \( p \)-starshaped with \( p \in S \) if for all \( x \in S \), the segment \([x, p]\) joining \( x \) to \( p \) is contained in \( S \) (that is, \( kx + (1-k)p \in S \) for all \( x \in S \) and all real \( k \) with \( 0 \leq k \leq 1 \)); (5) convex if \( S \) is \( p \)-starshaped for every \( p \in S \). The convex hull \( co(S) \) of \( S \) is the smallest convex set in \( X \) that contains \( S \), and the closed convex hull \( cco(S) \) of \( S \) is the closure of its convex hull. \( T \) and \( I \) are said to be (6) commuting on \( S \) if \( ITx = TIx \) for all \( x \in S \); (7) \( R \)-weakly commuting on \( S \) [7] if there exists a real number \( R > 0 \) such that \( \|ITx - ITx\| \leq R \|Tx - Ix\| \) for all \( x \in S \). Suppose \( S \subset X \) is \( p \)-starshaped with \( p \in F(I) \) and is both \( T \)- and \( I \)-invariant. Then \( T \) and \( I \) are called (8) \( R \)-subweakly commuting on \( S \) if there exists a real number \( R > 0 \) such
that \(||Txx - ITxx|| \leq Rd(Ixx, [Tx, p])|\) for all \(x \in S\), where \(d(y, A) = \inf\{||y - z|| : z \in A\}\) for \(A \subset S\) and \(y \in S\). It is clear from the definitions that commutativity implies \(R\)-subweak commutativity, but the converse is not true in general. To see this, we consider the following examples:

1. Let \(X = \mathbb{R}\) with norm \(||x|| = |x|\), and let \(T, I\) be given by \(Tx = 4x - 3, \quad Ix = 2x^2 - 1\) for all \(x \in X\). Then \(T\) and \(I\) are \(R\)-subweakly commuting on \(S = [1, \infty)\). However, they are not commuting on \(S\).

2. Let \(X = \mathbb{R}^2\) with norm \(||(x, y)|| = \max(|x|, |y|)\), and let \(T, I\) be defined by \(T(x, y) = (2x - 1, y^3), \quad I(x, y) = (x^2, y^2)\) for all \((x, y) \in X\). Then \(T\) and \(I\) are \(R\)-subweakly commuting on \(S = \{(x, y) : x \geq 1, y \geq 1\}\) but they are not commuting on \(S\).

Remark. Let \(T\) and \(I\) be \(R\)-subweakly commuting self-mappings of a \(p\)-starshaped subset \(S\) of \(X\) with \(p \in F(I)\).

1. If \(Tx = Ix\), then \(Txx = ITxx\).

2. Suppose \(T_{x_n}, I_{x_n} \to y\) for some \(y \in S\). (a) If \(T\) is continuous at \(y\), then \(ITx_n \to Ty\). (b) If \(T\) and \(I\) are continuous at \(y\), then \(Ty = Iy\) and \(Tyy = ITy\).

Suppose \(\hat{x} \in X\). An element \(x \in S\) is called the best \(S\)-approximant to \(\hat{x}\) if \(||x - \hat{x}|| = d(\hat{x}, S)\), where \(d(\hat{x}, S) = \inf\{||\hat{x} - y|| : y \in S\}\). We denote the set of all such elements by \(P_S(\hat{x})\) and define \(C_2(\hat{x}) = \{x \in S : Ix \in P_S(\hat{x})\}\). Let \(\mathfrak{S}_0\) represent the class of closed convex subsets of \(X\) containing \(0\). Then for \(S \in \mathfrak{S}_0\), we set \(S_\ell = \{x \in S : ||x|| \leq 2||\hat{x}||\}\). Obviously \(P_S(\hat{x}) \subset S_\ell \in \mathfrak{S}_0\).


**Theorem 1.1.** Let \(T\) be a nonexpansive self-mapping of \(X\), \(S\) a finite dimensional \(T\)-invariant subspace of \(X\), and \(\hat{x} \in F(T)\). Then \(P_S(\hat{x}) \cap F(T) \neq \emptyset\).

