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On well-covered triangulations: Part I

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Abstract

A graph G is said to be *well-covered* if every maximal independent set of vertices has the same cardinality. A planar (simple) graph in which each face is a triangle is called a *triangulation*. It is the aim of this paper to prove that there are no 5-connected planar well-covered triangulations.

Keywords: Independent set; Well-covered; Triangulation

1. Introduction

In this paper all graphs will be finite and simple.

In 1969, the fourth author first proposed the study of graphs in which each maximal independent set of vertices has the same size and suggested that the name *well-covered* be applied to them [9]. Although it is now well known that the independent set problem is NP-complete for graphs in general (cf. [5]), for certain interesting subfamilies of graphs, such as those called *claw-free,* the problem becomes polynomially solvable (cf. [8,12]). Clearly, the independent set problem has a polynomial solution for the class of well-covered graphs, but how does one recognize this class? It was shown independently by Chvátal and Slater [2] and by Sankaranarayana and Stewart [11] that the recognition problem for well-covered graphs is co-NP-complete. In contrast, if the

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graphs are claw-free, then the recognition problem becomes polynomial. (See [13,14].) For more comprehensive treatments of well-covered graphs, see [3,10].

A widely studied subclass of graphs are those which are maximal planar and which are commonly called (planar) *triangulations*. Any triangulation (larger than a single triangle) must have vertex connectivity 3, 4 or 5 since, by Euler's theorem it must contain a vertex of degree 3, 4 or 5. Lebesgue [7], Kotzig [6], Borodin [1] and Jendrol' [4] have extensively investigated what kind of configurations must always exist in any triangulation. In the next section, these results are used to prove that there is no 5-connected planar well-covered triangulation. Planar well-covered triangulations which are 3- and 4-connected will be treated in a subsequent paper.

Note that in this paper all graphs are simple and if v is a vertex of a graph, $N[v]$ will denote the closed neighborhood of the vertex v; namely, $N[v]=N(v)\cup\{v\}$.

2. There exists no 5-connected well-covered triangulation

In order to prove the main result of this section, as well as other results to follow, we shall make extensive use of the work of Kotzig [6], Borodin [1] and Jendrol' [4] on the structure of triangulations. After Jendrol', we shall call a triangle with vertex degrees a, b and c an (a,b,c)-*triangle*. The results of the above three authors which we shall need can be summarized as follows. (See [4, Theorem 4].)

Theorem 2.1. *Each* 5-*connected planar triangulation of order at least five contains an* (a,b,c)-*triangle*, *where*

(i) $a=5,\ b=5,\ 5\leqslant c\leqslant 7$, *or*
(ii) $a=5,\ b=6,\ c=6$.

First we remind the reader that if a graph is 5-connected, it has a unique embedding in the plane and that any face may be considered to be the outer or infinite face. We further note that in a 5-connected triangulation, no 4-cycle is induced. Finally we remind the reader of the following well known (see [10]) but very useful observation when working with well-covered graphs: if G is a well-covered graph and I is an independent set of vertices of G, then $G-N[I]$ is also well-covered.

A technical lemma was found to facilitate the proof of the main result.

Definition 2.2. The 5-tuple (F,L,B,C,d) is called an F-*configuration* in the graph G provided that F is a subgraph of G, $B\cup C\cup\{d\}\subseteq V(F)$, $B\cap C=\emptyset$, and there is an integer $k\geqslant 2$ such that the following are true:

$$\mathrm{Card}(B)=k \text{ and } \{b_1,b_2,\ldots,b_k\}=B\subseteq N(d)\cap V(F); \tag{2.1}$$

$$\mathrm{Card}(C)=k \text{ and } \{c_1,c_2,\ldots,c_k\}=C\subseteq V(F); \tag{2.2}$$

$$N[C]\subseteq V(F); \tag{2.3}$$

$$L \subseteq F \text{ is independent in } V(G) \text{ and is a maximal independent set in } V(F); \tag{2.4}$$

$$B \subseteq L; \tag{2.5}$$

$$(L-\{b_i\}) \cup \{c_i\} \text{ is independent in } V(G) \text{ and a maximal independent set in } V(F) \text{ for each } i \in \{1,2,\ldots,k\}; \tag{2.6}$$

$$(L-B) \cup \{d\} \cup (C-\{c_1,c_2\}) \text{ is independent in } V(G) \text{ and a maximal independent set in } V(F). \tag{2.7}$$

