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Near-Circularity for the Rational Zolotarev Problem in the Complex Plane

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We consider the *rational Zolotarev problem*

$$\min_{r \in \mathbf{R}_{ll}} \frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|}$$

for compact sets $E, F \subseteq \mathbb{C}$, where $\mathbf{R}_{ll}$ denotes the set of all rational functions of degree $\leqslant l$. This problem is of importance, e.g., for the determination of optimal parameters for the method of alternating directions (ADI method) which is used for the iterative solution of large linear systems. For E and F being real intervals, the solution of this problem was given explicitly in terms of elliptic functions by Zolotarev in the last century. For complex domains, however, little is known as yet about this problem. In this paper, after reviewing some results on the asymptotic behavior, we prove a result which is similar to the near-circularity criterion as it is well known in connection to classical approximation by polynomials or rational functions. If we assume that both sets E and F are bounded by Jordan curves, this gives us a lower bound for the minimal value in the rational Zolotarev problem. Moreover, we derive upper bounds for the modulus of the doubly connected region $D := \overline{\mathbb{C}} \backslash (E \cup F)$ and show how the near-circularity criterion can be used for the construction of the rational minimal solutions for small degrees.

1. INTRODUCTION

Various applications, e.g., the optimization of the ADI (alternating direction implicit) iterative method for the solution of large linear systems and the construction of digital filters, lead to the rational minimization problem

$$\min_{r \in \mathbf{R}_{ll}} \frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|}, \tag{1.1}$$

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where E and F are disjoint compact subsets of the complex plane and $\mathbf{R}_{ll}$ denotes the set of all rational functions of degree $\leqslant l$. If E and F are real intervals, the solution of Problem (1.1) has been intensively studied. For $E=[-1, 1]$, $F=(-\infty, -k] \cup [k, \infty)$ with $k>1$, (1.1) is the third of four approximation problems which were solved by Zolotarev in the years 1868, 1877, and 1878 (see, e.g. the review paper of Todd [15]). In this case, the solution can be given explicitly in terms of *elliptic functions* and, using bilinear transformations, also for arbitrary disjoint intervals $E, F \subseteq \mathbb{R}$ (Lebedev [10], Wachspress [18]). Because of the analogy to the problem stated and solved by Zolotarev for the real case, Gonchar [7] suggested that (1.1) be called the *rational Zolotarev problem* (in the complex plane).

From now on, we assume that E and F are both Jordan regions, since this is required in Section 3 for the near-circularity criterion. This is not a severe restriction if we think of applications, e.g., the determination of ADI parameters, where E and F are domains containing the spectra of the matrices in the ADI splitting.

In [7], Gonchar studied the asymptotic behavior of this problem for E and F being disjoint closed subsets of the extended complex plane $\bar{\mathbb{C}}$ such that each one has connected complement. Under these assumptions it can be deduced from Gonchar's result that, for the minimal value in the rational Zolotarev problem (1.1),

$$\sigma_l(E, F) = \min_{r \in \mathbf{R}_{ll}} \frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|},$$

$$\lim_{l \to \infty} (\sigma_l(E, F))^{1/l} = \rho(E, F)^{-1},$$

where $\rho(E, F)$ denotes the *modulus* of the doubly connected region $D := \bar{\mathbb{C}} \setminus (E \cup F)$. Examples of *asymptotically minimal* rational functions, i.e., sequences of rational functions $\{r_l\}_{l \in N}$ with the property

$$\lim_{l \to \infty} \left(\frac{\max_{z \in E} |r_l(z)|}{\min_{z \in F} |r_l(z)|} \right)^{1/l} = \rho(E, F)^{-1},$$

can be constructed by generalizations of the Fejér and Leja points which play an important role in connection with polynomial interpolation in the complex plane (see Gaier [5]). The *generalized Fejér points* are the "doubly connected special case" of a point set introduced by Walsh in [20]. Their construction requires the knowledge of the conformal map Ψ of the annulus $\{w \in \mathbb{C} : 1 < |w| < \rho(E, F)\}$ onto D. In contrast to this, the *generalized Leja points* which were introduced by Bagby [1] in 1969 are defined recursively. Their construction requires only the computation of maximum points on the boundaries ∂E and ∂F which makes them very useful in practice. Another sequence of asymptotically minimal rational functions which is based on the conformal map Ψ is given by the "Faber rationals" of Ganelius [6]. A different approach to the complex rational Zolotarev problem was given by Ellner and Wachspress in [4]. They show that the optimal rational functions for the real case are also optimal for some "elliptic function domains" in the complex plane. The results obtained using these rational functions are often rather promising even though they are, in general, neither optimal nor asymptotically optimal.