In 1981, Smoluk [14] made his contribution replacing the finite dimensionality of \(S\) by the requirement “\(cl(T(D))\) is compact for every bounded \(D \subset S\) and \(T\) is linear.” Afterwards, Habiniak [4] observed that Smoluk’s result remains valid if the linearity of \(T\) is dropped. Further generaliza-
tions of Meinardus’s result in various directions were obtained by Brosowski [2], Hicks and Humphries [5], Sahab et al. [8], Shahzad [9, 10], and Singh [12, 13]. Recently, Al-Thagafi [1] established the following extension of Habiniak’s result.

**THEOREM 1.2.** Let $T$ be a self-mapping of $X$ with $\hat{x} \in F(T)$ and $S \in \mathcal{S}_0$ such that $T(S_\hat{x}) \subseteq S$. If $T$ is nonexpansive on $S_\hat{x} \cup \{\hat{x}\}$ and $\text{cl}(T(S_\hat{x}))$ is compact, then

1. $P_S(\hat{x})$ is nonempty, closed, and convex,
2. $T(P_S(\hat{x})) \subseteq P_S(\hat{x})$, and
3. $F(T) \cap P_S(\hat{x}) \neq \emptyset$.

Using Theorem 1.2, he also obtained the following interesting result.

**THEOREM 1.3.** Let $I$ and $T$ be self-mappings of $X$ with $\hat{x} \in F(I) \cap F(T)$ and $S \in \mathcal{S}_0$ such that $T(S_\hat{x}) \subseteq I(S) \subseteq S$. Suppose that $I$ is linear and nonexpansive on $S_\hat{x}$, $\|x - \hat{x}\| = \|x - \hat{x}\|$ for all $x \in S$, $I$ and $T$ are commuting on $S_\hat{x}$, $T$ is $I$-nonexpansive on $S_\hat{x} \cup \{\hat{x}\}$, and one of the following two conditions is satisfied:

1. $\text{cl}(I(S_\hat{x}))$ is compact.
2. $\text{cl}(T(S_\hat{x}))$ is compact and $T$ is linear on $S_\hat{x}$.

Then

1. $P_S(\hat{x})$ is nonempty, closed, and convex,
2. $T(P_S(\hat{x})) \subseteq I(P_S(\hat{x})) \subseteq P_S(\hat{x})$, and
3. $F(I) \cap F(T) \cap P_S(\hat{x}) \neq \emptyset$.

We observe that the proof of Theorem 1.3 relies heavily on commutativity of $T$ and $I$. Naturally, one may raise the following question: Does the above theorem remain valid for a class of noncommuting maps? In this short note, we give a partial answer to this question. Thus we extend most of the known results to a class of noncommuting maps (e.g., [1, 4, 14, 15]).

Up to the present, inadequate efforts have been made in this direction. To the best of our knowledge, there have appeared a few articles (see, e.g., [10, 11]) which discuss the existence of invariant approximations for noncommuting mappings. The concept of $R$-subweak commutativity is a useful tool in establishing the existence of invariant approximations for a pair of mappings satisfying nonexpansive type conditions, as compared to other notions of noncommutativity, such as $R$-weak commutativity. This is evident from the proof of Lemma 2.1.
2. MAIN RESULTS

The following lemmas play a crucial role in the sequel.

**Lemma 2.1.** Let $S \subseteq X$ be closed and $T$ and $I$ self-mappings of $S$ such that $T(S) \subseteq I(S)$. If $cl(T(S))$ is complete, $T$ is $I$-contraction and continuous, and $T$, $I$ are $R$-weakly commuting, then $F(T) \cap F(I)$ is a singleton.