We note that (2.1), (2.5) and (2.7) combine to yield

$$N(d) \cap L = B \tag{2.8}$$

and that (2.6) together with $B \cap C = \emptyset$ implies that

$$L \cap C = \emptyset. \tag{2.9}$$

Lemma 2.3. *A 5-connected well-covered planar triangulation does not contain an F-configuration.*

Proof. Suppose for a contradiction that (F,L,B,C,d) is an F-configuration in a 5-connected well-covered planar triangulation G. Let I be a maximal independent set in $V(G)$ containing L such that $|I \cap N(d)|$ is a minimum.

Observe that for each $j \in \{1,2,\ldots,k\}$, $I_j = (I-\{b_j\}) \cup \{c_j\}$ is independent by (2.3) and (2.6). Further, since G is well-covered and since the cardinality of I and I_j are equal, I_j must be a maximal independent set in G.

As a result, we note that

$$\text{if } v \in G-F \text{ then } v \in N[I-\{b_j\}] \text{ for each } j \in \{1,2,\ldots,k\}. \tag{3.1}$$

We claim that $J = (I-N(d)) \cup \{d\} \cup (C-\{c_1,c_2\})$ is a maximal independent set in G. Note that (2.3) combined with (2.7) imply that J is independent; and (2.7) combined with (2.8) imply that $N[J] \supseteq F$.

To show that J is a maximal independent set in G assume that $v \in G-F$ and that $v \notin N[J]$. Then we note that v is not adjacent to d, but that v is adjacent to one or more elements of I (by the maximality of I), and all of these elements must be in $N(d)$ (by the construction of J).

However, if v were adjacent to α_1 and $\alpha_2 \in N(d) \cap I$, then G would contain the induced 4-cycle $\alpha_1 d \alpha_2 v$ (dv is not in G and $\alpha_1\alpha_2 \notin G$ since α_1 and α_2 are in the independent set I), which as noted above is impossible. Hence, v is adjacent to exactly one element of $N(d) \cap I$.

By (3.1) v is adjacent to no element of B, so v is adjacent to exactly one $\alpha \in (N(d) \cap I)-F$. But then $J' = (I-\{\alpha\}) \cup \{v\}$ is an independent set in G with the same cardinality

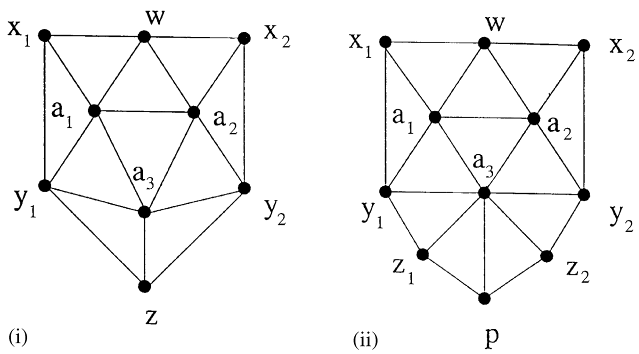

Fig. 1.

as I and hence by the well-covered property of G, J' is a maximal independent set in G containing L, but such that $|J' \cap N(d)| < |I \cap N(d)|$. This contradicts the minimum intersection property of I. This shows that no such v exists and hence that J is a maximal independent set in G.

But by (2.8), $|I| \geqslant |J| + 1$ and thus since G is well-covered, the hypothesis that G contains an F-configuration is false. □

We now turn to the main result of this section.

Theorem 2.4. *No* 5*-connected triangulation is well-covered.*

Proof. From the results of Jendrol' et al. above, we see that every 5-connected triangulation G must contain a triangle of type (5, 5, 5), (5, 5, 6), (5, 5, 7), or (5, 6, 6). We proceed to treat each of these possibilities in sequence.

