Finite volume methods for convection–diffusion problems

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Abstract

An overview of the nature of convection–diffusion problems and of the use of finite volume methods in their solution is given.

Keywords: Finite volume method; Convection–diffusion problem

1. Convection–diffusion problems

The simplest mathematical model of a convection–diffusion problem is a two-point boundary value problem of the following form:

$$- \varepsilon u''(x) + a(x)u'(x) = f(x) \quad \text{for } 0 < x < 1 \tag{1}$$

with $u(0)$ and $u(1)$ given, where ε is a small positive parameter and a and f are known functions. Here the term u'' corresponds to diffusion, u' represents convection, while f is a driving term. If the coefficient ε of u'' is small compared with the coefficient $a(\cdot)$ of u', then problem (1) is said to be of convection–diffusion type.

Example 1. Suppose that

$$- \varepsilon u''(x) + u'(x) = 1 \quad \text{for } 0 < x < 1 \tag{2}$$

with $u(0) = u(1) = 0$ and $0 < \varepsilon \ll 1$.

Clearly

$$u(x) = x + \frac{e^{-1/\varepsilon} - e^{-(1-x)/\varepsilon}}{1 - e^{-1/\varepsilon}} = x - e^{-(1-x)/\varepsilon} + O(e^{-1/\varepsilon}). \tag{3}$$

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These three terms have the following interpretations. The first is the solution of the *initial value problem*

$$v'(x) = 1 \quad \text{on } (0, 1) \qquad \text{subject to } v(0) = 0. \tag{4}$$

(This problem is obtained by formally setting ε equal to zero in (2) and taking one of the original boundary conditions.) The second term in (3) has a negligible influence on the solution when x is not near 1 (recall that ε is positive and small). It is essentially a correction to the solution of (4) which is required in order that the other boundary condition $u(1) = 0$ of the original problem be satisfied. The last term in (3) is of negligible size.

Thus from (3) we can see that a graph of $u = u(x)$ will closely approximate the straight line $u = x$ on almost all of $[0, 1)$. When x approaches 1, the graph (while, of course, remaining continuous) suddenly departs from this straight line and plunges downwards to satisfy the condition $u(1) = 0$. We say that the graph has a *boundary layer* at $x = 1$.

We summarize this behaviour as follows. Except on a narrow region near one of the boundaries, the solution of the original boundary value problem closely approximates the solution of an associated initial value problem.

In two dimensions the situation is similar, as we now explain.

Example 2. Consider the second-order problem

$$-\varepsilon \Delta u + \nabla \cdot (au) = f \quad \text{on } \Omega, \tag{5}$$

where Ω is the unit square in $\mathbb{R}^2$ and ε is a small positive parameter. We assume that $a = (a_1, a_2)$, with a_1 and a_2 smooth functions that are positive on $\bar{\Omega}$, and that $f \in L^2(\Omega)$.

We impose the following fairly general boundary conditions. Write Γ for the boundary of Ω and n for the outward-pointing unit normal to Γ. Set

$$\Gamma_- = \{p \in \Gamma : a \cdot n < 0 \text{ at } p\} \quad \text{and} \quad \Gamma_+ = \{p \in \Gamma : a \cdot n > 0 \text{ at } p\};$$

thus Γ_- consists of the two sides of Ω that lie on the coordinate axes. Following the terminology of fluid dynamics, we call Γ_- the inflow boundary and Γ_+ the outflow boundary. Suppose that Γ is the union of two disjoint sets, Γ_D and Γ_N, where $\Gamma_- \subseteq \Gamma_D$. Our boundary conditions are

$$u = g \text{ on } \Gamma_D \quad \text{and} \quad \frac{\partial u}{\partial n} = h \text{ on } \Gamma_N,$$

where g and h are given well-behaved functions.

These hypotheses together imply that (5) has a unique solution $u(x, y)$.