It is the main purpose of this paper to state and prove a result which is similar to the *near-circularity criterion*—which is well known in connection with polynomial and rational approximation of analytic functions in the complex plane (see Trefethen [16, 17]). Moreover, we show how this result can be used to solve the rational Zolotarev problem for small degrees l.

In the following section we summarize some known results on the asymptotic behavior of the rational Zolotarev problem and introduce Walsh's point set as generalized Fejér points. Furthermore, we present a technique for the computation of the generalized Leja points for piecewise differentiable boundary curves (using one-dimensional minimization algorithms instead of replacing the boundary by a discrete point set).

Section 3 contains the near-circularity criterion, its proof, and its geometric interpretation. Moreover, we deduce two corollaries on how improved lower bounds for the minimal value in the rational Zolotarev problem can be obtained. This leads to upper bounds for the modulus of the corresponding complementary region D. Finally, in Section 4, the exact minimal solutions for (1.1) are constructed for small degrees of the rational functions and the results are compared with the asymptotically minimal rational functions of higher degree.

2. THE ASYMPTOTICAL BEHAVIOR OF THE RATIONAL ZOLOTAREV PROBLEM

As pointed out in the introduction, it is reasonable to assume that the disjoint compact sets $E, F \subseteq \mathbb{C}$ are bounded by Jordan curves. Thus, the complementary region $D := \bar{\mathbb{C}} \setminus (E \cup F)$ is doubly connected and neither E nor F reduces to a single point. Under these assumptions it is well known that there exists a conformal map Φ of D onto a circular annulus $\{w \in \mathbb{C} : 1 < |w| < \rho(E, F)\}$ (see Henrici [8]). The number $\rho(E, F)$ is uniquely determined and is called the *modulus* of the doubly connected region D.

The main result about the asymptotic behavior of the rational Zolotarev problem is

THEOREM 2.1. *For the minimal value in the rational Zolotarev problem* (1.1),

$$\sigma_l(E, F) = \min_{r \in \mathbf{R}_l} \frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|}, \tag{2.1}$$

there hold

$$\sigma_l(E, F) \geqslant \rho(E, F)^{-l} \tag{2.2}$$

and

$$\lim_{\to \infty} (\sigma_l(E, F))^{1/l} = \rho(E, F)^{-1}. \tag{2.3}$$

The inequality (2.2) can be deduced directly from the results of Gonchar on the rate of growth of rational functions [7] which were proved for more general sets E and F there. To prove (2.3) one has to construct a sequence of rational functions $\{r_l\}_{l \in N}$, $r \in \mathbf{R}_{ll}$ which fulfills the condition

$$\lim_{l \to \infty} \left(\frac{\max_{z \in E} |r_l(z)|}{\min_{z \in F} |r_l(z)|} \right)^{1/l} = \rho(E, F)^{-1}. \tag{2.4}$$

We call sequences of rational functions fulfilling (2.4) *asymptotically minimal* for the rational Zolotarev problem.

We will now present two examples of such asymptotically minimal rational functions. They are both generated by point sets which are generalizations of *uniformly distributed nodes*. The concept of uniformly distributed nodes is important in connection with polynomial interpolation in the complex plane (cf. Gaier [5, Chap. II.2]) and semiiterative methods for the solution of linear systems (cf. Eiermann and Niethammer [3]). We start here with the corresponding generalization of the system of the Fejér nodes.

To define these generalized Fejér nodes, we need the conformal mapping function Ψ of the annulus $\{w \in \mathbb{C} : 1 < |w| < \rho(E, F)\}$ onto the doubly connected region $D = \bar{\mathbb{C}} \backslash (E \cup F)$, i.e., the inverse of the conformal mapping Φ introduced at the beginning of this section. Since ∂E and ∂F are given by Jordan curves, Ψ has a continuous extension onto $\bar{D}$ and we call the points

$$\varphi_j^{(l)} = \Psi(e^{2\pi i j/l}), \qquad \psi_j^{(l)} = \Psi(\rho(E, F)\, e^{2\pi i j/l}), \qquad j = 1, ..., l, \tag{2.5}$$

the lth *generalized Fejér nodes* for the rational Zolotarev problem.