Step 1: First we will show that G contains neither a (5, 5, 5)-triangle nor a (5, 5, 7)-triangle.

Suppose, to the contrary, that G does contain either a (5, 5, 5)-triangle or a (5, 5, 7)-triangle. Then G will contain as an induced subgraph the basic 9-vertex configuration (if it contains the (5, 5, 5)-triangle $a_1a_2a_3$) or the basic 11-vertex configuration (if it contains the (5, 5, 7)-triangle $a_1a_2a_3$) shown in Fig. 1(i) and (ii), respectively.

In either case, denote this configuration by F.

Observe that planarity and 5-connectedness imply that F is an induced subgraph of G (for example the edge zw cannot be in G for otherwise zwa_1a_3 forms an induced 4-cycle in G.) Then the set $L = \{x_1, x_2, z\}$ in case (i) (respectively, the set $L = \{x_1, x_2, z_1, z_2\}$ in case (ii)) is independent in G. But then setting $X = \{x_1, x_2\}$, and $A = \{a_1, a_2\}$, we see that (F, L, X, A, w) is an F-configuration in G in violation of Lemma 2.3. Thus, the hypothesis that G contains either a (5, 5, 5)-triangle or a (5, 5, 7)-triangle is false, completing step 1.

Step 2: Next we will show that G contains no (5, 5, 6)-triangle.

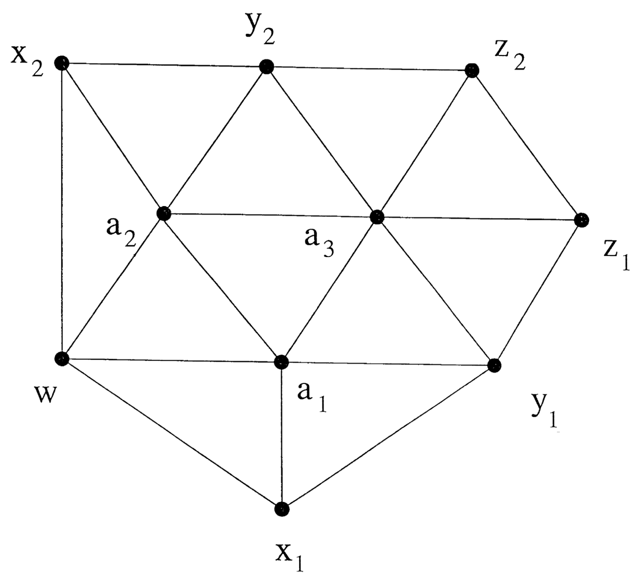

Fig. 2.

Suppose, to the contrary, that G does contain a (5, 5, 6)-triangle $a_1a_2a_3$. Then G must contain the 10-vertex configuration H of Fig. 2

As in the case of the subgraph F in step 1, H is an induced subgraph of G. We begin by noting that by the 5-connectivity and maximal planarity, vertices y_i and z_i must share a common neighbor $p_i \in G - H$, for $i = 1$ and 2 (see Fig. 3). Note that the absence of separating 4-cycles implies that $p_1 \neq p_2$, and that neither edge p_1z_2 nor edge p_2z_1 is in G. Furthermore, the edge p_1p_2 is not present, for if it were, then in the 4-cycle $p_1z_1z_2p_2$, one of the diagonals p_2z_1 or p_1z_2 would have to be in G (since no 4-cycle in G is induced). Finally note that the edge x_1x_2 is not in G since H is induced.

Claim 1. *Either the set* $\{p_1, z_2, x_1, x_2\}$ *or the set* $\{p_2, z_1, x_1, x_2\}$ *is independent.*

To prove Claim 1, first note that the edges x_1x_2, x_1z_2, and z_1x_2 are not in G since H is induced. Hence it will suffice to show that

$$\text{For some } i \text{ in } \{1,2\}, \text{ neither the edge } x_1p_i \text{ nor the edge } x_2p_i \text{ is in } G. \tag{4.1}$$

Suppose that G contains neither the edge x_1p_1 nor the edge x_2p_2. Then, since by planarity one of the edges x_1p_2 or x_2p_1 is not in G, (4.1) follows.