It is well-known (see, e.g. [7]) that, except near $\Gamma_+ \cap \Gamma_D$, the solution u of (5) is equal (modulo a little diffusion) to the solution v of the first-order hyperbolic problem

$$\nabla \cdot (av) = f \quad \text{on } \Omega \tag{6}$$

with initial data $v = 0$ on Γ_-. We call (6) the reduced problem; it is obtained by formally setting $\varepsilon = 0$ in (5).

layer in solution u

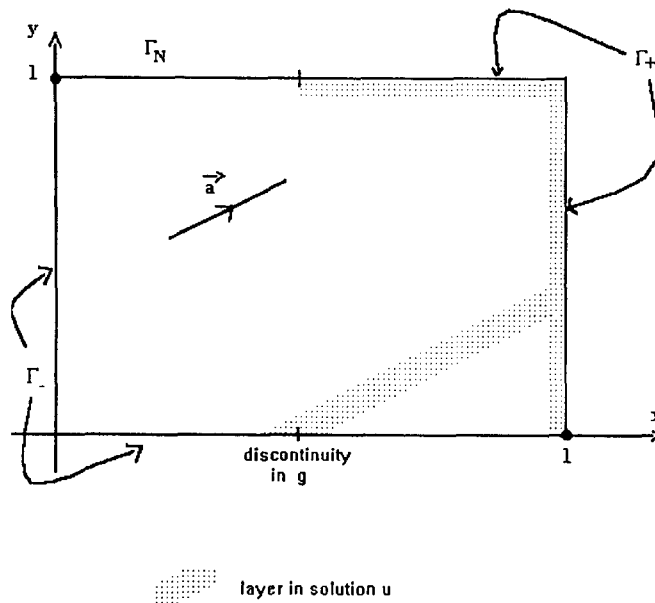Fig. 1.

At $\Gamma_+\cap\Gamma_D$ the function u will have a boundary layer; that is, close to $\Gamma_+\cap\Gamma_D$ the solution u changes rapidly in order to satisfy the boundary condition $u = g$ on Γ_D. This is analogous to the behaviour of (3). The situation is represented schematically in Fig. 1. We think of the solution as propagating or flowing from the inflow boundary Γ_- across Ω until it nears the Dirichlet outflow boundary, where it changes rapidly.

A discontinuity in the data on Γ_- will in general cause an *internal layer* in the solution; this is a narrow region, centred on one of the characteristic traces of the first-order hyperbolic problem (6) — i.e., following the direction of flow — in which the solution changes rapidly. That is, the solution of the reduced problem (6) is smoothed in this region by the diffusion term present in (5).

For further examples in two dimensions (and some graphs of solutions) see [2, p. 188].

2. Finite volume methods

The name "finite volume method" seems to date from 1979, but versions of these methods appear as early as the 1960s. They are widely used by engineers; for example, in the aerospace industry, they are the preferred method for the numerical simulation of complex problems such as modelling the airflow over an entire aircraft. Useful introductions to the technique are given in [1, 9].

The basic strategy of all finite volume methods is to write the differential equation in conservative form, integrate it over small regions (called "cells" or "finite volumes"), and convert each such integral into an integral over the boundary of the cell by means of Gauss's theorem.

More precisely, if the differential equation is

$$\frac{\partial f(u)}{\partial x} + \frac{\partial g(u)}{\partial y} = f \quad \text{on } \Omega \subset \mathbb{R}^2,$$

we partition Ω into a finite number of cells by means of a grid, then for each cell C set

$$\iint_C f = \iint_C \left[\frac{\partial f(u^h)}{\partial x} + \frac{\partial g(u^h)}{\partial y} \right] = \int_{\partial C} [f(u^h)\,dy - g(u^h)\,dx]. \tag{7}$$

Here u is the unknown solution and we use u^h to denote our discrete computed solution; h is a parameter which corresponds to the mesh diameter. In two dimensions the cells are usually rectangles or at least quadrilaterals.