**Proof.** Let $x_0 \in S$. Since $T(S) \subseteq I(S)$, we can define a sequence $\{x_n\}$ in $S$ by $Ix_n = Tx_{n-1}$ for $n \geq 1$. Then

$$
\|Ix_{n+1} - Ix_n\| = \|Tx_n - Tx_{n-1}\| \leq k\|Ix_n - Ix_{n-1}\|
$$

for some $k \in [0, 1)$. This implies that $\{Ix_n\}$ is a Cauchy sequence in $S$ and so the sequence $\{Tx_n\}$ is also Cauchy. Thus $Tx_n \to y \in S$. Consequently, $Ix_n \to y$. Following the proof of Pant [7, Theorem 1], we conclude that $F(T) \cap F(I)$ is a singleton.

**Lemma 2.2.** Let $S \subseteq X$ be closed and $T$ and $I$ self-mappings of $S$ such that $T(S) \subseteq I(S)$. Suppose $T$ is $I$-nonexpansive and continuous, $I$ is linear, and $p \in F(I)$. If $S$ is $p$-starshaped, $cl(T(S))$ is compact, and $T$, $I$ are $R$-subweakly commuting, then $F(T) \cap F(I) \neq \emptyset$.

**Proof.** Define for each $n \geq 1$, a mapping $T_n$ by $T_n x = k_n Tx + (1 - k_n)p$, where $\{k_n\}$ is a sequence with $0 < k_n < 1$ such that $k_n \to 1$ as $n \to \infty$. Then $T_n$ is a self-mapping of $S$ such that $T_n(S) \subseteq I(S)$ for each $n$. From the $R$-subweak commutativity of $T$, $I$ and the linearity of $I$, it follows that

$$
\|T_nIx - IT_nx\| = k_n\|Tnx - IT_nx\| \leq k_nR\|(k_nTx + (1 - k_n)p) -Ix\|
$$

$$
= k_nR\|T_nx - Ix\|
$$

for all $x \in S$. Thus $T_n$ and $I$ are $k_nR$-weakly commuting. Also

$$
\|T_nx - T_ny\| = k_n\|Tx - Ty\| \leq k_n\|Ix - Iy\|
$$

for all $x, y \in S$. Lemma 2.1 further implies that $F(T_n) \cap F(I) = \{x_n\}$ for each $n$. Since $cl(T(S))$ is compact and $S$ is closed, there exists a subsequence $\{x_{m}\}$ of $\{x_n\}$ such that $x_m \to x_0 \in S$ as $m \to \infty$. By the continuity of $T$, we have $x_0 \in F(T)$. Since $T(S) \subseteq I(S)$, it follows that $x_0 = Tx_0 = Iy$ for some $y \in S$. Moreover,

$$
\|Tx_m - Ty\| \leq \|Ix_m - Iy\| = \|x_m - x_0\|.
$$
Taking the limit as \( m \to \infty \) yields \( Tx_0 = Ty \). Thus \( x_0 = Tx_0 = Ty = Iy \).

Since \( T \) and \( I \) are \( R \)-subweakly commuting, it follows that
\[
\|Tx_0 - Ix_0\| = \|Ty - ITy\| \leq R\|Ty - Iy\| = 0.
\]

Hence \( x_0 \in F(T) \cap F(I) \).

We are now in a position to provide a partial answer to the above question.

**THEOREM 2.3.** Let \( I \) and \( T \) be self-mappings of \( X \) with \( \hat{x} \in F(I) \cap F(T) \) and \( S \subseteq \mathcal{S}_0 \) such that \( T(S_{\hat{x}}) \subseteq I(S) \subseteq S \). Suppose that \( I \) is linear and nonexpansive on \( S_{\hat{x}} \), \( \|Ix - \hat{x}\| = \|x - \hat{x}\| \) for all \( x \in S \), \( I \) and \( T \) are \( R \)-subweakly commuting on \( S_{\hat{x}} \), \( T \) is \( I \)-nonexpansive on \( S_{\hat{x}} \cup \{\hat{x}\} \), and \( \text{cl}(I(S_{\hat{x}})) \) is compact. Then

(i) \( P_S(\hat{x}) \) is nonempty, closed, and convex,

(ii) \( T(P_S(\hat{x})) \subseteq I(P_S(\hat{x})) \subseteq P_S(\hat{x}) \), and

(iii) \( F(I) \cap F(T) \cap P_S(\hat{x}) \neq \emptyset \).