Next suppose that G contains exactly one of the edges x_1p_1 or x_2p_2. Without loss of generality assume that the edge x_1p_1 is not in G and the edge x_2p_2 is in G. We obtain (4.1) in this case by establishing that the edge p_1x_2 is also not in G.

Indeed if p_1x_2 were in G, then let I be a maximal independent set in G containing $\{p_1, a_2, x_1, z_2\}$. Now since G is well-covered, the set $(I - \{z_2, a_2\}) \cup \{y_2\}$ fails to be

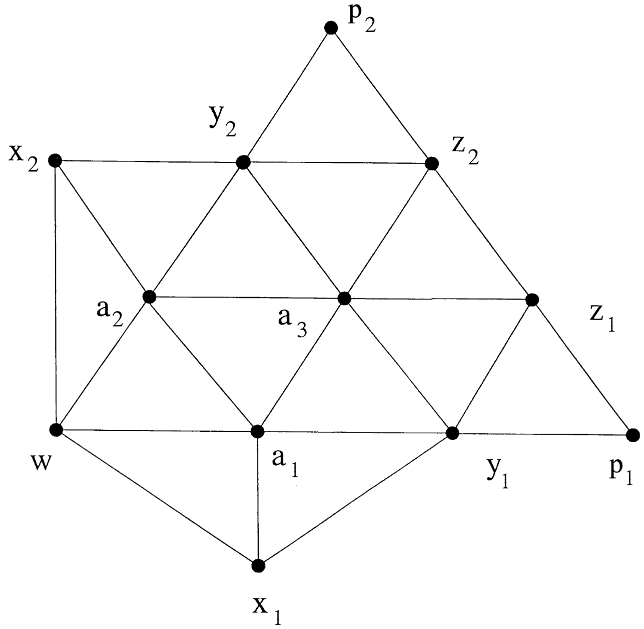

Fig. 3.

maximal and hence there exists a vertex z adjacent to z_2, but not adjacent to y_2 nor to any other vertex in $I-\{z_2\}$. Also since G is well-covered, the set $(I-\{x_1,a_2\})\cup\{a_1\}$ fails to be maximal and hence there exists a vertex x adjacent to x_1, but to no other vertex in I (see Fig. 4). But x and z are not adjacent by planarity and hence $J=(I-\{a_2,x_1,z_2\})\cup\{x,z,a_1,y_2\}$ is independent. Since $|J|=|I|+1$, this contradicts the fact that G is well-covered. Hence, the edge p_1x_2 is not in G and (4.1) is established in this case.

Finally suppose that G contains both the edge x_1p_1 and the edge x_2p_2. Since G is maximal planar and since p_1 and p_2 are not adjacent, x_1 and p_1 share a common neighbor u_1 in $G-H$ and by symmetry x_2 and p_2 share a common neighbor u_2 in $G-H$. Note that planarity and 5-connectivity prevent the u_i from being any of the vertices in $\{a_1,a_2,a_3,y_1,y_2,p_1,p_2,x_1,x_2,z_1,z_2,w\}$.

Now if $u_1=u_2$ or if the edge u_1u_2 is not in G, then let I be a maximal independent set in G containing $\{u_1,u_2,y_1,y_2\}$ and let $J=(I-\{y_1,y_2\})\cup\{a_3\}$. Since I and J are maximal independent sets of different sizes, this contradicts the fact that G is well-covered. Hence, $u_1\neq u_2$ and the edge u_1u_2 is in G as shown in Fig. 5.

Further, neither u_1 nor u_2 can be adjacent to w, for suppose edge u_1w is in G. Then by 5-connectivity, there are no vertices external to the 4-cycle $u_1wx_2u_2$. But then $G-N[\{p_1,p_2\}]$ would contain as a component the subgraph induced by $\{a_1,a_2,a_3,w\}$ which is not well-covered. Hence G is not well-covered, a contradiction.