That the rational functions

$$r_l(z) = \prod_{j=1}^{l} \frac{z - \varphi_j^{(l)}}{z - \psi_j^{(l)}}$$

generated by the generalized Fejér nodes of (2.5) form an asymptotically minimal sequence can be deduced from the fact that, for the doubly connected case, they coincide with the points constructed by Walsh in the proof of Theorem 9 in Chapter 8 of [19] (cf. [13, Theorem 2.4]). In [20], Walsh shows the asymptotical minimality of these points.

With the generalized Fejér nodes, rational functions which are asymptotically minimal for the minimization problem (1.1) can be constructed for arbitrary compact sets E and F where the complementary region $D = \bar{\mathbb{C}} \backslash (E \cup F)$ is doubly connected. However, one needs the conformal map Ψ of the annulus $\{w \in \mathbb{C} : 1 < |w| < \rho(E, F)\}$ onto D, which is known explicitly only for very rare special cases. Moreover, the numerical determination of this mapping function is, in general, very expensive, which restricts the use of these points in practice drastically. In particular, for polygonal boundary curves—for example, those the rectangular domains containing the spectra of the operators in the ADI splitting as they are obtained using Bendixson's theorem (cf., e.g., [14, Theorem 6.9.15])—we could make use of the implementation of the Schwarz–Christoffel map for the doubly connected case (cf. Henrici [8, Paragraph 17.5]) as it is described in [2].

Another system of uniformly distributed nodes is given by the Leja nodes. The following generalization for the Leja points to the rational Zolotarev problem is due to Bagby [1].

Given $\varphi_1 \in E$ and $\psi_1 \in F$ arbitrarily, for $l = 1, 2, \ldots$ the new points $\varphi_{l+1} \in E$, $\psi_{l+1} \in F$ are chosen recursively in such a way that with

$$r_l(z) = \prod_{j=1}^{l} \frac{z - \varphi_j}{z - \psi_j}$$

the two conditions

$$\begin{aligned} \max_{z \in E} |r_l(z)| &= |r_l(\varphi_{l+1})| \\ \min_{z \in F} |r_l(z)| &= |r_l(\psi_{l+1})| \end{aligned} \tag{2.6}$$

are fulfilled.

Bagby shows in [1] that the rational functions r_l obtained by this procedure are asymptotically minimal for the rational Zolotarev problem. However, this is still true if we start with a set of points $\varphi_j \in \bar{\mathbb{C}} \backslash F$, $\psi_j \in \bar{\mathbb{C}} \backslash E$, $j = 1, \ldots, k$ and then carry out the recursive procedure described in (2.6). This more general formulation requires only slight modifications of the original proof (cf. [13, Theorem 2.5]). For piecewise differentiable boundary curves the following strategy for the determination of the generalized Leja points in practice is near at hand: First, all the points on ∂E and ∂F, respectively, where the boundary is not differentiable have to be chosen as zeros, respectively as poles. After ensuring that the degrees in the numerator and denominator are equal, one computes the further points by the recursive procedure of (2.6). In practice this is now done by finding the local maximum of the function $|r(z)|^2$ on the boundary curve between two Leja points and then choosing the maximum of all these points as the new Leja point. Between two Leja nodes one can now determine the local maximum numerically by using the derivative of $|r(z)|^2$ with respect to the corresponding parametrization of the boundary, for example with the algorithm described in Section 10.3 in [11]. The generalized Leja points can be determined numerically in a relatively efficient way for a large class of boundary curves ∂E and ∂F and they have the very advantageous property that once computed points remain Leja points for all larger degrees. Moreover, the recursive construction automatically yields the value

$$\frac{\max_{z \in E} |r_l(z)|}{\min_{z \in F} |r_l(z)|}$$

in each step and with this information one can increase the degree of the rational function until this value is less than a given bound.

The generalized Leja points as well as the generalized Fejér points are asymptotically minimal, indeed, but, in general, this behavior becomes significant only for very large values of l. Roughly speaking, this can be explained with the fact that the location of the zeros and poles of the corresponding rational function is restricted to ∂E and ∂F, respectively. This will become clear in the following example, where the rational minimal solutions are known explicitly.

