Journal of Pure and Applied Algebra 93 (1994) 231–249
North-Holland

Some consequences of a Fatou property of the tropical semiring

Daniel Krob

LITP, Université Paris 7, 75251 Paris Cedex 05, France

Communicated by M. Barr
Received 6 August 1992
Revised 15 March 1993

Abstract

Krob, D., Some consequences of a Fatou property of the tropical semiring, Journal of Pure and Applied Algebra 93 (1994) 241–249.

We show that the equatorial semiring $\mathcal{Z}_{\min} = (\mathbb{Z} \cup \{+\infty\}, \min, +)$ is a Fatou extension of the tropical semiring $\mathcal{M} = (\mathbb{N} \cup \{+\infty\}, \min, +)$. This property allows us to give partial decidability results for the equality problem for rational series with multiplicities in the tropical semiring. We also deduce from it the decidability of the limitedness problem for the equatorial semiring, solving therefore a question of I. Simon.

1. Introduction

The tropical semiring is the semiring denoted by $\mathcal{M}$ which has support $\mathbb{N} \cup \{+\infty\}$ and operations $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$. It is currently used in the context of cost minimization in operations research. However, it appeared that $\mathcal{M}$ plays in fact an important role in several problems concerning rational languages (see [8] for a survey of the tropical semiring theory and of its applications). For instance, Simon showed that the finite power property for recognizable languages can be reduced to the limitedness problem for the tropical semiring (cf. [8]). In the same way, Simon used the tropical semiring to study the non-deterministic complexity of a usual finite automaton (cf. [9]).

An important problem in the tropical semiring theory was to see if it is possible to decide whether two $\mathcal{M}$-rational series are equal or not. Recently

Correspondence to: D. Krob, LITP, Université Paris 7, 2, place Jussieu, 75251 Paris Cedex 05, France. Email: dk@mustang.ibp.fr.

SSDI 0022-4049(93)E0107-F

we solved this question by proving that this equality problem was undecidable (cf. [4,5]). Our proof was strongly based on the introduction of the equatorial semiring $\mathcal{Z}_{\min}$ which is just the extension of $\mathcal{M}$ to arbitrary integers. Indeed it can be shown that the equality problem for $\mathcal{Z}_{\min}$ is undecidable and that the above two decidability problems are equivalent (cf. [4,5]).

Hence the question remains to see if partial decidability results can be given for special kinds of equality problems in $\mathcal{M}$ or in $\mathcal{Z}_{\min}$. This is one of the purposes of this paper. Indeed we show that the equality problem is decidable for $\mathcal{M}$-rational or $\mathcal{Z}_{\min}$-rational series which are both min and max recognizable. This gives in particular a new class of $\mathcal{M}$-rational series for which the equality problem is decidable and which contains several classes for which this problem was known to be decidable.

It is also worth noticing that the above decidability result in fact highly depends on a Fatou property of $\mathcal{M}$. Indeed we can show that $\mathcal{Z}_{\min}$ is an effective Fatou extension of $\mathcal{M}$. This allows us to show that it is decidable whether a $\mathcal{Z}_{\min}$-rational series belongs to $\mathcal{M}\langle\langle A\rangle\rangle$ and this last result is the fundamental tool in the above decidability result. It is also interesting to observe that our study points out a great lack of symmetry between min and max. For instance, if $\mathcal{N}$ and $\mathcal{Z}_{\max}$ denote respectively the semirings $\mathbb{N} \cup \{-\infty\}$ and $\mathbb{Z} \cup \{-\infty\}$ equipped with max and $+$ as sum and product, it is not true anymore that $\mathcal{Z}_{\max}$ is a Fatou extension of $\mathcal{N}$.

On another hand, it appears that the above effective Fatou property has other applications. Indeed, using again this result, it can be shown that the limitedness problem for the equatorial semiring is decidable. This was also an open question of I. Simon (see [8]). We devote the last section of this paper to this result.

2. Preliminaries

2.1. *Min and max semirings*

The *equatorial semiring* is the commutative semiring denoted by $\mathcal{Z}_{\min}$ which has $\mathbb{Z} \cup \{+\infty\}$ as support, whose addition $\oplus$ is defined by $a \oplus b = \min\{a, b\}$ and whose product $\otimes$ is given by $a \otimes b = a + b$. The operations of $\mathbb{Z}$ are extended to $\mathcal{Z}_{\min}$ in the usual natural way and the units for $\oplus$ and $\otimes$ are respectively $+\infty$ and 0. The *tropical semiring* is the subsemiring of $\mathcal{Z}_{\min}$ denoted by $\mathcal{M}$ which has $\mathbb{N} \cup \{+\infty\}$ as support. Let us also introduce the semiring $\mathcal{Z}_{\max}$ which is the semiring whose support is $\mathbb{Z} \cup \{-\infty\}$, whose addition, denoted also $\oplus$, is given by $a \oplus b = \max\{a, b\}$ and whose product $\otimes$ is defined by $a \otimes b = a + b$. We can now also consider the *polar semiring* $\mathcal{N}$ which is the subsemiring of $\mathcal{Z}_{\max}$ which has $\mathbb{N} \cup \{-\infty\}$ as support. It is interesting to notice that $\mathcal{Z}_{\min}$ and $\mathcal{Z}_{\max}$ are isomorphic, an effective isomorphism being obtained by the mapping $x \to -x$ from $\mathcal{Z}_{\min}$ into $\mathcal{Z}_{\max}$. Observe that this isomorphism maps the subsemiring $\mathcal{M}^-$ of $\mathcal{Z}_{\min}$ whose support is $\mathbb{Z}^- \cup \{+\infty\}$ onto $\mathcal{N}$ and that therefore $\mathcal{M}^-$ is clearly effectively isomorphic to $\mathcal{N}$.

2.2. *Rational and recognizable series*

We refer to [1], [2], [6] or [7] for all details concerning series, rational series and recognizable series with multiplicities in an arbitrary semiring K. We will denote here by $K\langle\langle A\rangle\rangle$ the K-algebra of *series* over A with multiplicities in K. Let us recall that an element S of $K\langle\langle A\rangle\rangle$ is a formal sum of the form

$$S = \sum_{w\in A^*} (S|w)w \tag{1}$$

where $(S|w) \in K$ denotes the coefficient of the series S on $w \in A^*$. The sum and the multiplication by an element of K are defined componentwise on $K\langle\langle A\rangle\rangle$. The product is defined on $K\langle\langle A\rangle\rangle$ by the usual Cauchy rule. We can also define the star S^* of every proper series[1] S of $K\langle\langle A\rangle\rangle$ by the relation

$$S^* = \sum_{w\in A^*} \Big(\sum_{k=0}^{+\infty} \sum_{w_1\ldots w_k = w} (S|w_1)\ldots(S|w_k)\Big)w.$$

The K-algebra of *K-rational series* is then the smallest sub-K-algebra of $K\langle\langle A\rangle\rangle$, denoted by $K\mathrm{Rat}(A)$, that contains all letters $a \in A$ and that is stable by the star operation (defined on proper series).