Now, for $i=1,2$, let q_i be the external common neighbor of p_i and z_i. Observe that $q_i=z_j$ and $q_i=u_j$, where i and j range over $\{1,2\}$, are all excluded by 5-connectivity.

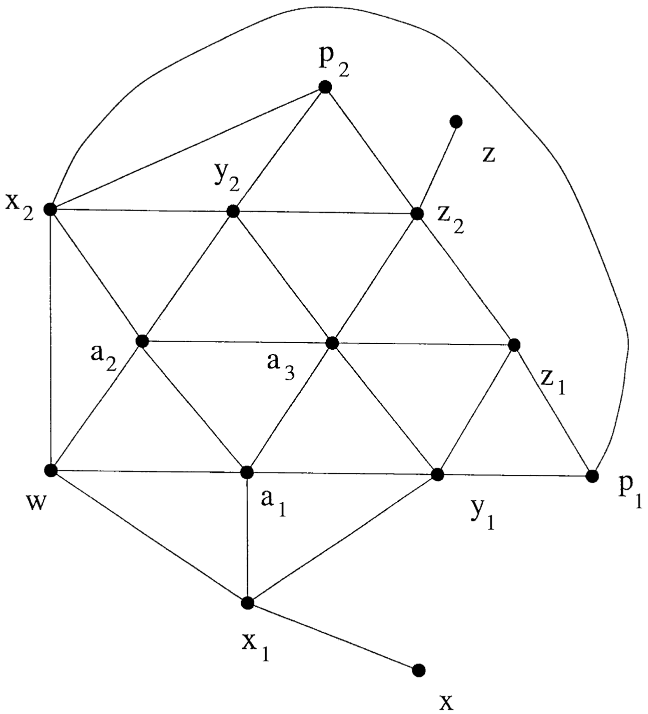

Fig. 4.

We further observe that if $q_1 = q_2$ or if the edge q_1q_2 is not in G, then $G - N[\{q_1, q_2, w\}]$ would contain as a component the subgraph of G induced by $\{y_1, y_2, a_3\}$ which is not well-covered. Hence again G is not well-covered, a contradiction. Hence $q_1 \neq q_2$ and the edge $q_1q_2 \in E(G)$, as shown in Fig. 6.

But now by planarity, one of the edges u_1q_2 or q_1u_2 is not in G; without loss of generality suppose edge u_1q_2 is not in G. Note that then $\{u_1, q_2, w\}$ is independent in G and $G - N[\{u_1, q_2, w\}]$ contains as a component the graph induced by $\{y_2, a_3, y_1, z_1\}$ which is not well-covered. Therefore again G is not well-covered, a contradiction.

Hence Claim 1 is proved.

Without loss of generality, suppose that the set $K = \{p_1, x_1, x_2, z_2\}$ is independent.

Let F be the subgraph of G depicted in Fig. 3 and set $X = \{x_1, x_2\}$, and $A = \{a_1, a_2\}$. Since w can be adjacent to neither p_1 nor z_2, we see that (F, K, X, A, w) is an F-configuration in G in violation of Lemma 2.3. Thus, the hypothesis that G contains a (5, 5, 6)-triangle is false completing step 2.

Step 3: Next we will show that G contains no (5, 6, 6)-triangle.

Suppose, to the contrary, that G does contain a (5, 6, 6)-triangle $a_3a_2a_1$. Then G must contain the 11-vertex configuration H of Fig. 7.