EXAMPLE. We consider disks which are symmetric with respect to the origin, i.e.,

$$E = \{z \in \mathbb{C} : |z - \alpha| \leqslant \rho\}, \qquad F = -E$$

with $0 < \rho < \alpha$. In Section 3 it will be shown that the exact minimal solutions are given by

$$r_l^*(z) = \left(\frac{z - \sqrt{\alpha^2 - \rho^2}}{z + \sqrt{\alpha^2 - \rho^2}}\right)^l.$$

The values

$$\tau_l = \left(\frac{\max_{z \in E} |r_l(z)|}{\min_{z \in F} |r_l(z)|}\right)^{1/l}$$

for the rational functions generated by the generalized Fejér and Leja points (τ_l^{Fej} and τ_l^{Lej}) are compared with the exact minimal solution (τ_l^*) in

TABLE 2.1

$\rho = 0.9\alpha$

l	τ_l^{Fej}	τ_l^{Lej}	τ_l^*
2	0.6807	0.6807	0.3929
4	0.5491	0.5491	0.3929
8	0.4671	0.4871	0.3929
16	0.4284	0.4584	0.3929
32	0.4103	0.4357	0.3929
∞	0.3929	0.3929	0.3929

Table 2.1. The generalized Fejér points can also be calculated explicitly here since the conformal mapping Ψ is given by

$$\Psi(w) = \delta \frac{w+\gamma}{w-\gamma} \tag{2.7}$$

with

$$\gamma = \frac{\sqrt{\alpha+\rho}+\sqrt{\alpha-\rho}}{\sqrt{\alpha+\rho}-\sqrt{\alpha-\rho}}, \qquad \delta = -\sqrt{\alpha^2-\rho^2},$$

and $\rho(E, F) = \gamma^2$. For the determination of the Leja points we started there with $\varphi_1 = \alpha + \rho$, $\varphi_2 = \alpha - \rho$ and $\psi_1 = -\varphi_1$, $\psi_2 = -\varphi_2$.

From Theorem 2.1 we know that

$$\lim_{l \to \infty} \tau_l^{Fej} = \lim_{l \to \infty} \tau_l^{Lej} = 0.3929$$

holds but, as we can see here, the convergence is rather slow.

3. A Near-Circularity Criterion for the Rational Zolotarev Problem

The *near-circularity criterion* for the rational approximaton of an analytic function on a compact subset of the complex plane as it was proved by Klotz [9] for the case of the unit circle and by Trefethen [16, 17] for arbitrary Jordan domains has the following generalization for the rational Zolotarev problem.

Theorem 3.1. *Let the boundaries ∂E and ∂F of the disjoint sets E and F be given by Jordan curves and assume further that the rational function*

$r \in \mathbf{R}_{ll}$ possesses l zeros in E and l poles in F. If $r^* \in \mathbf{R}_{ll}$ denotes the minimal solution for the rational Zolotarev problem (1.1), *then*

$$\frac{\min_{z \in \partial E} |r(z)|}{\max_{z \in \partial F} |r(z)|} \leqslant \frac{\max_{z \in E} |r^*(z)|}{\min_{z \in F} |r^*(z)|} \leqslant \frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|} \tag{3.1}$$

holds.

Proof. The second inequality is trivial.

Moreover, the first inequality is automatically true if one of the zeros of r lies on ∂E or one of the poles of r is located on ∂F. Therefore, we can assume in the sequel that these points are all in the interior of the corresponding sets.

Assume that the first inequality is not fulfilled. This implies that there is a rational function $\tilde{r} \in \mathbf{R}_{ll}$ with

$$\frac{\max_{z \in \partial E} |\tilde{r}(z)|}{\min_{z \in \partial F} |\tilde{r}(z)|} < \frac{\min_{z \in \partial E} |r(z)|}{\max_{z \in \partial F} |r(z)|}.$$

By multiplying $\tilde{r}$ by a positive constant c which fulfills

$$\frac{\max_{z \in \partial E} |\tilde{r}(z)|}{\min_{z \in \partial E} |r(z)|} < \frac{1}{c} < \frac{\min_{z \in \partial F} |\tilde{r}(z)|}{\max_{z \in \partial F} |r(z)|},$$

we obtain

$$\max_{z \in \partial E} |\tilde{r}(z)| < \min_{z \in \partial E} |r(z)|,$$

$$\max_{z \in \partial F} \left| \frac{1}{\tilde{r}(z)} \right| < \min_{z \in \partial F} \left| \frac{1}{r(z)} \right|,$$

and from this

$$|\tilde{r}(z)| < |r(z)| \qquad \text{for} \quad z \in \partial E,$$

$$\left| \frac{1}{\tilde{r}(z)} \right| < \left| \frac{1}{r(z)} \right| \qquad \text{for} \quad z \in \partial F.$$

By Rouché's Theorem the functions $\tilde{r} - r$ and r have the same number of zeros in E°, namely l, and, similarly, $1/\tilde{r} - 1/r$ and $1/r$ have l zeros in F°. So we have found $2l$ points now (l in E° and l in F°) where the functions r and $\tilde{r}$ have the same value.