Let us also recall that a *K-representation* of order n of a free monoid A^* is just a monoid morphism μ from A^* into the monoid of square matrices of order n with entries in K. Note therefore that a K-representation of order n of A^* is completely defined by the images $(\mu(a))_{a\in A}$ of all letters $a \in A$. We will therefore often identify in the sequel a K-representation μ with the matrix family $(\mu(a))_{a\in A}$.

A *K-automaton* of order n is then a triple (I, μ, T) where μ is a K-representation of order n of A^* and where I and T are respectively a row and a column vector of order n with entries in K (see [1], [2], [6] or [7] for more details). Note that one can always graphically represent any K-automaton $\mathcal{A} = (I, \mu, T)$ of order n by a graph $\mathcal{G}(\mathcal{A})$ defined as follows (see also Fig. 1):

- the set of vertices of $\mathcal{G}(\mathcal{A})$ is $[1, n]$,
- for every letter $a \in A$ and every pair (i, j) of vertices in $[1, n]$, there is an oriented edge in $\mathcal{G}(\mathcal{A})$ labelled by the pair $\mu(a)_{i,j}\, a \in K \times A$ going from i to j,
- for every vertex $i \in [1, n]$, there is an input-arrow labeled by $I_i \in K$ that points onto the vertex i,

[1] A series S of $K\langle\langle A\rangle\rangle$ is said to be proper iff its constant term $(S|1)$ is zero.

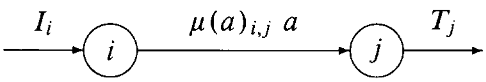

Fig. 1.

- for every vertex $i \in [1, n]$, there is an output-arrow labeled by $T_i \in K$ which is issued of the vertex i.

When an element $\mu(a)_{i,j}$, I_i or T_i is equal to 0_K, one do not usually write the corresponding edge or arrow in the graph $\mathcal{G}(\mathcal{A})$. The family $(\mu(a))_{a\in A}$ appears now in this graphical representation as the family of transition matrices of the automaton $\mathcal{A}$. Note finally that one obtains the usual notion of automaton when K is equal to the boolean semiring $\mathcal{B} = \{0, 1\}$ where $1 + 1 = 1$.

A series S of $K\langle\langle A\rangle\rangle$ is then said to be *K-recognizable* iff there exists a K-automaton $\mathcal{A} = (I, \mu, T)$ such that $(S|w) = I\,\mu(w)\,T$ for every $w \in A^*$. In the previous case, we say that the series S is recognized by $\mathcal{A}$. Note finally that the Kleene–Schützenberger theorem claims that a series of $K\langle\langle A\rangle\rangle$ is K-recognizable iff it is K-rational. We will often use this result in the sequel without mentioning it explicitly.

For every series S of $K\langle\langle A\rangle\rangle$ and for every word $u \in A^*$, the *left derivative* of S with respect to u is the series $u^{-1}S$ of $K\langle\langle A\rangle\rangle$ defined by $(u^{-1}S|w) = (S|uw)$ for every $w \in A^*$. We recall that we have

$$S = (S|1) + \sum_{a\in A} a(a^{-1}S) \tag{2}$$

for every $S \in K\langle\langle A\rangle\rangle$. We also recall that a series S of $K\langle\langle A\rangle\rangle$ is recognizable iff there exists a K-semimodule $M = \langle S_1, \ldots, S_n\rangle$ of finite type that contains S and that is stable by left derivative[2] (cf. Proposition 1.5.1 of [1] or Theorem 2.3.1 of [7] for more details). This is exactly equivalent to asking that there exists a finite family $(S_i)_{i=1,n}$ of series of $K\langle\langle A\rangle\rangle$ such that

$$a^{-1}S_i = \sum_{p=1}^{n} k_p(a, i)S_p \tag{3}$$

for every $a \in A$ and every $i \in [1, n]$, where every $k_p(a, i)$ is in K, and such that S is a K-linear combination of the series S_i.

2.3. Other definitions concerning series

Let us now explicit some notions concerning K-rational series that will be useful. First we denote by $\underline{L}$ the *characteristic* series of any language $L \subset A^*$

[2] This last property is expressed more simply by saying that the K-semimodule M is stable.

which is the series of $K\langle\langle A\rangle\rangle$ defined by

$$\forall\, w \in A^*, \quad (\underline{L}|w) = \begin{cases} 1_K & \text{if } w \in L, \\ 0_K & \text{if } w \notin L. \end{cases} \tag{4}$$

$\underline{L}$ is always a K-rational series when L is a rational language (cf. [1] or [7] for instance). When K is equal to $\mathcal{Z}_{\min}$ or to $\mathcal{M}$, one should observe that $\underline{L}$ is then defined by $(\underline{L}|w) = 0$ if $w \in L$ and $(\underline{L}|w) = +\infty$ if $w \notin L$.

We also denote here as usually by $S \odot T$ the *Hadamard product* of two series S, T of $K\langle\langle A\rangle\rangle$. It is the series defined by $(S \odot T|w) = (S|w)(T|w)$ for every $w \in A^*$. We recall that $S \odot T$ is a K-rational series when S and T are K-rational series (see [1] or [7] for more details). The *constant K-rational* series whose every coefficient is equal to some element $k \in K$, will be always denoted by k.

Let K be a semiring and let L be a subsemiring of K. Then K is said to be a *Fatou extension* of L iff every K-rational series which belongs to $L\langle A\rangle$ is in fact an L-rational series. We refer the reader to [1] or [7] for more details on Fatou extensions.

Let K be a semiring, let m be an element of K and let S be a series in $K\langle\langle A\rangle\rangle$. Then the *$m$-support* of S is exactly the language $\{w \in A^* \mid (S|w) = m\}$. In the same way, the support of S is the language $\{w \in A^* \mid (S|w) \neq 0_K\}$.

2.4. Series over min or max semirings

When K is equal to $\mathcal{Z}_{\min}$ or $\mathcal{M}$, the K-recognizable series can be interpreted in a very simple way. Indeed let $\mathcal{A} = (I, \mu, T)$ be a K-automaton with $K = \mathcal{Z}_{\min}$ or $K = \mathcal{M}$. The element of $\mathbb{Z} \cup \{+\infty\}$ or $\mathbb{N} \cup \{+\infty\}$ that labels every edge of the associated graph $\mathcal{G}(\mathcal{A})$ is then said to be the *cost* of the corresponding edge. We also associate with every path p going from some vertex i to some other vertex j in $\mathcal{G}(\mathcal{A})$ its cost which is the sum of I_i, of all costs of the edges used by p and of T_j. For every word $w \in A^*$, the coefficient $(S|w) = I\mu(w)T$ of the series S recognized by $\mathcal{A}$ is then equal to the minimal cost of all paths indexed by w in $\mathcal{G}(\mathcal{A})$. The same kind of interpretation (replacing "minimal" by "maximal") holds also when K is equal to $\mathcal{Z}_{\max}$ or to $\mathcal{N}$.