As in the case of the subgraph F in step 1, H is an induced subgraph of G. We begin by noting that by 5-connectivity and maximal planarity, vertices w_i and x_i must

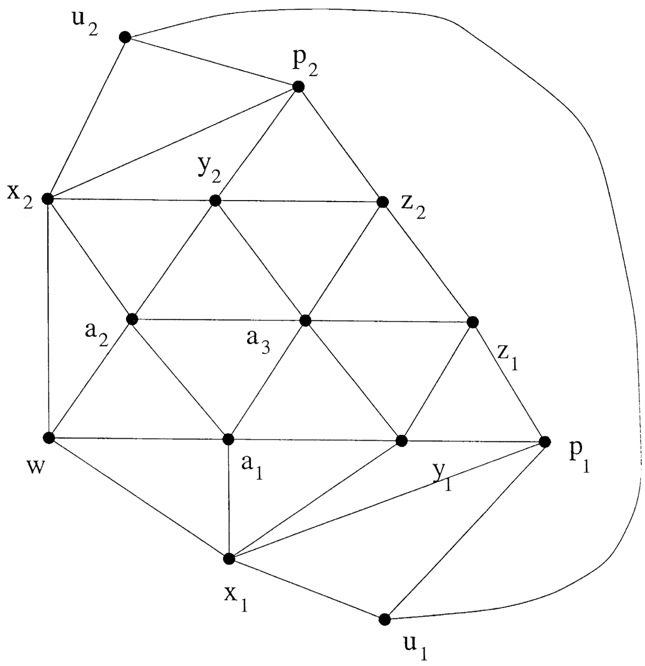

Fig. 5.

share a common neighbor $p_i \in G-H$, for both $i=1$ and 2. Furthermore, vertices z and w_i must have a common neighbor $r_i \in G-H$, for $i=1$ and 2. Note that it is possible that $r_1=r_2$, but since separating 4-cycles are forbidden, $r_i \neq p_1 \neq p_2 \neq r_i$, for $i=1$ and 2 (see Fig. 8).

Claim 2. *Each of* r_1 *and* r_2 *is adjacent to* x_3 *and thus* $N(z)-H=\{r_1,r_2\}$, *where either* $r_1=r_2$ *or edge* r_1r_2 *is in* G.

To prove Claim 2, first suppose that the edge r_1x_3 is not in G. Thus, the set $K=\{x_1,x_2,x_3,r_1\}$ is independent in G. Let F be the subgraph of G depicted in Fig. 8 and set $F_1=(F\cap N[K])-\{p_1\}$, $X=\{x_1,x_3\}$, and $A=\{a_1,a_3\}$. Since y_1 can be adjacent to neither r_1 nor x_2, we see that (F_1,K,X,A,y_1) is an F-configuration in G in violation of Lemma 2.3. Thus, the hypothesis that the edge r_1x_3 is not in G is false. Similarly, the hypothesis that the edge r_2x_3 is not in G is false.

If $r_1=r_2$, then by planarity and 5-connectivity, $N(z)-H=\{r_1\}=\{r_1,r_2\}$. Otherwise, each $r\in N(z)-(H\cup\{r_1,r_2\})$ lies outside the 4-cycle $r_1zr_2x_3$ (that is, is separated from the vertices a_1,a_2 and a_3) and thus cannot exist. Hence, $N(z)-H=\{r_1,r_2\}$ and by maximal planarity, the edge r_1r_2 is in G. This completes the proof of Claim 2.

In summary then, the graph G contains a subgraph as shown in Fig. 9 where $r_1=r_2$ is possible.

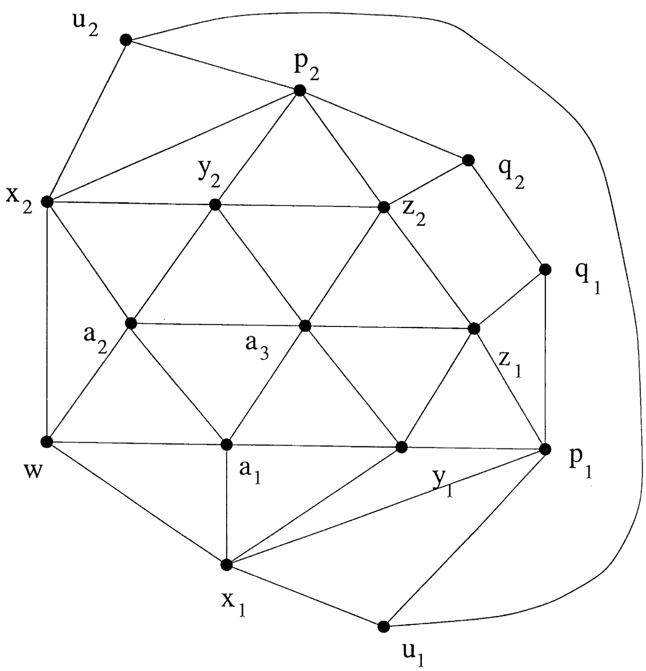
u2
p2
q2
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Fig. 6.

