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The Recursive Sequence $x_{n+1} = g(x_n, x_{n-1})/(A + x_n)$

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Abstract—In this note, we investigate the periodic character of solutions of the nonlinear, second-order difference equation

$$x_{n+1} = \frac{g(x_n, x_{n-1})}{A + x_n},$$

where the parameter A and the initial conditions x_0 and x_1 are positive real numbers. We give sufficient conditions under which every positive solution of this equation converges to a period two solution.

Keywords—Difference equation, Period, Positive solution.

1. INTRODUCTION

In this note, we consider a nonlinear difference equation and deal with the question of whether every solution of that difference equation converges to a period two solution. Recently, there has been a lot of interest in studying the global attractivity, the boundedness character, and the periodic nature of nonlinear difference equations. For some recent results concerning, among other problems, the periodic nature of scalar nonlinear difference equations, see, for example, [1–6]. In [3,7], two similar results were established that can be applied in considerations of nonlinear difference equations for proving that every solution of these difference equations converges to a period two solution. The main theorem in this note is motivated by Conjecture 5.2.5 in [8]. See also [4,5]. In this paper, we consider the following nonlinear difference equation:

$$x_{n+1} = \frac{g(x_n, x_{n-1})}{A + x_n}, \tag{1}$$

where the parameter A and the initial conditions x_0 and x_1 are positive real numbers.

2. MAIN RESULT

Here we formulate and prove the main result of this note.

THEOREM 1. *Let the sequence (x_n) of positive numbers satisfies equation (1). Assume that the function $g : \mathbf{R}_+^2 \to \mathbf{R}_+$ is continuous and satisfies the following functional equation:*

$$g(x, y) - g(y, z) = (x - z)h(x, y, z) - A(x - y), \tag{2}$$

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for some continuous function $h : \mathbf{R}_+^3 \to \mathbf{R}_+$ *such that*

$$\frac{h(x,y,z)}{x} \to 0, \qquad \text{as } x,y,z \to \infty, \tag{3}$$

and

$$\sup_{(x,y,z)\in\mathbf{R}_+^3} \frac{h(x,y,z)}{A+x} = M_h < +\infty. \tag{4}$$

Assume that equation (1) has a period two solution.

Then, every positive solution of equation (1) converges to a period two solution.

PROOF. Subtracting x_{n-1} from the left- and the right-hand side in equation (1), we obtain

$$x_{n+1} - x_{n-1} = \frac{g(x_n, x_{n-1}) - x_{n-1}x_n - Ax_{n-1}}{A + x_n}.$$

Since

$$x_{n-1}x_n = g(x_{n-1}, x_{n-2}) - Ax_n,$$

and by (2), we obtain

$$\begin{aligned} x_{n+1} - x_{n-1} &= \frac{g(x_n, x_{n-1}) - g(x_{n-1}, x_{n-2}) + Ax_n - Ax_{n-1}}{A + x_n} \\ &= \frac{(x_n - x_{n-2})\, h(x_n, x_{n-1}, x_{n-2})}{A + x_n}. \end{aligned}$$

From this, we see that the signum of $x_n - x_{n-2}$ remains invariant for all $n \geq 2$. Also, the following formula:

$$x_{n+1} - x_{n-1} = (x_2 - x_0) \prod_{k=2}^{n} \frac{h(x_k, x_{k-1}, x_{k-2})}{A + x_k} \tag{5}$$

holds.

Thus, the sequences (x_{2n}) and (x_{2n-1}) are monotone. If one of them is nonincreasing, then it must be convergent. Suppose that (x_{2n}) is nondecreasing and unbounded. Then applying (3), we have that for every $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$0 < \frac{h(x_{2k}, x_{2k-1}, x_{2k-2})}{A + x_{2k}} < \varepsilon, \qquad \text{for all } n \geq 2n_0.$$

Let $\varepsilon \in (0, 1/M_h)$. Then from (4) and (5), we have

$$|x_{2n} - x_{2n-2}| \leq |x_2 - x_0|\, \varepsilon^{n-2n_0} M_h^{n+2n_0-2} = \frac{|x_2 - x_0|\, M_h^{2n_0-2}}{\varepsilon^{2n_0}} (\varepsilon M_h)^n.$$

From that, and since $\varepsilon M_h < 1$, we obtain that the sequence (x_{2n}) is bounded, which is a contradiction.

The case when the sequence (x_{2n-1}) is nondecreasing and unbounded is similar and will be omitted. Thus, both sequences (x_{2n}) and (x_{2n-1}) converge, as desired.