Let us consider now an arbitrary curve Γ which connects the sets E and F. Since $|\tilde{r}(z)| < |r(z)|$ on ∂E and $|\tilde{r}(z)| > |r(z)|$ on ∂F, by continuity arguments there is a point $z_0 \in \Gamma$ with $|\tilde{r}(z_0)| = |r(z_0)|$. The rational function $\tilde{r}$ was only determined up to a constant of absolute value 1 at this stage. So, by multiplying $\tilde{r}$ by $q := r(z_0)/\tilde{r}(z_0)$ (and again denoting this function by $\tilde{r}$), we obtain $\tilde{r}(z_0) = r(z_0)$, too.

With this, we have found $2l+1$ zeros in total of the rational function $\tilde{r} - r \in \mathbb{R}_{2l,2l}$ (note that z_0 is neither contained in E nor in F) which was assumed not to vanish identically. This is, of course, a contradiction. ∎

Now is the time to justify the notation "*near-circularity criterion.*" If we consider, as a generalization of the *error curves* introduced by Trefethen [16, 17], the "ring-shaped" domains

$$C(r) := \left\{ \frac{r(\lambda)}{r(\mu)} : \lambda \in \partial E, \mu \in \partial F \right\},$$

then our "generalized near-circularity criterion" asserts that the minimum distance of $C(r)$ to the origin gives a lower and the maximum distance gives an upper bound for the best possible value in the rational Zolotarev problem. This means that if $C(r)$ reduces to a perfect circle around the origin we can be sure that we have found a minimal solution for the rational Zolotarev problem.

That such situations, where we can apply our generalized near-circularity criterion to prove the minimality of a rational function, really occur, is shown by the following example, which already appeared in the last section.

EXAMPLE. We consider again the case that E and F are disjoint disks which are symmetric with respect to the real line, e.g.,

$$E = \{ z \in \mathbb{C} : |z - \alpha_1| \leqslant \rho_1 \}, \qquad F = \{ z \in \mathbb{C} : |z - \alpha_2| \leqslant \rho_2 \}$$

with $\alpha_1 - \rho_1 > \alpha_2 + \rho_2, \rho_1, \rho_2 > 0$. We show now that the solution of the rational Zolotarev problem (1.1) is given by

$$r^*(z) = \left(\frac{z - \varphi}{z - \psi} \right)^l$$

with

$$\varphi = \frac{\alpha_1^2 - \rho_1^2 - (\alpha_2^2 - \rho_2^2) + \sqrt{\xi}}{2(\alpha_1 - \alpha_2)} \tag{3.2}$$

and

$$\psi = \frac{\alpha_1^2 - \rho_1^2 - (\alpha_2^2 - \rho_2^2) - \sqrt{\xi}}{2(\alpha_1 - \alpha_2)}, \tag{3.3}$$

where we set

$$\xi = [(\alpha_1-\alpha_2)^2-(\rho_1^2+\rho_2^2)]^2-4\rho_1^2\rho_2^2.$$

For $z \in \partial E$, i.e., $z=\alpha_1+\rho_1 w$ with $|w|=1$, there holds

$$\begin{aligned} r^*(z) &= \left(\frac{\alpha_1+\rho_1 w-\varphi}{\alpha_1+\rho_1 w-\psi}\right)^l \\ &= \left(\frac{\rho_1}{\alpha_1-\psi}\left(w+\frac{\alpha_1-\varphi}{\rho_1}\right)\Big/\left(1+\frac{\rho_1}{\alpha_1-\psi}\,w\right)\right)^l \end{aligned} \tag{3.4}$$

and, since

$$\begin{aligned} \frac{\alpha_1-\varphi}{\rho_1} &= \frac{(\alpha_1-\alpha_2)^2+\rho_1^2-\rho_2^2-\sqrt{\xi}}{2\rho_1(\alpha_1-\alpha_2)} \\ &= \frac{2\rho_1(\alpha_1-\alpha_2)}{(\alpha_1-\alpha_2)^2+\rho_1^2-\rho_2^2+\sqrt{\xi}} = \frac{\rho_1}{\alpha_1-\psi}, \end{aligned}$$

we obtain

$$|r(z)| = \left(\frac{\rho_1}{\alpha_1-\psi}\right)^l \qquad \text{for} \quad z \in \partial E.$$