Let now K be a semiring. Then a series $S \in K\langle\langle A\rangle\rangle$ is said to be *limited* iff the set $\{(S|w) \mid w \in A^*\}$ is a finite subset of K. Let now K be equal to $\mathcal{M}$ or $\mathcal{N}$. Then we say that a K-rational series $S \in K\mathrm{Rat}(A)$ is *$\mathbb{N}$-limited* iff the set $\{(S|w) \mid w \in A^*\}$ is a finite subset of $\mathbb{N}$. We denote by $\mathbb{N}\text{-}\mathrm{Lim}(K)$ the set of $\mathbb{N}$-limited K-rational series.

Let us also define two important classes of $\mathcal{Z}_{\min}$ and $\mathcal{M}$-rational series. A series S of $\mathbb{Z}\langle\langle A\rangle\rangle$ (resp. of $\mathbb{N}\langle\langle A\rangle\rangle$) is said to be a *$\mathbb{Z}$-MinMax series* (resp. an *$\mathbb{N}$-MinMax series*) iff S belongs both to $\mathcal{Z}_{\min}\mathrm{Rat}(A)$ and $\mathcal{Z}_{\max}\mathrm{Rat}(A)$ (resp. $\mathcal{M}\mathrm{Rat}(A)$ and $\mathcal{N}\mathrm{Rat}(A)$). We denote respectively by $\mathbb{Z}\text{-}\mathrm{MinMax}(A)$ and by

$\mathbb{N}$-MinMax(A) these two classes which are therefore equal to

$$\mathbb{N}\text{-MinMax}(A) = \mathcal{M}\mathrm{Rat}(A) \cap \mathcal{N}\mathrm{Rat}(A) \subset \mathbb{N}\langle\langle A\rangle\rangle,$$

$$\mathbb{Z}\text{-MinMax}(A) = \mathcal{Z}_{\min}\mathrm{Rat}(A) \cap \mathcal{Z}_{\max}\mathrm{Rat}(A) \subset \mathbb{Z}\langle\langle A\rangle\rangle.$$

Let us finally equip $\mathcal{Z}_{\min}$, $\mathcal{Z}_{\max}$, $\mathcal{M}$ and $\mathcal{N}$ with the order induced by the usual order of $\mathbb{Z}$, extended in a natural way by setting $-\infty < z < +\infty$ for every $z \in \mathbb{Z}$. This order is also transported to series by defining for every series S and T with multiplicities in these semirings, $S \leq T$ if and only if $(S|w) \leq (T|w)$ for every $w \in A^*$.

3. Structure of *m*-supports

We give in this section some basic results concerning the structure of m-supports of $\mathcal{Z}_{\min}$, $\mathcal{M}$ and $\mathcal{N}$-recognizable series that will be useful in the sequel. Let us first recall the following result which is folklore (it is in fact a general property of positive semirings). We refer to [4] for the proof.

Proposition 3.1. *Let S be a rational series of $\mathcal{Z}_{\min}\mathrm{Rat}(A)$ (resp. of $\mathcal{Z}_{\max}\mathrm{Rat}(A)$). Then the set*

$$\{w \in A^* \mid (S|w) = +\infty\} \quad (\text{resp. } \{w \in A^* \mid (S|w) = -\infty\})$$

is a constructable rational language of Rat(A). □

The following proposition gives the structure of m-supports of series with multiplicities in $\mathcal{M}$ or in $\mathcal{N}$.

Proposition 3.2. *Let S be a rational series of $\mathcal{M}\mathrm{Rat}(A)$ (resp. of $\mathcal{N}\mathrm{Rat}(A)$) and let m be an element of $\mathbb{N} \cup \{+\infty\}$ (resp. of $\mathbb{N} \cup \{-\infty\}$). Then the set*

$$\{w \in A^* \mid (S|w) = m\}$$

is a constructable rational language of Rat(A).

Proof. We will argue only in the tropical semiring case, since the other case is similar. Note first that the result follows from Proposition 3.1 when $m = +\infty$. Let us suppose now that $m \in \mathbb{N}$. Then we can write

$$\begin{aligned}&\{w \in A^* \mid (S|w) = m\}\\ &\quad = \{w \in A^* \mid (S|w) \geq m\} - \{w \in A^* \mid (S|w) \geq m + 1\}.\end{aligned}$$

It follows from this equality that it suffices to prove that the set

$$\{w \in A^* \mid (S|w) \geq m\}$$

is a constructable rational language in order to conclude. Let us now introduce the semiring $\mathcal{M}_m$ whose support is $\{0, \ldots, m\}$ and which is equipped with min as addition and with a product defined by $x \otimes y = x + y$ if $x + y \leq m$ and $x \otimes y = m$ if $x + y \geq m$. We can clearly consider the projection π_m from $\mathcal{M}$ into $\mathcal{M}_m$ defined by $\pi_m(x) = x$ if $x \leq m$ and $\pi_m(x) = m$ if $x \geq m$. We also denote by π_m its natural extension as a morphism of algebras from $\mathcal{M}\langle\langle A \rangle\rangle$ into $\mathcal{M}_m\langle\langle A \rangle\rangle$. With these notations, we now have

$$\{w \in A^* \mid (S|w) \geq m\} = \{w \in A^* \mid (\pi_m(S)|w) = m\}.$$

Since recognizable series are effectively preserved by morphisms (cf. [2,7] for instance), $\pi_m(S)$ is an $\mathcal{M}_m$-recognizable series. Hence we are now led to prove that the set

$$\{w \in A^* \mid (T|w) = m\}$$

is a constructable rational language for every $\mathcal{M}_m$-rational series T. However, since $\mathcal{M}_m$ is a finite semiring, such a series T has a finite number N of left derivatives (cf. Corollary 2.3.2 of [7] for instance). Hence there exists an effective finite family $(w_i)_{i=1,N}$ of words of A^* such that $\mathcal{T} = (T_i)_{i=1,N}$ is exactly equal to the finite family of the left derivatives of T, if we set $T_i = w_i^{-1}T$ for every $i \in [1, N]$. We can always suppose that $w_1 = 1$ in such a way that $T_1 = T$. Since $\mathcal{T}$ is by construction stable by left derivative, there exists for every $i \in [1, N]$ and for every $a \in A$ another (effectively computable) integer $i(a) \in [1, N]$ such that $a^{-1}T_i = T_{i(a)}$. Applying now relation (2) to every T_i, we get the relations