Note how the conservative form of the original equation lends itself to an easy application of Gauss's theorem. Many physical models (e.g., the Navier–Stokes equations) are of this form.

The finite volume method has certain advantages over competing methods such as finite element and finite difference methods: it is conceptually simple, yields a conservative discretization, is easily used on nonuniform grids, and facilitates multilevel solution (see [9]). Some disadvantages are that its precise implementation is not universally agreed (see below), and there is a lack of underlying theory in the case of convection–diffusion problems (consequently it is sometimes unclear how to proceed in complex situations).

The two main variants of the finite volume method compute an approximation of the solution of the differential equation at the nodes of the grid. In the *cell-vertex* finite volume method, these nodes are the vertices of the cells, while in the *cell-centre* variant the nodes are (approximately) at the cell centres, as we explain below.

3. Finite volume methods applied to convection–diffusion problems

Consider the problem

$$\begin{aligned} - \varepsilon \Delta u + \nabla \cdot (au) &= f \quad \text{on } \Omega \equiv (0,1)^2, \\ u &= 0 \quad \text{on } \Gamma \equiv \partial \Omega. \end{aligned} \tag{8}$$

As in (5), we assume that $a = (a_1, a_2)$ with a_1 and a_2 smooth positive functions, that $f \in L^2(\Omega)$, and that ε is a small positive parameter.

Place a rectangular grid on $\bar{\Omega}$ with sides parallel to the coordinate axes. Let h denote the mesh diameter. Our computed approximate solution u^h is piecewise bilinear on this grid. We enforce the boundary condition by requiring $u^h = 0$ on Γ.

We begin with the *cell-centre method*. In order to have the nodes of the grid at the centres of cells, we introduce a new rectangular grid (the dashed lines in Fig. 2) whose nodes are the cell centres of the original grid. We integrate over each new-grid mesh rectangle C (these are the cells of (7)):

$$\iint_C f = \iint_C (-\varepsilon \Delta u^h + \nabla \cdot (au^h)) = \int_{\partial C} \left(-\varepsilon \frac{\partial u^h}{\partial n} + u^h (a \cdot n) \right), \tag{9}$$

old grid: solid lines

new grid: dashed lines
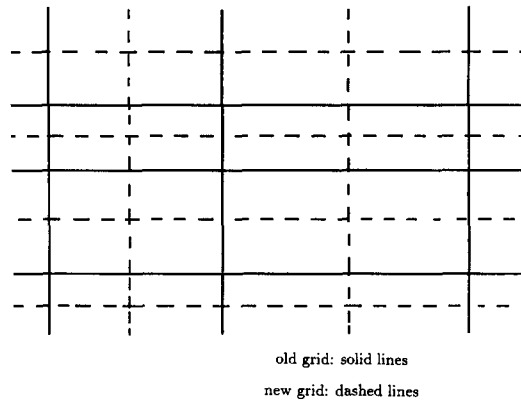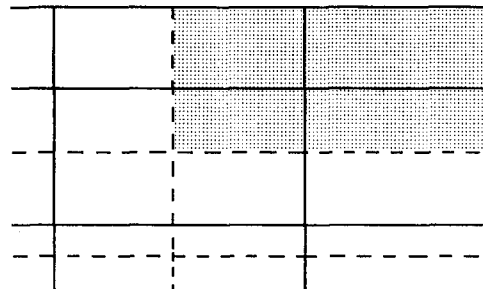
Fig. 2.

Fig. 3.

where we have now used n as the outward-pointing unit normal to the boundary ∂C of C. Note that the partial derivatives in the final integral of (9) are well-defined almost everywhere on each ∂C, as ∂C lies almost entirely in the interiors of the original rectangles.

The cells of the new mesh omit a narrow strip bordering on Γ. If we wish to retain the conservative property of the basic method, we can choose enlarged cells near Γ in order to cover all of Ω, as in Fig. 3.