Analogously,

$$|r(z)| = \left(\frac{\rho_2}{\psi-\alpha_2}\right)^l \qquad \text{for} \quad z \in \partial F;$$

hence,

$$\frac{|r(\lambda)|}{|r(\mu)|} = \left(\frac{\rho_1}{\rho_2}\frac{\psi-\alpha_2}{\alpha_1-\psi}\right)^l = \left(\frac{(\alpha_1-\alpha_2)^2-(\rho_1^2+\rho_2^2)-\sqrt{\xi}}{2\rho_1\rho_2}\right)^l$$

for each $\lambda \in \partial E$, $\mu \in \partial F$. By

$$\frac{\alpha_1-\varphi}{\alpha_1-\psi} = \frac{(\alpha_1-\alpha_2)^2+\rho_1^2-\rho_2^2-\sqrt{\xi}}{(\alpha_1-\alpha_2)^2+\rho_1^2-\rho_2^2+\sqrt{\xi}} < 1$$

and

$$\frac{\alpha_2-\varphi}{\alpha_2-\psi} = \frac{-(\alpha_1-\alpha_2)^2+\rho_1^2-\rho_2^2-\sqrt{\xi}}{-(\alpha_1-\alpha_2)^2+\rho_1^2-\rho_2^2+\sqrt{\xi}} > 1$$

it follows from (3.4) that the zero φ of multiplicity l is contained in E° and the pole ψ of multiplicity l in F°, which enables us to apply Theorem 3.1.

Unfortunately, these situations—where the set $C(r)$ represents a perfect circle around the origin—occur only very rarely. For instance, consider the case where E and F are given by real intervals. There, the minimal solution is characterized by an *alternation condition* (Wachspress [18]), i.e., $C(r)$ is also a real interval. Thus, from Theorem 3.1 we get zero as a lower bound for the minimal deviation which is trivially true. Indeed, a real interval is not a Jordan curve but it can be interpreted as the limiting case of *ellipses*. So, it can be expected that, roughly speaking, we obtain error curves that become more and more "flat" the more the sets E and F tend to intervals.

To enable us to show that we have a rational function which is not far away from the solution of the minimization problem (1.1) the set $C(r)$ has not necessarily to be very circular. If we look a little closer at Theorem 3.1 we see that we can admit all rational functions which have all their zeros in E and all their poles in F. From this, we obtain

COROLLARY 3.2. *Let the sets E and F fulfill the assumptions of Theorem* 3.1. *Further denote by $\tilde{\mathbf{R}}_{ll}$ the set of all rational functions of degree l which have all their zeros in E and all their poles in F; then there holds*:

$$\max_{r \in \tilde{\mathbf{R}}_{ll}} \frac{\min_{z \in \partial E} |r(z)|}{\max_{z \in \partial F} |r(z)|} \leqslant \min_{r \in \mathbf{R}_{ll}} \frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|}. \tag{3.5}$$

An immediate consequence of (2.3), of Corollary 3.2, and of the fact that, for any rational function $r \in \tilde{\mathbf{R}}_{ll}$, we have $r^k \in \tilde{\mathbf{R}}_{kl,kl}$ is

COROLLARY 3.3. *Using the same assumptions and notations as in Corollary* 3.2, *we have, for each $l \in \mathbf{N}$*,

$$\left(\max_{r \in \mathbf{R}_{ll}} \frac{\min_{z \in \partial E} |r(z)|}{\max_{z \in \partial F} |r(z)|}\right)^{1/l} \leqslant \rho(E, F)^{-1} \leqslant \left(\min_{r \in \mathbf{R}_{ll}} \frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|}\right)^{1/l}. \tag{3.6}$$

This means that, using the near-circularity criterion, we obtain upper bounds for the modulus of the doubly-connected region $D := \bar{\mathbb{C}} \setminus (E \cup F)$. Lower bounds for the modulus were established before using Gonchar's result (2.2).