$$\forall i \in [1, N], \quad T_i = (T_i|1) + \sum_{a \in A} aT_{i(a)},$$

which can also be expressed as a linear system

$$\mathcal{T} = (\mathcal{T}|1) + \Big(\sum_{a \in A} aM_a\Big)\mathcal{T},$$

where $\mathcal{T}$ denotes the column vector $(T_i)_{i=1,N}$, $(\mathcal{T}|1)$ denotes the column vector $(T_i|1)_{i=1,N} \in \mathcal{M}_m^N$ and M_a denotes the square matrix of order N defined by $(M_a)_{i,i(a)} = 0$ and by $(M_a)_{i,j} = m$ if $j \neq i(a)$ for every $i \in [1, N]$. Using now Propositions 7.6.1 and 7.6.2 of [2], it follows easily from our computations that we can write

$$T = (0\,m\,m\,\ldots\,m)\Big(\sum_{a \in A} M_a a\Big)^* (T_i|1)_{i=1,N}$$

in an effective way. Observe that all the entries of any matrix M_a belong to $\{0, m\}$. Hence, since the subsemiring $\{0, m\}$ of $\mathcal{M}_m$ is isomorphic to the boolean semiring, it follows easily from this last relation that

$$T = \min_{i=1,N} ((T_i|1) \otimes \underline{I_i}),$$

where I_i denotes for every $i \in [1, N]$ the rational language recognized by the boolean (up to the previous isomorphism) automaton (I, μ, T) with $I = (0\, m \, \ldots \, m)$, where the representation μ is defined by the matrices $(M_a)_{a \in A}$ and where T is the column vector whose every entry is equal to m, excepted the ith which is 0. Note that $(I_i)_{i=1,N}$ is a partition of A^* that consists in constructable rational languages. Hence it follows that

$$\{w \in A^* \mid (T|w) = m\} = \bigcup_{(T_i|1)=m} I_i.$$

Our proposition follows now immediately from this last relation. □

Notes. (1) Using the effective isomorphism between $\mathcal{N}$ and $\mathcal{M}^-$ mentioned in the preliminary section, it follows easily from the above proposition that m-supports are also effectively rational for every $\mathcal{M}^-$-rational series.

(2) It follows easily from Propositions 3.1 and 3.2 that every limited $\mathcal{M}$-rational series S can be effectively written as follows

$$\text{(Lim)} \quad S = \min_{i=1,M} (a_i \otimes \underline{I_i}),$$

where M is a positive integer, $(I_i)_{i=1,M}$ is a family of rational languages and $(a_i)_{i=1,M}$ is a family of positive integers. When S is a $\mathbb{N}$-limited series, relation (Lim) is in particular still valid but the family $(I_i)_{i=1,M}$ is now a partition of A^*. In this last case, relation (Lim) holds also when we replace min by max in it. It follows that every $\mathbb{N}$-limited $\mathcal{M}$-rational series is also a ($\mathbb{N}$-limited) $\mathcal{N}$-rational series and it can be easily checked that the converse also holds. Thus we proved that

$$\mathbb{N}\text{-Lim}(\mathcal{M}) = \mathbb{N}\text{-Lim}(\mathcal{N}) \subset \mathbb{N}\text{-MinMax}(A) \subset \mathbb{N}\langle\langle A \rangle\rangle.$$

All the above inclusions are in fact strict (see the notes following Corollary 5.4).

(3) Note finally that the situation in the tropical semiring is completely different from the situation in $\mathbb{N}$ for instance. Indeed m-supports are not rational in general for arbitrary $\mathbb{N}$-rational series (cf. [1]).

The following result follows immediately from the two previous propositions.

Corollary 3.3. *Let m be a constant in $\mathbb{N} \cup \{+\infty\}$ (resp. in $\mathbb{N} \cup \{-\infty\}$). Then it is decidable whether a series in $\mathcal{M}\mathrm{Rat}(A)$ (resp. in $\mathcal{N}\mathrm{Rat}(A)$) is equal to* *the constant series m or has a coefficient (or an infinite number of coefficients) equal to m.* $\square$

Note. The same result holds obviously for $\mathcal{M}^-$-rational series.

4. Fatou properties

The purpose of this section is to show a lack of symmetry between min and max with respect to Fatou properties. Indeed we can prove that $\mathcal{Z}_{\min}$ is a Fatou extension of $\mathcal{M}$, but that $\mathcal{Z}_{\max}$ is not a Fatou extension of $\mathcal{N}$. We will see later that this result is strongly connected with decidability results for special sorts of equality problems.

4.1. A Fatou property of $\mathcal{M}$

Proposition 4.1. *$\mathcal{Z}_{\min}$ is a Fatou extension of $\mathcal{M}$.*

Proof. Let S be a $\mathcal{Z}_{\min}$-recognizable series of $\mathcal{M}\langle\langle A\rangle\rangle$. To prove our proposition, we must show that S is $\mathcal{M}$-recognizable. Note first that we can suppose that S is not equal to the constant series $+\infty$ since there is nothing to prove in this case. Then, according to Proposition 1.5.1 of [1] or to Theorem 2.3.1 of [7], there exists a stable $\mathcal{Z}_{\min}$-module $Z = \langle S_1, \ldots, S_n\rangle$ of finite type which contains S. Since $S \neq +\infty$, we can always suppose that S_i is not equal to the constant series $+\infty$ for every $i \in [1, n]$. Hence there exists a family $(k_i)_{i=1,n}$ of elements of $\mathcal{Z}$ such that

$$S = \bigoplus_{k=1}^{n} k_i \otimes S_i = \min_{k=1,n} (k_i \otimes S_i). \tag{5}$$

Since $S \in \mathcal{M}\langle\langle A\rangle\rangle$, it follows from relation (5) that every series $k_i \otimes S_i$ belongs to $\mathcal{M}\langle\langle A\rangle\rangle$. Let us now introduce for every $i \in [1, n]$ the integer

$$l_i = \min\{l \in \mathbb{Z} \mid l \otimes S_i \in \mathcal{M}\langle\langle A\rangle\rangle\} \tag{6}$$

which exists since the set considered in (6) is non-empty and has obviously a least element. Then $T_i = l_i \otimes S_i \in \mathcal{M}\langle\langle A\rangle\rangle$ and $l_i \leq k_i$ for every $i \in [1, n]$. Let now M be the $\mathcal{M}$-module generated by the series $T_1, \ldots, T_n$. It follows from relation (5) that

$$S = (k_1 - l_1) \otimes T_1 \oplus \cdots \oplus (k_n - l_n) \otimes T_n \in M = \langle T_1, \ldots, T_n\rangle.$$