If Neumann boundary conditions were present in this problem, we would of course no longer specify u^h on that part of Γ, and would approximate the Neumann condition by means of finite differencing (see [9, p. 23]).

We now move on to the *cell-vertex method* for (8). Here no new grid is introduced; we simply integrate over each cell formed by the original grid. Thus once again we have (9), where the C's are now different from before. There is an immediate question of interpretation: since each ∂C is precisely where the piecewise bilinear solution u^h may lose smoothness, it is not clear what is meant by

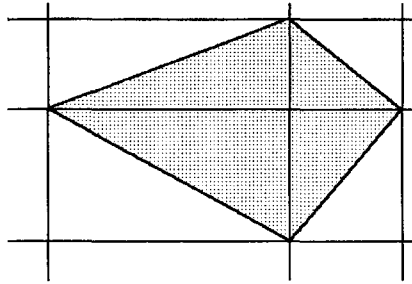$$\int_{\partial C} -\varepsilon \frac{\partial u^h}{\partial n}. \tag{10}$$

Fig. 4.

This integral is handled by some form of averaging. Mackenzie and Morton [3] assign a value to, say, $\partial u^h/\partial x$ at each node by setting it equal to the average of $\mathrm{div}(u^h, 0)$ over the diamond-shaped region in Fig. 4 (which surrounds the node in question), then applying Gauss's theorem to this integral of $\mathrm{div}(u^h, 0)$, which expresses the average in terms of values of u^h. Armed with nodal values of the derivatives, we then interpolate to them by piecewise bilinears and can now evaluate (10) easily. In [6], the value of (10) on each face of ∂C is got by taking the obvious finite difference approximation over the parallel faces of the two neighbouring cells. While this approximation is simpler than that of [3] in the present rectangular case, it is not clear how it should be generalized to distorted quadrilateral meshes.

4. Cell-centre or cell-vertex?

Arguments rage to and fro over this vexed question. We shall not take a stand, but merely compare the two finite volume variants in several ways.

For brevity in this section, we write CCM for the cell-centre method and CVM for the cell-vertex approach. We shall consider the solution of problem (8) by piecewise bilinear functions on a rectangular grid as in Section 3.

(i) *Does the number of unknowns equal the number of equations?* For the CCM, yes — as each unknown is surrounded by a cell which yields a single equation (9). For the CVM, possibly no! Suppose that we use the grid of Fig. 5. Its 12 cells will yield 12 equations of the form (9). As $u = 0$ on Γ, the only unknown values are those at the six interior nodes of the grid. It turns out [3, 5] that the correct procedure in this problem is to discard equations corresponding to those cells that are adjacent to the outflow boundary Γ_+. This leaves six cells — the correct number.

(ii) *How compact is the scheme?* This is an important consideration when it comes to solving the linear system of discrete equations. For the convection term, the CVM uses four points (the vertices of a single cell) while the CCM uses nine (since each new cell in (9) overlaps with four original mesh rectangles, all of whose nodes play a part). On the other hand, for the diffusion term, the CVM needs 12 points (whether we follow [3] or [6], each of the four nodes of the cell uses its four immediate neighbours) while the CCM works with the same nine points that were used for the convection term.
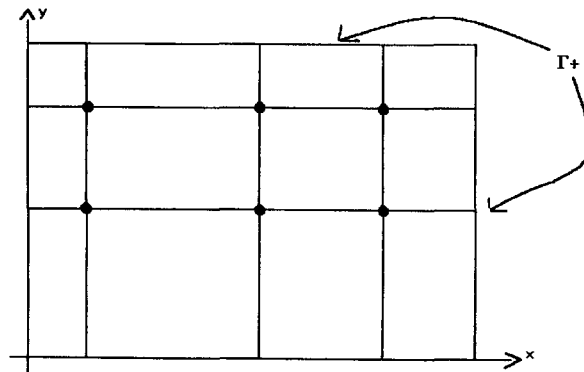
Fig. 5.