4. COMPUTATIONAL RESULTS

As an example, let us consider the minimization problem (1.1) on rectangles

$$E = \{z \in \mathbb{C} : \alpha \leqslant \operatorname{Re} z \leqslant \beta, \quad |\operatorname{Im} z| \leqslant \gamma\};$$

$$F = \{z \in \mathbb{C} : -\beta \leqslant \operatorname{Re} z \leqslant -\alpha, \quad |\operatorname{Im} z| \leqslant \gamma\}$$

with $0<\alpha<\beta$. This problem arises from the determination of optimal ADI parameters if we use Bendixson's theorem to get rectangles E and F which contain the eigenvalues of the parts in the ADI splitting.

We computed the rational minimal solutions for $l=1$, $l=2$ and $l=4$ using standard minimization algorithms (cf. [11, Sect. 10]). For $l=1$ and $l=2$ this can be reduced to several *one-dimensional* minimization problems as described in [12]; for $l=4$ we minimized

$$\frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|}$$

for

$$r(z)=\frac{z^4-\sigma_3 z^3+\sigma_2 z^2-\sigma_1 z+\sigma_0}{z^4+\sigma_3 z^3+\sigma_2 z^2+\sigma_1 z+\sigma_0}$$

with respect to the parameters $\sigma_3, \sigma_2, \sigma_1, \sigma_0 \in \mathbb{R}$. It should be remarked that the symmetry, i.e., $r(-z)=1/r(z)$, as well as the fact that the coefficients are all real is justified by our computational results but not proved in general.

Although the near-circularity criterion cannot be applied directly here to show that the constructed rational functions are indeed the minimal solutions, we can use it to give some insight to the problem. Let

$$\eta_1 := \max_{z \in E} |r(z)|, \qquad \eta_2 := \min_{z \in F} |r(z)|;$$

then, clearly, the solution of the rational Zolotarev problem is also the solution for the sets $\tilde{E}, \tilde{F}$ enclosed by the *rational lemniscates* $\Gamma_1 := \{z \in \mathbb{C} : |r(z)| = \eta_1\}$ and $\Gamma_2 := \{z \in \mathbb{C} : |r(z)| = \eta_2\}$. On the other hand, the rational Zolotarev problem has the geometrical interpretation that we have to enclose the compact sets E and F by rational lemniscates in such a way that η_1/η_2 is minimized.

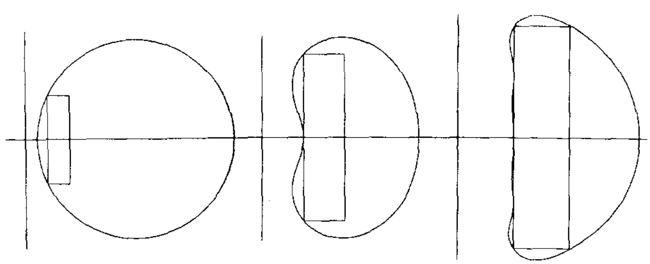

FIG. 4.1. $\beta=\gamma=2\alpha$, $l=1, 2, 4$.

TABLE 4.1

τ_l^*, $\beta = 2\alpha$

l	$\gamma = 0.25\alpha$	$\gamma = 0.5\alpha$	$\gamma = \alpha$	$\gamma = 2\alpha$	$\gamma = 4\alpha$
1	0.0374	0.0627	0.1716	0.3820	0.6096
2	0.0338	0.0603	0.1086	0.2560	0.4825
4	0.0276	0.0547	0.1086	0.2069	0.4243

For $l = 1$ the rational lemniscates are obtained from

$$\left|\frac{z-\varphi}{z-\psi}\right| = \eta$$

or

$$\left|z - \frac{\varphi - \eta^2\psi}{1-\eta^2}\right| = \frac{\eta}{1-\eta^2}\,|\varphi - \psi|. \tag{4.1}$$

This means that, for the case $l = 1$, the geometrical interpretation of the rational Zolotarev problem is to find a circle with midpoint $(\varphi - \eta_1^2\psi)/(1-\eta_1^2)$ and radius $(\eta_1/(1-\eta_1^2))\,|\varphi - \psi|$ enclosing the set E and a circle with midpoint $(\varphi - \eta_2^2\psi)/(1-\eta_2^2)$ and radius $(\eta_2/(1-\eta_2^2))\,|\varphi-\psi|$ enclosing the set F in such a way that η_1/η_2 is minimal. With this illustrative formulation, the rational Zolotarev problem can be solved for $l = 1$ for a large number of exemplary regions.