To end our proof, it suffices to show that M is stable. Let then a be in A and k be in $[1,n]$. Since Z is stable, there exists a family of integers $(z_i(a,k))_{i=1,n} \in \mathbb{Z}^n$ such that

$$a^{-1}T_k = l_k \otimes a^{-1}S_k = l_k \otimes \Big(\bigoplus_{i=1}^{n} z_i(a,k) \otimes S_i\Big)$$
$$= \min_{i=1,n}\big((l_k + z_i(a,k)) \otimes S_i\big) \in \mathcal{M}\langle\langle A\rangle\rangle.$$

Hence every series $(l_k + z_i(a,k)) \otimes S_i$ belongs to $\mathcal{M}\langle\langle A\rangle\rangle$ and according to definition (6), it follows that $l_k + z_i(a,k) \geq l_i$ for every $i \in [1,n]$. Hence we have

$$a^{-1}T_k = (l_k + z_1(a,k) - l_1) \otimes T_1$$
$$\oplus \cdots \oplus (l_k + z_n(a,k) - l_n) \otimes T_n \in M.$$

This proves that M is stable. Thus S is $\mathcal{M}$-recognizable and this ends our proof. $\square$

It should be noticed that $\mathcal{Z}_{\min}$ is in fact a *constructive* Fatou extension of $\mathcal{M}$ as shows the following corollary of the previous result.

Proposition 4.2. *$\mathcal{Z}_{\min}$ is a constructive Fatou extension of $\mathcal{M}$.*

Proof. Let $S \in \mathcal{M}\langle\langle A\rangle\rangle$ be a $\mathcal{Z}_{\min}$-recognizable series given by a $\mathcal{Z}_{\min}$-automaton. Let us then show that the proof of Proposition 4.1 enables us to effectively construct an $\mathcal{M}$-automaton recognizing S. Note first that it is decidable according to Proposition 1 whether $S = +\infty$ and that it is immediate to conclude in this case. Hence we can suppose that S is not equal to the constant series $+\infty$. Then, according to the proof of Proposition 1.5.1 of [1] or of Theorem 2.3.1 of [7], we can construct explicitly the series S_i and the elements k_i of $\mathcal{Z}_{\min}$ involved in relation (5) of the proof of Proposition 4.1. Moreover, according again to Proposition 3.1, we can also effectively suppose that every S_i is distinct from the constant series $+\infty$ and it is possible to choose for every $i \in [1,n]$ a coefficient $A_i \neq +\infty$ of S_i. Using the notations of the proof of Proposition 4.1, we get immediately that $-A_i - 1 \leq l_i \leq k_i$ for every $i \in [1,n]$.

Let us now define the finite set $P = [-A_1 - 1, k_1] \times \cdots \times [-A_n - 1, k_n]$. For every $p = (p_i)_{i=1,n} \in P$, let us then define the series $T_i(p) = p_i \otimes S_i$ where $i \in [1,n]$. Arguing as in the proof of Proposition 4.1, it is easy to show that

$$S = \bigoplus_{i=1}^{n} (k_i - p_i) \otimes T_i(p) \tag{7}$$

and that we have for every $a \in A$ and every $i \in [1,n]$,

$$a^{-1}T_i(p) = \bigoplus_{j=1}^{n} (p_i + z_j(a,i) - p_j) \otimes T_k(p). \tag{8}$$

Since $(T_i(p)|1) = p_i + (S_i|1)$ for every $i \in [1,n]$, it is now easy to deduce from the proofs of Proposition 1.5.1 of [1] or of Theorem 2.3.1 of [7] and from relations (7) and (8) that S is recognized by the $\mathcal{Z}_{\min}$-automaton

$$\begin{aligned}\mathcal{A}(p) = (&(k_i - p_i)_{i=1,n}; ((p_i + z_j(a,i) - p_j)_{1\le i,j\le n})_{a\in A};\\ &(p_i + (S_i|1))_{i=1,n})\end{aligned}$$

for every $p = (p_i)_{i=1,n} \in P$. Moreover, the proof of Proposition 4.1 shows that there exists $p \in P$ such that $\mathcal{A}(p)$ is an $\mathcal{M}$-automaton. Therefore it follows that we can effectively find an $\mathcal{M}$-automaton recognizing S by looking at every automaton $\mathcal{A}(p)$. $\square$

The following result is an easy consequence of the previous proposition.

Corollary 4.3. *It is decidable whether a $\mathcal{Z}_{\min}$-recognizable series belongs to $\mathcal{M}\langle\langle A\rangle\rangle$ or not.*

Proof. The corollary follows easily from the proof of Proposition 4.2 which shows that a $\mathcal{Z}_{\min}$-rational series S belongs to $\mathcal{M}\langle\langle A\rangle\rangle$ iff it is a $\mathcal{M}$-rational series and iff there exists $p \in P$ such that $\mathcal{A}(p)$ is an $\mathcal{M}$-automaton. $\square$

4.2. The case of $\mathcal{Z}_{\max}$

We have the following negative result for $\mathcal{Z}_{\max}$. Note that it claims equivalently that $\mathcal{Z}_{\min}$ is not a Fatou extension of $\mathcal{M}^-$.

Proposition 4.4. *$\mathcal{Z}_{\max}$ is not a Fatou extension of $\mathcal{N}$.*

Proof. Let μ and ν be the two $\mathcal{Z}_{\max}$-representations of A^* of order 1 defined by

$$\mu(a) = (1), \quad \mu(b) = (0) \quad \text{and} \quad \nu(a) = (0), \quad \nu(b) = (-1).$$

Using these two representations, it is easy to see that the series

$$r_a = \sum_{w\in(a+b)^*} |w|_a w \quad \text{and} \quad r_b = \sum_{w\in(a+b)^*} -|w|_b w$$

Fig. 2.

are $\mathcal{Z}_{\max}$-rational since they are respectively recognized by the two $\mathcal{Z}_{\max}$-automata of order 1 associated with μ and ν and shown in Fig. 2. Hence the series

$$R = r_a \odot r_b = \sum_{w \in (a+b)^*} (|w|_a - |w|_b) w$$

is $\mathcal{Z}_{\max}$-rational. By symmetry, the series

$$S = \sum_{w \in (a+b)^*} (|w|_b - |w|_a) w$$

is also $\mathcal{Z}_{\max}$-rational. It follows that the series T defined by

$$T = R \oplus S = \max(S, R) = \sum_{w \in (a+b)^*} \big||w|_a - |w|_b\big| \, w$$

is $\mathcal{Z}_{\max}$-rational. Moreover, T belongs to $\mathcal{N}\langle\langle A\rangle\rangle$. But T cannot belong to $\mathcal{N}\mathrm{Rat}(A)$ since it were the case, we would have according to Proposition 3.2

$$\begin{aligned} &\{w \in (a+b)^* \mid (T|w) = 0\} \\ &\quad = \{w \in (a+b)^* \mid |w|_a = |w|_b\} \in \mathrm{Rat}(a,b), \end{aligned}$$

which is clearly not the case. Hence it follows that $\mathcal{Z}_{\max}$ is not a Fatou extension of $\mathcal{N}$. This ends our proof. $\square$

