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On chromatic roots of large subdivisions of graphs

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Abstract

Given a graph G, we derive an expression for the chromatic polynomials of the graphs resulting from subdividing some (or all) of its edges. For special subfamilies of these, we are able to describe the limits of their chromatic roots. We also prove that for any $\varepsilon>0$, all sufficiently large subdivisions of G have their chromatic roots in $|z-1|<1+\varepsilon$. A consequence of our work will be a characterization of the graphs having a subdivision whose chromatic polynomial has a root with negative real part. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The *chromatic polynomial*, $\pi(G,x)$, of a graph G is the polynomial whose value at each positive integer x is the number of functions $f: V \to \{1,\dots,x\}$ such that $uv\in E$ implies $f(u)\neq f(v)$, where V and E are the sets of vertices and edges of G, respectively. The roots of $\pi(G,x)$ are the *chromatic roots* of G, and in general a chromatic root is any (complex) number which is a root of some chromatic polynomial. The study of chromatic roots has emerged as a rather fascinating topic in its own right, having attracted considerable attention (cf. [2,3,5–8,12,13,16,17]). We are interested here in the chromatic roots of large subdivisions of graphs. If e is an edge of a graph G, then by subdividing e we mean replacing e by a path, the *length* of which is its number of edges. A *subdivision* of G is any graph formed by subdividing (one or more) edges in G. We shall derive an expression for the resulting chromatic polynomials; this expression simplifies considerably in the case of *uniform* subdivisions of G (cf. Section 4). Their chromatic polynomials form what is known as a recursive family of polynomials (cf. Section 2), and we will apply a theorem of Beraha et al. [3] to describe the limits of their roots. We will see that the circle $|z-1|=1$ plays a key role in describing the

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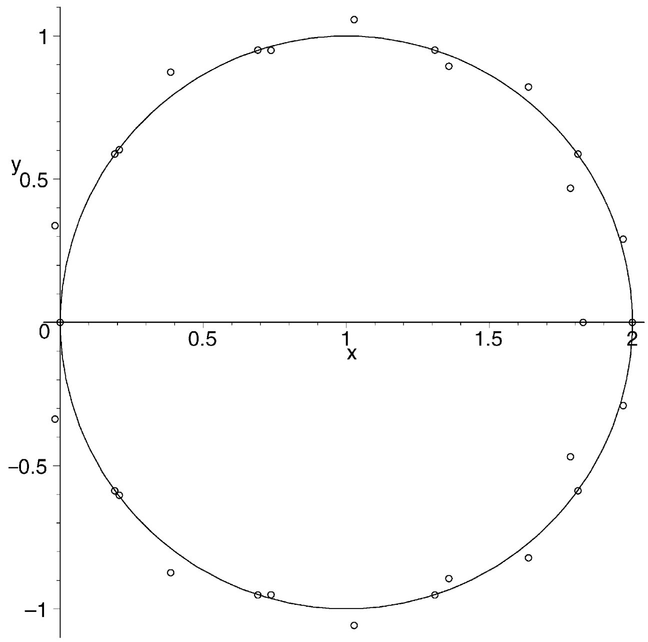

Fig. 1. The chromatic roots of a subdivision of $K_4 - e$ with $|z-1| = 1$.

limits of chromatic roots of uniform subdivisions of a graph, and is itself among those limits (Fig. 1 shows the roots of a subdivision (a theta graph with paths of length 10, 10 and 11) of $K_4 - e$ along with the circle $|z-1| = 1$).

We will derive two interesting consequences of our work here. Firstly, Farrell [10] conjectured in 1980 that there are no chromatic roots with negative real part, but recent results [8,13–15] have provided families of graphs that do have chromatic roots of negative real part. A corollary of our expansion of chromatic polynomials of uniform subdivisions of graphs leads to, in fact, a complete characterization of those graphs which have a subdivision having a chromatic root with negative real part.