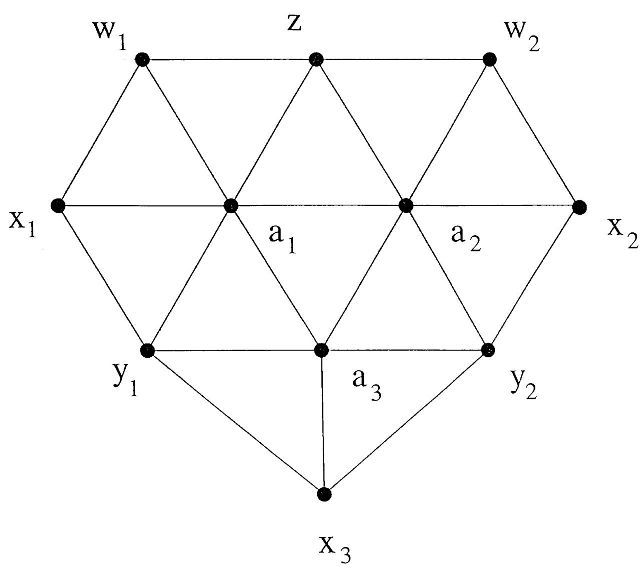
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Fig. 7.

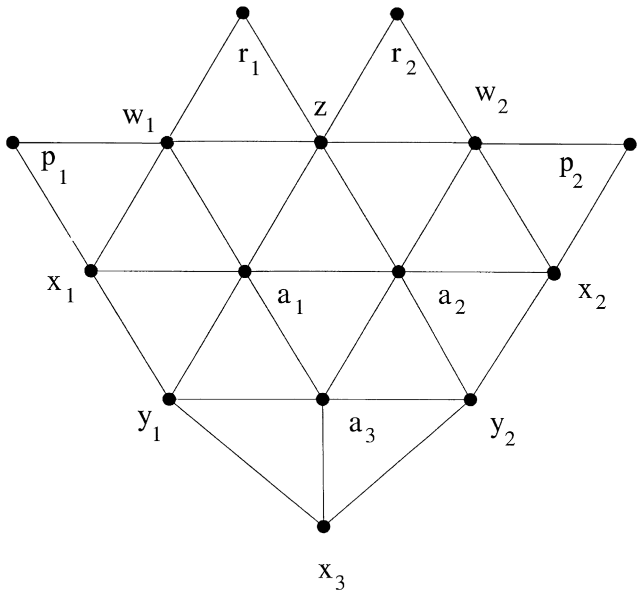


Fig. 8.

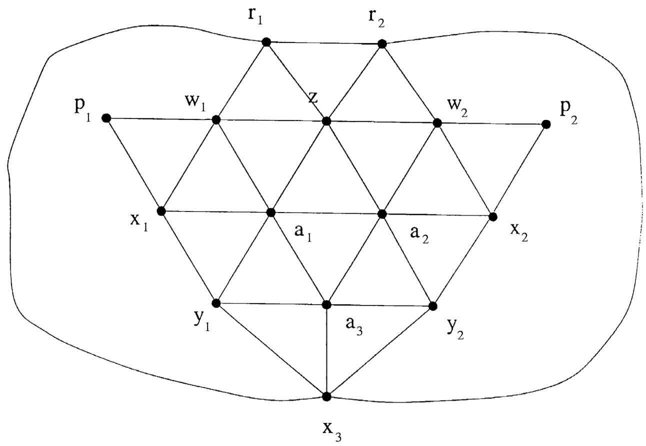
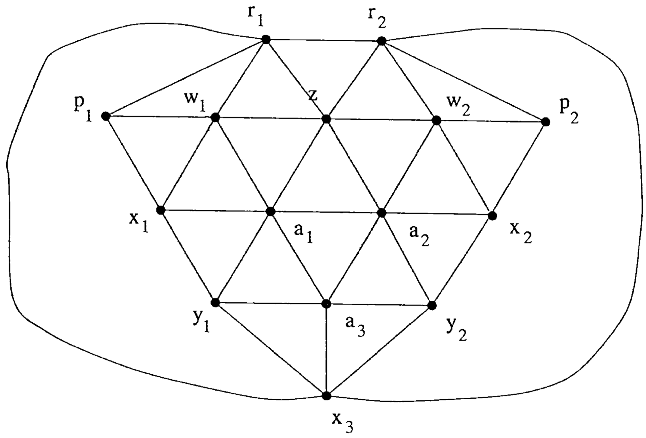