The rational lemniscates Γ_1 are shown in Fig. 4.1. This figure seem to indicate that the results obtained using rational functions of degree 2 and 4 are far better than those with only 1 parameter. In Table 4.1 we have listed the numbers

$$\tau_l^*(E, F) = \left(\min_{r \in \mathbf{R}_{ll}} \frac{\max_{z \in E} |r(z)|}{\min_{z \in F} |r(z)|}\right)^{1/l} \tag{4.2}$$

TABLE 4.2

$\tilde{\tau}_l$, $\beta = 2\alpha$

l	$\gamma = 0.25\alpha$	$\gamma = 0.5\alpha$	$\gamma = \alpha$	$\gamma = 2\alpha$	$\gamma = 4\alpha$
1	0.0102	0.0294	0.0294	0.0294	0.0294
2	0.0165	0.0294	0.0700	0.1263	0.1587
4	0.0170	0.0322	0.0722	0.1552	0.2640

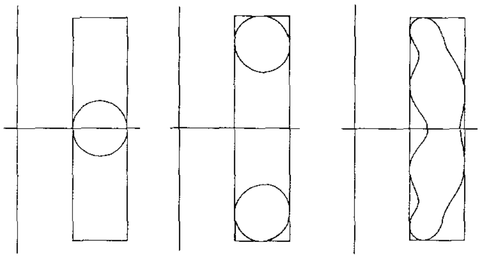

FIG. 4.2. $\beta=\gamma=2\alpha$, $l=1, 2, 4$.

for $l=1, 2, 4$ and some rectangles with $\beta=2\alpha$ and different values of γ. By Corollary 3.2 the values

$$\tilde{\tau}_l=\left(\max_{r\in\tilde{\mathbf{R}}_{ll}}\frac{\min_{z\in\partial E}|r(z)|}{\max_{z\in\partial F}|r(z)|}\right)^{1/l} \tag{4.3}$$

given in Table 4.2 are lower bounds for these quantities. The geometrical interpretation of the determination of the values $\tilde{\tau}$ in Table 4.2 is that we have to put a rational lemniscate into the considered rectangle in an optimal way. This is illustrated in Fig. 4.2.

To get more detailed information about the asymptotic convergence behavior we compute the generalized Leja points by the method given in Section 2 starting with

$$\varphi_{1/2}=\alpha\pm i\gamma, \qquad \varphi_{3/4}=\beta\pm i\gamma,$$

and $\psi_j=-\varphi_j, j=1, ..., 4$.

Comparing the values of Table 4.1 with those of Table 4.3 shows that the asymptotic behavior is surprisingly well approximated for $l=4$ and sometimes even for $l=2$. Note that the numbers listed in Table 4.3 as well as in Table 4.1 give upper bounds for $\rho(E, F)^{-1}$, whereas the numbers of Table 4.2 are lower bounds for this quantity. This means that, using only

TABLE 4.3

τ_l^{Lej}, $\beta=2\alpha$

l	$\gamma=0.25\alpha$	$\gamma=0.5\alpha$	$\gamma=\alpha$	$\gamma=2\alpha$	$\gamma=4\alpha$
4	0.0425	0.0892	0.2000	0.4385	0.7376
8	0.0336	0.0607	0.1382	0.3013	0.6288
16	0.0282	0.0539	0.1066	0.2243	0.4239
32	0.0262	0.0489	0.0982	0.2109	0.4002
64	0.0251	0.0453	0.0927	0.1963	0.3706

rational functions of degree 4, we obtain the bounds $0.1552 \leqslant \rho(E, F)^{-1} \leqslant 0.2069$ for the example of Fig. 4.1 and 4.2. If we also use the computational results for the Leja points, this is improved by $0.1552 \leqslant \rho(E, F)^{-1} \leqslant 0.1963$.

It is clear from (2.3) that the upper bounds are sharp for $l \to \infty$. An interesting open question would be: Is the same true for the lower bounds, i.e., is the identity

$$\lim_{l \to \infty} \left(\max_{r \in \tilde{\mathbf{R}}_{ll}} \frac{\min_{z \in \partial E} |r(z)|}{\max_{z \in \partial F} |r(z)|} \right)^{1/l} = \rho(E, F)^{-1} \tag{4.4}$$

correct?

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