Secondly, experimental evidence of chromatic roots of subdivisions leads to the observation that the roots tend to be drawn towards the unit circle centered at $z = 1$. We show that in fact for any $\varepsilon > 0$, the chromatic roots of *all* large subdivisions of a graph have their roots in $|z-1| < 1 + \varepsilon$ (herein improving a recent result [6] which only proves *some* subdivision has its roots in this region).

2. Background: recursive families of polynomials

Before we proceed onto a discussion of the roots of chromatic polynomials of subdivisions of graphs, we need to state (in detail) an analytic results on particular families of polynomials (namely, *recursive families*). We begin with the following definition.

Definition 2.1. If $\{f_n(x)\}$ is a family of (complex) polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $\{f_n(x)\}$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathscr{R}(\{f_n(x)\})$, where $\mathscr{R}(\{f_n(x)\})$ is the union of the roots of the $f_n(x)$.

Now (as in [3]) a family $\{f_n(x)\}$ of polynomials is a *recursive family of polynomials* if the $f_n(x)$ satisfy a homogeneous linear recurrence

$$f_n(x) = \sum_{i=1}^{k} a_i(x) f_{n-i}(x), \tag{1}$$

where the $a_i(x)$ are fixed polynomials, with $a_k(x) \not\equiv 0$. The number k is the *order* of the recurrence.

We can form from (1) its associated *characteristic equation*

$$\lambda^k - a_1(x)\lambda^{k-1} - a_2(x)\lambda^{k-2} - \cdots - a_k(x) = 0, \tag{2}$$

whose roots $\lambda = \lambda(x)$ are algebraic functions, and there are exactly k of them counting multiplicity (cf. [1,11]).

If these roots, say $\lambda_1(x), \lambda_2(x), \ldots, \lambda_k(x)$, are distinct, then the general solution to (1) is known [3] to be

$$f_n(x) = \sum_{i=1}^{k} \alpha_i(x) \lambda_i(x)^n, \tag{3}$$

with the 'usual' variant (cf. [3]) if some of the $\lambda_i(x)$ are repeated. The functions $\alpha_1(x), \alpha_2(x), \ldots, \alpha_k(x)$ are determined from the initial conditions, that is, the k linear equations in the $\alpha_i(x)$ obtained by letting $n = 0, 1, \ldots, k-1$ in (3) or its variant. The details are found in [3].

Beraha et al. [3] proved the result below on recursive families of polynomials and their roots.

Theorem 2.2 (Beraha et al. [3]). *If* $\{f_n(x)\}$ *is a recursive family of polynomials, then a complex number* z *is a limit of roots of* $\{f_n(x)\}$ *if and only if there is a sequence* $\{z_n\}$ *in* $\mathbb{C}$ *such that* $f_n(z_n) = 0$ *for all* n *and* $z_n \to z$ *as* $n \to \infty$.

The main result of their paper characterizes precisely the limits of roots of a recursive family of polynomials.

Theorem 2.3 (Beraha et al. [3]). *Under the non-degeneracy requirements that in* (3) *no* $\alpha_i(x)$ *is identically zero and that for no pair* $i \neq j$ *is it true that* $\lambda_i(x) \equiv \omega\lambda_j(x)$ *for some complex number* ω *of unit modulus, then* $z \in \mathbb{C}$ *is a limit of roots of* $\{f_n(x)\}$ *if and only if either*

(i) *two or more of the* $\lambda_i(z)$ *are of equal modulus, and strictly greater* (*in modulus*) *than the others*; *or*

(ii) *for some j,* $\lambda_j(z)$ *has modulus strictly greater than all the other* $\lambda_i(z)$ *have, and* $\alpha_j(z) = 0$.

This result has found application to the chromatic roots of *recursive families of graphs* [5], that is, families of graphs whose Tutte (and therefore chromatic) polynomials satisfy a homogeneous linear recurrence; see [2,13] for some examples. It is also proved in [3] that the first non-degeneracy requirement in the statement of the theorem is equivalent to $f_n(x)$ satisfying no lower order (homogeneous, linear) recurrence. What we shall need in this paper is the following.