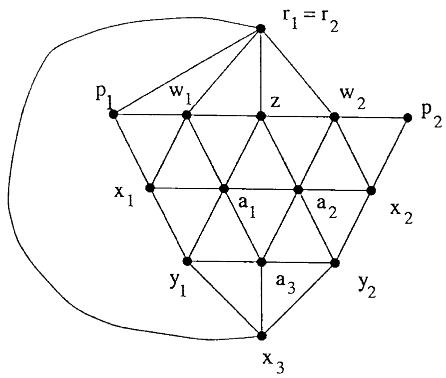
Fig. 9.

Claim 3. *Each vertex in* $\{r_1, r_2\}$ *is adjacent to either* p_1 *or* p_2.

To prove Claim 3, let us begin by assuming that r_1 is adjacent to neither p_1 nor p_2. Then the set $K = \{y_1, y_2, p_1, p_2, r_1\}$ is independent in G. (Indeed, edges $p_i y_j$, $i \neq j$, are prohibited by planarity combined with Claim 2, whereas the edges $p_i y_i$ and $r_1 y_j$ are prohibited by 5-connectivity.)



H1



H2

Fig. 10.

Let F be the subgraph of G depicted in Fig. 9 and set $Y=\{y_1,y_2,r_1\}$, and $A=\{a_1,a_2,z\}$. We note that for $i=1$ and 2, edge p_ix_3 would yield the 4-cycle $x_3p_ix_iy_i$. But both of the edges x_3x_i and p_iy_i are prohibited by the 5-connectivity and hence this 4-cycle would be induced in G, a contradiction. Thus, we see that (F,K,Y,A,x_3) is an F-configuration in G in violation of Lemma 2.3.

Thus, the hypothesis that r_1 is adjacent to neither p_1 nor p_2 is false. Similarly, the hypothesis that r_2 is adjacent to neither p_1 nor p_2 is false. This completes the proof of Claim 3.

Now we observe that, because of planarity, if $r_1 \neq r_2$, the only way that Claim 3 can be satisfied is for both edges r_1p_1 and r_2p_2 to be in G. Hence, the graph must contain one of the two configurations H_1 and H_2 shown in Fig. 10 where in H_2 the edge $r_2 p_2$ may or may not exist.

Let $K = \{a_3, p_1, p_2, z\}$ and note that this set is independent in G. Let I be a maximal independent set in G containing K and observe that it contains no vertices of Fig. 10 except those in K. Now both the sets $J_i = (I - \{z, p_i\}) \cup \{w_i\}$, for $i = 1$ and 2 are independent. Since $|I| = |J_i| + 1$, and G is well-covered, there exists a vertex q_i such that $q_i \neq N[J_i]$ because J_i is not maximal. But $q_i \in N[I]$ and hence q_i is adjacent to p_i, but q_i is adjacent to no other vertex in I. However, $q_1 q_2$ is not in G by planarity and hence $J = (I - \{z, p_1, p_2\}) \cup \{q_1, q_2, w_1, w_2\}$ is independent. But $|J| = |I| + 1$, violating the well-covered property of G and completing the proof of step 3 and thence the theorem. □

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