Notes. (1) It follows easily from the above result that Proposition 3.2 cannot be extended to $\mathcal{Z}_{\max}$-rational series or equivalently to $\mathcal{Z}_{\min}$-rational series. Even worse, there exists in fact m-supports of $\mathcal{Z}_{\min}$ or $\mathcal{Z}_{\max}$-rational series which are not recursive (cf. [4,5]). Hence Proposition 3.1 gives really the only kind of m-support which is rational for every $\mathcal{Z}_{\min}$ or $\mathcal{Z}_{\max}$-rational series.

(2) The "negative part" of a $\mathcal{Z}_{\min}$-rational series S, that is to say the series S^- defined by $(S^-|w) = (S|w)$ when $(S|w) \leq 0$ and by $(S^-|w) = 0$ when $(S|w) > 0$, is always $\mathcal{Z}_{\min}$-rational since we clearly have $S^- = S \oplus 0$. On the other hand, we can also define the positive part S^+ of a $\mathcal{Z}_{\min}$-rational series S by setting $(S^+|w) = 0$ when $(S|w) < 0$ and $(S^+|w) = (S|w)$ when $(S|w) \geq 0$. One should notice that S^+ is not in general a $\mathcal{Z}_{\min}$-rational series. Indeed if it was the case, S^+ would be $\mathcal{M}$-rational according to Proposition 4.1.

Hence the 1-support (cf. Section 2.3) of S, which is the 1-support of S^+, would always be a rational language according to Proposition 3.2.

Let us now consider the series R and S constructed in the proof of Proposition 4.4. They are also $\mathcal{Z}_{\min}$-rational series that are recognized by the automata of Fig. 2 interpreted as $\mathcal{Z}_{\min}$-automata. It suffices then to consider the $\mathcal{Z}_{\min}$-rational series $T = 1 \odot \min(R, S)$ in order to conclude to a contradiction since the 1-support of T is the set of words over $\{a, b\}^*$ with the same number of a and of b which is clearly not rational. The same kind of result holds also for $\mathcal{Z}_{\max}$-rational series by interchanging the terms "positive" and "negative".

5. Some partial decidability results for the equality problem in $\mathcal{Z}$

We will give in this section some decidability results for special kinds of $\mathcal{Z}_{\min}$-series. These results can always be translated for $\mathcal{Z}_{\max}$-rational series, using the effective isomorphism mentioned in Section 2.1.

5.1. Constant series

This short section is devoted to the generalization of Corollary 3.3 to $\mathcal{Z}_{\min}$-rational series.

Proposition 5.1. *It is decidable whether a $\mathcal{Z}_{\min}$-rational series is equal to a given constant series $m \in \mathbb{Z} \cup \{+\infty\}$.*

Proof. Let $k \in \mathbb{Z} \cup \{+\infty\}$ be a given constant and let S be a $\mathcal{Z}_{\min}$-rational series. If $k = +\infty$, the decidability of $S = +\infty$ follows easily from Poposition 3.1. Let us now suppose that $k \in \mathbb{Z}$. In this case, the decidability of the problem $S = k$ is clearly equivalent to the decidability of the problem $S \odot (-k) = 0$. Hence we can suppose that $k = 0$. Observe now that $S = 0$ implies obviously that S is an $\mathcal{M}$-rational series. Hence, according to Corollary 4.3, we can effectively reduce the problem $S = 0$ for $\mathcal{Z}_{\min}$-rational series to the same problem for $\mathcal{M}$-rational series. The decidability of this last problem follows now immediately from Corollary 3.3. This ends our proof. $\square$

5.2. Min-max series

Let now S be a series of $\mathcal{Z}_{\min}\langle\langle A \rangle\rangle$ (resp. of $\mathcal{Z}_{\max}\langle\langle A \rangle\rangle$). Let us then denote by $-S$ the series of $\mathcal{Z}_{\max}\langle\langle A \rangle\rangle$ (resp. of $\mathcal{Z}_{\min}\langle\langle A \rangle\rangle$) defined by $(-S|w) = -(S|w)$ for every $w \in A^*$ where we set $-(+\infty) = -\infty$ and $-(-\infty) = +\infty$. This new operation allows to directly connect $\mathcal{Z}_{\min}$-rational series with $\mathcal{Z}_{\max}$-rational series.

Proposition 5.2. *Let S be a $\mathcal{Z}_{\min}$-rational (resp. $\mathcal{Z}_{\max}$-rational) series. Then the series $-S$ is a $\mathcal{Z}_{\max}$-rational (resp. $\mathcal{Z}_{\min}$-rational) series.*

Proof. It is an obvious consequence of the fact that $-S$ is just the image of S in the isomorphism (which is an involution) $z \to -z$ from $\mathcal{Z}_{\min}$ into $\mathcal{Z}_{\max}$. □

We are now able to give our main decidability result.

Proposition 5.3. *Let S be a $\mathcal{Z}_{\min}$-rational series and let T be a $\mathcal{Z}_{\max}$-rational series. Then it is decidable whether $S = T$ or whether $S \geq T$.*

Proof. Let S be a $\mathcal{Z}_{\min}$-rational series and let T be a $\mathcal{Z}_{\max}$-rational series. Let now I be the $+\infty$-support of S and J be the $-\infty$-support of T (cf. Section 2.3). According to Proposition 3.1, these two languages are constructable rational languages.

Let us now first study the equality problem $S = T$. If I or J are non-empty, then S is clearly distinct from T. Hence we can suppose that $I = J = \emptyset$. In this case, the problem $S = T$ can be reduced to the problem $S \odot (-T) = 0$. But it follows immediately from Propositions 5.2 and 5.1 that this last problem is decidable.