Corollary 2.4. *Suppose* $\{f_n(x)\}$ *is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \cdots + \alpha_k(x)\lambda_k(x)^n, \tag{4}$$

where the $\alpha_i(x)$ *and* $\lambda_i(x)$ *are fixed non-zero polynomials, such that for no pair* $i \neq j$ *is* $\lambda_i(x) \equiv \omega\lambda_j(x)$ *for some* $\omega \in \mathbb{C}$ *of unit modulus. Then the limits of roots of* $\{f_n(x)\}$ *are exactly those z satisfying* (i) *or* (ii) *of Theorem* 2.3.

Proof. It is enough to show that $f_n(x)$ satisfies a kth-order homogeneous linear recurrence, for then Theorem 2.3 applies as the non-degeneracy requirements are satisfied here. Such a recurrence is

$$f_n(x) = a_1(x)f_{n-1}(x) + a_2(x)f_{n-2}(x) + \cdots + a_k(x)f_{n-k}(x) \quad (n \geqslant k)$$

together with the initial polynomials

$$f_j(x) = \sum_{i=1}^{k} \alpha_i(x)\lambda_i(x)^j \quad (j = 0, \ldots, k-1),$$

where the $a_i(x)$ are such that

$$(\lambda - \lambda_1(x)) \cdots (\lambda - \lambda_k(x)) \equiv \lambda^k - a_1(x)\lambda^{k-1} - a_2(x)\lambda^{k-2} - \cdots - a_k(x).$$

This completes the proof. □

In the next section, we derive an expression for the chromatic polynomial of subdivisions of a graph, that when restricted to the uniform case, provides a recursive family of polynomials. The Beraha–Kahane–Weiss Theorem will then allow us to derive some precise information on the limit points of these chromatic roots.

3. An expression for the chromatic polynomials of subdivisions

Before we proceed onto the main chromatic expansion, we will need a few basic facts about chromatic polynomials (all of which are discussed in [4]). Let G be a graph, possibly containing parallel edges or loops. Parallel edges have no effect on the chromatic polynomial, and it follows directly from the definition (cf. Section 1) that if G has a loop then indeed $\pi(G,x) \equiv 0$. The well-known *deletion–contraction* reduction states that for e any edge of G, $\pi(G,x)=\pi(G-e,x)-\pi(G\cdot e,x)$, where $G\cdot e$ is the contraction of e in G, and is obtained by removing e and identifying its end vertices. It is not hard to verify that the formula holds even if e is a parallel edge or loop. We sometimes rewrite the deletion–contraction $\pi(G-e,x)=\pi(G,x)+\pi(G\cdot e,x)$. Also, if G and H intersect exactly on a complete graph, K_p, of order p, then $\pi(G\cup H,x)=\pi(G,x)\pi(H,x)/\pi(K_p,x)$; this is sometimes referred to as the *Complete Cutset Theorem*. Finally, the chromatic polynomials of K_n, T_n (any tree of order n), and C_n (the cycle of order n) are given by $x(x-1)\cdots(x-n+1)$, $x(x-1)^{n-1}$, and $(-1)^n(1-x)((1-x)^{n-1}-1)$, respectively. Now let us assume, for the remainder of the paper, that G is a graph, *without loops*, which may indeed have parallel edges; its vertex and edge sets V and E have cardinalities n and m, the order and size of G, respectively. Also, any parallel edges and/or loops resulting from the contraction of an edge at any time are *not* to be thrown away.

We derive now a rather technical expression of the chromatic polynomial of a general subdivision of a graph. What is crucial is the expansion of this polynomial in terms of powers of $1-x$ and coefficients that depend only on the underlying graph (and *not* the exact subdivision we have taken).