While the CVM uses more points overall, iterative solvers may mimic the behaviour of u in seeking to follow the convection-driven flow, and then the more compact convection stencil of the CVM is helpful.

(iii) *Sensitivity to mesh deformation.* As adaptive procedures are often used in convection–diffusion problems, deformed meshes are commonplace. Numerical evidence and some truncation error analysis indicate that the CVM is more accurate than the CCM on certain nonuniform meshes.

(iv) *Parasitic modes and stability.* Both CCM and CVM solutions may exhibit nonphysical oscillations, which may be triggered by discontinuities in the boundary conditions or changes in the local mesh size. Consider the case of pure convection (i.e., $\varepsilon = 0$), with $\boldsymbol{a} = (1, 1)$, on a uniform square mesh. Then the null space of the discrete CVM operator includes the unwelcome *chequerboard mode*; this oscillation takes the value $(-1)^{i+j}$ at the node (x_i, y_j). The CCM is worse; it has the same chequerboard mode and in addition the two *washboard modes* given, respectively, by $(-1)^i$ and $(-1)^j$ at (x_i, y_j).

In convection–diffusion problems, the same parasitic modes may occur because the diffusion coefficient ε is so small.

Instability must always be guarded against in convection–diffusion problems; conventional numerical techniques, whether finite volume or other methods, will often be unstable if applied carelessly. Finite difference and finite element methods achieve stability by some form of *upwinding*; this means that difference approximations of the first-order derivatives are not symmetric but are biased in the upstream direction (i.e., in the direction of $-\boldsymbol{a}$). See [7] for a detailed discussion of this issue.

When we discarded certain cells from the CVM in (i) above, we were retaining the cells that lay immediately upstream of the nodes where the value of u^h was unknown. This strategy is how the CVM upwinds. With the CCM, the cells are centred on the nodes, and upwinding is accomplished by choosing a quadrature rule that evaluates the convection term in ∂C integral of (9) in terms of the value of u^h at some upstream point(s). This is very similar to finite difference upwinding. See [1, 9].

(v) *Analysis.* There is extensive numerical experience with finite volume methods, but the analysis of such methods for convection–diffusion problems (i.e., proofs of realistic error estimates)

has lagged far behind. Much less is known than for finite difference or finite element methods (see [7]).

In [5] the authors analyse the CVM in one dimension and, by reformulating it as a Petrov–Galerkin finite element method, prove an energy norm error bound. This result is extended to certain problems in two dimensions in [6]. Shishkin [8] has proved a pointwise convergence result for the CCM on a special mesh in two dimensions.

References

[1] C. Hirsch, *Numerical Computation of Internal and External Flows, Vol. 1* (Wiley, Chichester, 1988).
[2] C. Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method* (Cambridge Univ. Press, Cambridge, 1987).
[3] J.A. Mackenzie and K.W. Morton, Finite volume solutions of convection–diffusion test problems, *Math. Comp.* **60** (1992) 189–220.
[4] K.W. Morton, Finite volume methods and their analysis, in: J.R. Whiteman, Ed., *The Mathematics of Finite Elements and Applications VII, Proc. MAFELAP 1900, Brunel*, 1990 (Academic Press, London, 1991) 189–214.
[5] K.W. Morton and M. Stynes, An analysis of the cell vertex scheme, *Math. Model. Anal. Numér.* **28** (1994) 699–724.
[6] K.W. Morton, M. Stynes and E. Süli, Analysis of cell-vertex methods for convection–diffusion problems, Oxford Univ. Computing Laboratory Report, in preparation.
[7] H.-G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations* (Springer, Berlin, to appear).
[8] G.I. Shishkin, Methods of constructing grid approximations for singularly perturbed boundary-value problems. Condensing grid methods, *Russian J. Numer. Anal. Math. Modelling* **7** (1992) 537–562.
[9] P. Wesseling, *An Introduction to Multigrid Methods* (Wiley, Chichester, 1992).