Let us finally study the inequality problem $S \geq T$. In this case, let us introduce the two respectively $\mathcal{Z}_{\min}$-rational and $\mathcal{Z}_{\max}$-rational series

$$\tilde{S} = (S \odot \underline{I \cup J}) \min \underline{A^* - (I \cup J)} \quad \text{and}$$
$$\tilde{T} = (T \odot \underline{I \cup J}) \max \underline{A^* - (I \cup J)}.$$

We clearly have

$$(\tilde{S}|w) = \begin{cases} (S|w) & \text{if } w \notin I \cup J \\ 0 & \text{if } w \in I \cup J \end{cases}$$

and

$$(\tilde{T}|w) = \begin{cases} (T|w) & \text{if } w \notin I \cup J \\ 0 & \text{if } w \in I \cup J \end{cases}$$

for every $w \in A^*$. Hence the inequality problem $S \geq T$ is now clearly equivalent to the inequality problem $\tilde{S} \geq \tilde{T}$ which is itself equivalent to the inequality problem $\tilde{S} \odot (-\tilde{T}) \geq 0$. But it follows immediately from Proposition 5.2 and Corollary 4.3 that this last problem is decidable. This ends therefore the proof of our proposition. □

Note. Note that the problem $S \leq 0$ is undecidable when S is a $\mathcal{Z}_{\min}$-rational series (see [4] or [5]). It follows that the problem $S \leq T$ is undecidable when S is a $\mathcal{Z}_{\min}$-rational series and T a $\mathcal{Z}_{\max}$-rational series.

Corollary 5.4. *The equality problem is decidable for the* $\mathbb{Z}$-MinMax *series of* $\mathcal{Z}_{\min}\langle\langle A\rangle\rangle$ *and for the* $\mathbb{N}$-MinMax *series of* $\mathcal{M}\langle\langle A\rangle\rangle$. □

Notes. (1) In a note following Proposition 3.2, we showed that the $\mathbb{N}$-MinMax series contain the $\mathbb{N}$-limited series. It can also be shown that every one-letter $\mathcal{M}$-rational series in $\mathbb{N}\langle A\rangle$ is a $\mathbb{N}$-MinMax series. Hence the above corollary contains in particular the two corresponding known decidability results.

(2) The restriction to series which have not $+\infty$ as coefficient is not very important in our context since it can easily be shown using Proposition 3.1 that the decidability of $S = T$ can always be reduced for every $\mathcal{Z}_{\min}$-rational series S and T to the decidability of $\tilde{S} = \tilde{T}$ where $(\tilde{S}|w)$ is equal to $(S|w)$ if $(S|w) \neq +\infty$ and to 0 if not.

(3) It follows clearly from the above result and from the undecidability of the equality problem for $\mathcal{M}$-rational series that there exists $\mathcal{M}$-rational series which are not $\mathcal{N}$-rational series. In fact using all the results of this paper and of [4] or [5], it is not difficult to deduce such a series from the series HD involved in the proof of the main theorem of [4] or [5]. However, the problem remains to characterize $\mathbb{N}$-MinMax series and to construct less tricky examples.

6. Decidability of the limitedness problem for $\mathcal{Z}_{\min}$

In this section, we show how to apply our results for solving the limitedness problem for $\mathcal{Z}_{\min}$-rational series. This will answer to a question of Simon (cf. [8]).

6.1. Negatively limited $\mathcal{Z}_{\min}$-rational series

Let us first give the following definition.

Definition 6.1. A $\mathcal{Z}_{\min}$-rational series S is said to be a *negatively limited* series iff there exists an integer $K \in \mathbb{Z}$ such that $(S|w) \geq K$ for every $w \in A^*$.

Note. A $\mathcal{Z}_{\min}$-rational series S is clearly not a negatively limited series if and only if there exists an infinite sequence $(w_i)_{i\in\mathbb{N}}$ of words of A^* such that

$$(S|w_0) > (S|w_1) > \cdots > (S|w_n) > \cdots.$$

Let us now give some important lemmas.

Lemma 6.2. *Let S be a proper $\mathcal{Z}_{\min}$-rational series. Then the two following conditions are equivalent*:

(1) *S^* is negatively limited.*

(2) $\forall\, w \in A^*$, $(S|w) \geq 0$.

Proof. Let us suppose first that the second condition holds. Then we clearly have $(S^*|w) \geq 0$ for every $w \in A^*$ and hence S^* is obviously negatively limited.

Let us now consider a proper $\mathcal{Z}_{\min}$-rational series S such that S^* is negatively limited and let us suppose that the second condition is not satisfied. Then, since S is a proper series, there exists a non-empty word u such that $(S|u) < 0$. Therefore we clearly have

$$(S^*|u^n) \leq \underbrace{(S|u) + \cdots + (S|u)}_{n \text{ times}} = n(S|u) \xrightarrow{n \to +\infty} -\infty$$

for every $n \in \mathbb{N} - \{0\}$. It follows clearly from this last inequality that S^* cannot be negatively limited. Hence we obtained a contradiction and the second condition must hold. This ends the proof of our lemma. □

Lemma 6.3. *Let S, T be two $\mathcal{Z}_{\min}$-rational series. Then the two following assertions are equivalent:*

(1) *$S \oplus T$ is negatively limited.*

(2) *S and T are negatively limited.*

Proof. Let us first suppose that assertion (1) holds. Then there exists $K \in \mathbb{Z}$ such that $\min((S|w), (T|w)) = (S \oplus T|w) \geq K$ for every $w \in A^*$. It follows clearly from this last inequality that $(S|w) \geq K$ and $(T|w) \geq K$ for every $w \in A^*$. Hence S and T are negatively limited.

Let us suppose conversely that S and T are negatively limited. Then there exists $M, N \in \mathbb{Z}$ such that $(S|w) \geq M$ and $(T|w) \geq N$ for every $w \in A^*$. It follows clearly from these two inequalities that we have $(S \oplus T|w) = \min((S|w), (T|w)) \geq \min(M, N)$ for every $w \in A^*$. Hence $S \oplus T$ is a negatively limited series. This ends therefore our proof. □

Lemma 6.4. *Let S be a $\mathcal{Z}_{\min}$-rational series and let $k \in \mathbb{Z}$. Then the two following assertions are equivalent:*

(1) *$k \otimes S$ is negatively limited.*

(2) *S is negatively limited.*

Proof. The first assertion is clearly equivalent to the fact that there exists some constant $K \in \mathbb{Z}$ such that $(S|w) \geq K - k$ for every word $w \in A^*$ and this last condition is also obviously equivalent to the fact that S is negatively limited. This proves our lemma. □

Lemma 6.5. *Let S, T be two $\mathcal{Z}_{\min}$-rational series which are different from the constant series $+\infty$. Then the two following assertions are equivalent:*

(1) *$S \otimes T$ is negatively limited.*
(2) *S and T are negatively limited.*

Proof. Let us suppose first that $S \otimes T$ is negatively limited. Then there exists a constant $K \in \mathbb{Z}$ such that

$$(S \otimes T|w) = \min_{uv=w} ((S|u) + (T|v)) \geq K$$

holds for every $w \in A^*$. Let us suppose for instance that S is not negatively limited. Then there would exist an infinite sequence $(u_i)_{i \in \mathbb{N}}$ of words such that $(S|u_i) \xrightarrow{i \to +\infty} -\infty$. Since $T \neq +\infty$, there exists a word v_0 such that $(T|v_0) \neq +\infty$. It follows that we have

$$(S \otimes T|u_i v_0) \leq (S|u_i) + (T|v_0) \xrightarrow{i \to +\infty} -\infty$$

for every $i \in \mathbb{N}$. Hence $(S \otimes T|u_i v_0) \xrightarrow{i \to +\infty} -\infty$. This contradiction shows that S must be negatively limited. Arguing in the same way, we can also show that T must be negatively limited.

