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Note

Counting maximal cycles in binary matroids

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Abstract

It is shown that each binary matroid contains an odd number of maximal cycles and, as a result of this, that each element of an Eulerian binary matroid is contained in an odd number of circuits.

Let M be a binary matroid with circuits $\mathscr{C}(M)$ and cycles $\mathscr{Z}(M)$, and let $\mathscr{C}_e(M)$ be the set of circuits containing an element e in the underlying set $E(M)$. (For matroid-theoretical background see [1].) Then M is Eulerian ($:\Leftrightarrow E(M) \in \mathscr{Z}(M)$) iff $\# \mathscr{C}_e(M)$ is odd for each e. Woodall [2] lists earlier occurrences of this statement and gives a new proof. The non-trivial part ('only if') can be reformulated ('each binary matroid contains an odd number of maximal cycles'), allowing another (simpler) proof.

For $S \subseteq E(M)$ let $M \backslash S$ denote the 'subtraction' of S with element set $E(M)\backslash S$ and $\mathscr{C}(M\backslash S) = \mathscr{C}(M) \cap \mathscr{P}(E(M)\backslash S)$, and for $e \in E(M)$ let $M\backslash e = M\backslash\{e\}$. Finally, let $\mathscr{Z}_p(M)$ be the set of maximal (prime) cycles of M and $\mathscr{Z}_0(M) = \mathscr{Z}(M)\backslash(\mathscr{Z}_p(M) \cup \{\emptyset\})$ the set of 'proper' cycles.

Lemma 1. *For an Eulerian binary matroid M and $e \in E(M)$ we have*

$$\# \mathscr{C}_e(M) = \# \mathscr{Z}_p(M\backslash e).$$

Proof. The mapping

$$\varphi : \begin{Bmatrix} \mathscr{C}_e(M) \to \mathscr{Z}_p(M\backslash e) \\ C \mapsto E(M)\backslash C \end{Bmatrix}$$

is bijective.

Theorem 2. *Let M be a binary matroid. Then $\# \mathscr{Z}_p(M)$ is odd.*

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This statement is in general not true for non-binary-matroids, as the uniform matroid $\mathcal{U}_{2,4}$ contains 4 maximal cycles.

Proof. Since the cycles of M form an $\mathbb{F}_2$-vectorspace, either $\mathscr{Z}_p(M) = \{\emptyset\}$ or $\#\mathscr{Z}_p(M) + \#\mathscr{Z}_0(M) + 1 = \#\mathscr{Z}(M) \in 2\mathbb{Z}$. To show $\#\mathscr{Z}_0(M) \in 2\mathbb{Z}$, we count the pairs of maximal cycles C and proper cycles D in C.

$$\mathscr{S} := \{(C,D) \in \mathscr{Z}_p(M) \times \mathscr{Z}_0(M) / D \subseteq C\}.$$

Clearly,

$$\varphi : \begin{Bmatrix} \mathscr{S} \to \mathscr{S} \\ (C,D) \mapsto (C, C \backslash D) \end{Bmatrix}$$

is an involution without fixed points, thus $\#\mathscr{S} \in 2\mathbb{Z}$.

Now it suffices to show that for each $D_0 \in \mathscr{Z}_0(M)$ the set

$$\mathscr{S}_{D_0} := \{(C,D) \in \mathscr{S} / D = D_0\}$$

is odd. Observe that

$$\varphi_{D_0} : \begin{Bmatrix} \mathscr{S}_{D_0} \to \mathscr{Z}_p(M \backslash D_0) \\ (C, D_0) \mapsto C \backslash D_0 \end{Bmatrix}$$

is bijective and $\mathscr{Z}_p(M \backslash D_0)$ is odd by induction, which completes the proof. □

Corollary 3. *Let M be an Eulerian binary matroid and $e \in E(M)$. Then $\#\mathscr{C}_e(M)$ is odd.*

In fact, this corollary is equivalent to the above theorem, since for each binary matroid M there is a (unique) Eulerian binary matroid M' with element set $E(M) \cup \{e\}$ such that $M' \backslash e = M$.

References

[1] D.J.A. Welsh, Matroid Theory (Academic Press, London, 1976).
[2] D.R. Woodall, A proof of McKee's Eulerian-bipartite characterization, Discrete Math. 84 (1990) 217–220.