3. APPLICATIONS

COROLLARY 1. *Consider the difference equation*

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + f(x_n, x_{n-1})}{A + x_n}, \tag{6}$$

where $\alpha, \beta, \gamma, A \in (0, \infty)$ *and* $f : \mathbf{R}_+^2 \to \mathbf{R}_+$ *is continuous and satisfies the following functional equation:*

$$f(x, y) - f(y, z) = (x - z) h_1(x, y, z), \tag{7}$$

for some continuous function $h_1 : \mathbf{R}_+^3 \to \mathbf{R}_+$ *such that*

$$\frac{h_1(x, y, z)}{x} \to 0, \qquad \text{as } x, y, z \to \infty, \tag{8}$$

and

$$\sup_{(x,y,z) \in \mathbf{R}_+^3} \frac{h_1(x, y, z)}{A + x} < +\infty. \tag{9}$$

Then we have the following.

(a) *The condition* $\gamma = \beta + A$ *is necessary for the existence of a prime period two solution of equation (6).*

(b) *Assume* $\gamma = \beta + A$ *and that equation (6) has a period two solution, then every positive solution of equation (6) converges to a period two solution.*

PROOF.

(a) Let

$$x, y, x, y, x, y \dots$$

be a prime period two solution of equation (6). Then we have

$$x = \frac{\alpha + \beta y + \gamma x + f(y, x)}{A + y} \qquad \text{and} \qquad y = \frac{\alpha + \beta x + \gamma y + f(x, y)}{A + x},$$

i.e.,

$$xy = \alpha + \beta y + (\gamma - A) x + f(y, x) \qquad \text{and} \qquad xy = \alpha + \beta x + (\gamma - A) y + f(x, y). \tag{10}$$

If we set $z = x$ in (7), we obtain that the function $f(x, y)$ is symmetric, i.e., $f(x, y) = f(y, x)$ for all $x, y \in \mathbf{R}_+$. Hence, subtracting the last two equalities, we obtain

$$(\gamma - A - \beta)(x - y) = 0.$$

Since $x \neq y$, we obtain the result.

(b) Set $g(x, y) = \alpha + \beta x + (A + \beta) y + f(x, y)$. Then

$$g(x, y) - g(y, z) = (x - z)(A + \beta + h_1(x, y, z)) - A(x - y).$$

Hence, condition (2) of Theorem 1 is satisfied for $h(x, y, z) = A + \beta + h_1(x, y, z)$. By (8), we have

$$\frac{A + \beta + h_1(x, y, z)}{x} \to 0, \qquad \text{as } x, y, z \to \infty.$$

On the other hand, by (9), we obtain

$$\sup_{(x,y,z) \in \mathbf{R}_+^3} \frac{A + \beta + h_1(x, y, z)}{A + x} \le \frac{A + \beta}{A} + \sup_{(x,y,z) \in \mathbf{R}_+^3} \frac{h_1(x, y, z)}{A + x} < +\infty.$$

Thus, all the conditions of Theorem 1 are satisfied, from which the result follows.

REMARK 1. Note that the problem and the conjecture in [8] are established by setting the function f in Corollary 1 equal to zero. In this case, we obtain the difference equation which was investigated in [5].

EXAMPLE 1. Consider the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + (A+\beta)x_{n-1} + \delta\sqrt{x_n^2 + x_{n-1}^2}}{A + x_n}, \tag{11}$$

where $\alpha, \beta, \delta, A \in (0, \infty)$.

Then every positive solution of equation (11) converges to a period two solution.

It is easy to see that, for all $\alpha, \beta, \delta, A \in (0, \infty)$, system (10) consists of only one equation,

$$xy = \alpha + \beta(x+y) + \delta\sqrt{x^2 + y^2}.$$

It is easy to see that the above equation has a solution on $\mathbf{R}_+^2$. Hence, equation (11) has solutions of period two.

By some calculations, as in Corollary 1, we obtain

$$x_{n+1} - x_{n-1} = (x_n - x_{n-2})\frac{\alpha + \beta + (\delta\,(x_n + x_{n-2}))\,/\left(\sqrt{x_n^2 + x_{n-1}^2} + \sqrt{x_{n-1}^2 + x_{n-2}^2}\right)}{A + x_n}.$$

Since

$$h_1(x,y,z) = \frac{x+z}{\sqrt{x^2+y^2} + \sqrt{y^2+z^2}}$$

is a bounded function on $\mathbf{R}_+^3$, we see that conditions (8) and (9) are satisfied. Hence, the result follows from Corollary 1.

In the same manner, we can consider the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + (A+\beta)x_{n-1} + \delta\sqrt[k]{x_n^k + x_{n-1}^k}}{A + x_n}, \tag{12}$$

where $k \in \mathbf{N}$ and $\alpha, \beta, \delta, A \in (0, \infty)$. Also, we can prove that every positive solution of equation (12) converges to a period two solution.

We finish this note with the following natural question.

QUESTION 1. Describe the class of functions containing all continuous functions $f : \mathbf{R}_+^2 \to \mathbf{R}_+$ which satisfy functional equation (7), for some continuous function $h_1 : \mathbf{R}_+^3 \to \mathbf{R}_+$ which satisfies conditions (8) and (9).

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