Let us now suppose that both S and T are negatively limited. Hence there exists two constants $M, N \in \mathbb{Z}$ such that $(S|w) \geq M$ and $(T|w) \geq N$ for every $w \in A^*$. It follows clearly from these two inequalities that

$$(S \otimes T|w) = \min_{uv=w} ((S|u) + (T|v)) \geq M + N.$$

Hence $S \otimes T$ is negatively limited. This ends therefore our proof. □

We can now give our first decidability result for this section.

Proposition 6.6. *It is decidable whether a $\mathcal{Z}_{\min}$-rational series S is negatively limited or not.*

Proof. Let us first notice that a proper $\mathcal{Z}_{\min}$-rational expression—i.e. a $\mathcal{Z}_{\min}$-rational expression which involves only proper series enclosed under a star—is always effectively computable from a $\mathcal{Z}_{\min}$-automaton recognizing a $\mathcal{Z}_{\min}$-rational series S. Note also that conversely a $\mathcal{Z}_{\min}$-automaton is always effectively computable from a $\mathcal{Z}_{\min}$-rational expression. We can then consider the algorithm shown in Fig. 3 that works with a proper $\mathcal{Z}_{\min}$-rational expression E as input and answers "true" or "false", where Positive(S) denotes an algorithm that answers "true" or "false" according to the fact that S is a positive series or not. Note that such an algorithm exists according to Corollary 4.3. Observe also that the tests $S = +\infty$ or $T = +\infty$ can effectively be made according to Proposition 5.1.

```
NegLimited(E)
Begin
    Case E = a ∈ A:
      "true"; End ;
    Case E = S*:
      Positive(S); End;
    Case E = S ⊕ T:
      NegLimited(S) and NegLimited(T); End;
    Case E = k ⊗ S:
      If k = +∞
        Then "true"
        Else NegLimited(S);
      End;
    Case E = S ⊗ T:
      If S = +∞ or T = +∞
        Then"true";
        Else NegLimited(S) and NegLimited(T);
      End;
End;
```

Fig. 3.

Notice now that the above algorithm ends clearly always. Moreover, all the previous lemmas shows that it answers "true" if and only if the $\mathcal{Z}_{\min}$-rational series represented by the $\mathcal{Z}_{\min}$-rational proper expression E is negatively limited. This ends therefore our proof. □

Note. When S is a negatively limited $\mathcal{Z}_{\min}$-rational series, it is not difficult using Lemmas 6.2–6.5 to adapt the previous algorithm in order to effectively compute a constant $K \in \mathbb{Z}$ such that $(S|w) \geq K$ for every $w \in A^*$.

6.2. Limited $\mathcal{Z}_{\min}$-rational series

We can now give our decidability result for the limitedness problem for $\mathcal{Z}_{\min}$-rational series.

Proposition 6.7. *It is decidable whether a $\mathcal{Z}_{\min}$-rational series is limited.*

Proof. Let S be a $\mathcal{Z}_{\min}$-rational series. If S is limited, S is also in particular negatively limited. But, according to Proposition 6.6, this last property can be decided. Hence the decidability of limitedness for $\mathcal{Z}_{\min}$-rational series can be reduced to the decidability of the same problem for negatively limited series. Let now S be such a series. According to the note following Proposition 6.6, we can effectively compute $K \in \mathbb{Z}$ such that $(S|w) \geq K$ for every $w \in A^*$. Let us then consider the $\mathcal{Z}_{\min}$-rational series $T = S \odot (-K)$. Then it is easy to see that S is limited if and only if T is limited. But this last series is positive. Hence according to Proposition 4.2, T is an effective $\mathcal{M}$-rational series. Our result follows now from the decidability of the limitedness problem for $\mathcal{M}$-rational series (cf. [3]). This ends the proof of our proposition. $\square$

Notes. (1) It follows clearly from the above proof that a $\mathcal{Z}_{\min}$-rational series S is limited iff there exists a $\mathcal{M}$-rational limited series T and a constant $k \in \mathbb{Z}$ such that $S = k \otimes T$.

(2) An easy consequence of the above proposition is that it is decidable for every $\mathcal{M}$-rational series S whether there exists a constant $M \in \mathbb{N}$ such that

$$\forall\, w \in A^*, \quad k|w| - M \leq (S|w) \leq k|w| + M,$$

where k denotes a given positive integer.

Acknowledgement

I want to thank here Professor I. Simon for the personal papers he communicated me and for several e-conversations we had. I must also thank the anonymous referee for several valuable comments that allow to highly improve the final version of this paper. Special thanks are finally due to the LACIM—and especially to P. Leroux and C. Reutenauer—for their kind support during the preparation of this paper.

References

[1] J. Berstel and C. Reutenauer, Rational Series and Their Languages (Springer, Berlin, 1986).
[2] S. Eilenberg, Automata, Languages and Machines, Vol. A (Academic Press, New York, 1974).
[3] K. Hashiguchi, Limitedness theorem on finite automata with distances functions, J. Comput. System. Sci. 24 (2) (1982) 233–244.
[4] D. Krob, The equality problem for rational series with multiplicities in the tropical semiring is undecidable, in: W. Kuich, ed., Proceedings of ICALP'92, Lecture Notes in Computer Science, Vol. 623 (Springer, Berlin.1992) 101–112.
[5] D. Krob, The equality problem for rational series with multiplicities in the tropical semiring is undecidable, LITP Technical Report 92-63, Paris, 1992; Internat. J. Algebra and Comput., to appear.
[6] W. Kuich and A. Salomaa, Semirings, Automata, Languages (Springer, Berlin, 1986).
[7] A. Salomaa and M. Soittola, Automata-Theoretic Aspects of Formal Power Series (Springer, Berlin, 1978).
[8] I. Simon, Recognizable sets with multiplicities in the tropical semiring, in: M.P. Chytil et al., eds., Proceedings of MFCS'88, Lecture Notes in Computer Science, Vol. 324 Springer, Berlin, 1988) 107–120.
[9] I. Simon, The nondeterministic complexity of finite automata, in: M. Lothaire, ed., Mots (Hermès, Paris, 1990) 384–400.