**Proof.** We follow the arguments used in [1]. Since \( I \) is nonexpansive on \( S_{\hat{x}} \cup \{\hat{x}\} \), (i) follows from Theorem 1.2. Also, we have \( I(P_S(\hat{x})) \subseteq P_S(\hat{x}) \) (again by Theorem 1.2). Let \( y \in T(P_S(\hat{x})) \). Since \( T(S_{\hat{x}}) \subseteq I(S) \) and \( P_S(\hat{x}) \subseteq S_{\hat{x}} \), there exist \( z \in P_S(\hat{x}) \) and \( x_1 \in S \) such that \( y = Tz =Ix_1 \). Further, since \( T \) is \( I \)-nonexpansive on \( S_{\hat{x}} \cup \{\hat{x}\} \) and \( \|Ix - \hat{x}\| = \|x - \hat{x}\| \) for all \( x \in S \), it follows that
\[
\|Ix_1 - \hat{x}\| = \|Tz - T\hat{x}\| \leq \|z - \hat{x}\| = d(\hat{x}, S).
\]

Thus \( x_1 \in C_S(\hat{x}) = P_S(\hat{x}) \) and so (ii) holds.

Clearly, by Theorem 1.2, \( F(I) \cap P_S(\hat{x}) \neq \emptyset \) and so there exists \( p \in P_S(\hat{x}) \) such that \( p \in F(I) \). Hence (iii) follows immediately from Lemma 2.2.

The following theorem contains Theorem 1.3(b) as a special case.

**THEOREM 2.4.** Let \( I \) and \( T \) be self-mappings of \( X \) with \( \hat{x} \in F(I) \cap F(T) \) and \( S \subseteq \mathcal{S}_0 \) such that \( T(S_{\hat{x}}) \subseteq I(S) \subseteq S \). Suppose that \( I \) is linear and nonexpansive on \( S_{\hat{x}} \), \( \|Ix - \hat{x}\| = \|x - \hat{x}\| \) for all \( x \in S \), \( I \) and \( T \) are commuting on \( S_{\hat{x}} \), \( T \) is \( I \)-nonexpansive on \( S_{\hat{x}} \cup \{\hat{x}\} \), and \( \text{cl}(T(S_{\hat{x}})) \) is compact and convex. Then

(i) \( P_S(\hat{x}) \) is nonempty, closed, and convex,

(ii) \( T(P_S(\hat{x})) \subseteq I(P_S(\hat{x})) \subseteq P_S(\hat{x}) \), and

(iii) \( F(I) \cap F(T) \cap P_S(\hat{x}) \neq \emptyset \).
Proof. We prove only (iii). Let $D_0 = \text{clco}(T(P_s(\hat{x})))$. Then $D_0$ is convex and compact. Consequently, $D_1 = I(D_0)$ is convex and compact. Also

$I(D_0) \subset \text{clco}(IT(P_s(\hat{x}))) = \text{clco}(TI(P_s(\hat{x}))) \subset \text{clco}(T(P_s(\hat{x}))) = D_0$.

This shows that $I(D_1) \subset D_1$. By Theorem 4 of [4], $I$ have a fixed point $q \in D_1 \subset P_s(\hat{x})$. Hence (iii) follows from Lemma 2.2.

Remark. (1) In Theorem 2.3, $cl(I(S))$ is convex because $S_\delta \in S$ and $I$ is linear.

(2) In Theorem 2.4, the assumption “$cl(T(S))$ is convex” may be replaced by any condition that guarantees that $\text{clco}(T(P_s(\hat{x})))$ is compact. For example, if $X$ is a Banach space then $\text{clco}(T(P_s(\hat{x})))$ is compact by Mazur’s theorem [3, p. 416]. In this case, Theorem 2.4 provides a positive answer to Al-Thagafi’s open question [1, p. 323]. For details, we refer the reader to [9].

Question. Does Theorem 2.4 hold if the commutativity is replaced by $R$-subweak commutativity?

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