Theorem 3.1. *Let $E'=\{e_1,\ldots,e_k\}\subseteq E$, and $G^{e_1,\ldots,e_k}_{l_1,\ldots,l_k}$ be the graph obtained from G by subdividing edge e_i into a path of length l_i $(i=1,\ldots,k)$. Then*

$$\pi(G^{e_1,\ldots,e_k}_{l_1,\ldots,l_k},x) = \frac{(-1)^{\sum_{i=1}^{k} l_i}}{x^k}\Bigg\{\pi(G-E',x)(1-x)^{\sum_{i=1}^{k} l_i} - \sum_{1\leqslant i_1<\cdots<i_{k-1}\leqslant k} f_{i_1,\ldots,i_{k-1}}(x)(1-x)^{\sum_{j=1}^{k-1} l_{i_j}} + \sum_{1\leqslant i_1<\cdots<i_{k-2}\leqslant k} f_{i_1,\ldots,i_{k-2}}(x)(1-x)^{\sum_{j=1}^{k-2} l_{i_j}} - \cdots + (-1)^{k-1}\sum_{1\leqslant i_1\leqslant k} f_{i_1}(x)(1-x)^{l_{i_1}} + (-1)^k g_{E'}(x)\Bigg\},$$

where

$$\begin{aligned} g_{E'}(x) = {} & \pi(G,x)+(1-x)\sum_{i=1}^{k}\pi(G\cdot e_i,x) \\ & +(1-x)^2\sum_{1\leqslant i_1<i_2\leqslant k}\pi(G\cdot e_{i_1}\cdot e_{i_2},x)+\cdots \\ & +(1-x)^{k-1}\sum_{1\leqslant i_1<\cdots<i_{k-1}\leqslant k}\pi(G\cdot e_1\cdot\cdots\cdot e_{k-1},x) \\ & +(1-x)^k\pi(G\cdot e_1\cdot\cdots\cdot e_k,x) \end{aligned} \tag{5}$$

and the f's (and clearly $g_{E'}$) are polynomials that depend on G and $E' = \{e_1, \ldots, e_k\}$, but not *on $l_1, \ldots, l_k$.*

This can be proved by induction on k. Let us examine the cases $k = 1$ and 2, the latter being sufficiently descriptive of the general argument which is tedious but no more difficult. Because of the degree of symbolism involved, it will be convenient, for the remainder of this section only, to denote the chromatic polynomial $\pi(H, x)$ of a graph H by the symbol H itself, and it will be clear from the context whether we are actually referring to the graph or its chromatic polynomial.

For $k = 1$, we are subdividing a single edge, e, of G into a path of length l, say. Tossing e into the graph G_l^e, and contracting, we have

$$G_l^e = (G_l^e + e) + (G_l^e + e) \cdot e.$$

Now, $G_l^e + e$ creates a cycle of length $l+1$, intersecting G exactly on e, while $(G_l^e + e) \cdot e$ produces a cycle of length l which intersects $G \cdot e$ on a single vertex. Hence,

$$\begin{aligned} G_l^e &= \frac{C_{l+1} \cdot G}{x(x-1)} + \frac{C_l \cdot G \cdot e}{x} \\ &= \frac{(-1)^{l+1}(1-x)((1-x)^l - 1)G}{x(x-1)} + \frac{(-1)^l(1-x)((1-x)^{l-1} - 1)G \cdot e}{x} \\ &= \frac{(-1)^l}{x}\{(G + G \cdot e)(1-x)^l - (G + (1-x)G \cdot e)\}, \end{aligned} \tag{6}$$

which establishes the result for $k = 1$.

Now for $k = 2$, we want an expression for the chromatic polynomial of $G_{l,s}^{e,f}$, the graph resulting from subdividing edges e and f of G into paths of length l and s, respectively. This we derive from the case $k = 1$ (more specifically, from (6)):

$$G_{l,s}^{e,f} = (G_l^e)_s^f = \frac{(-1)^s}{x}\{(G_l^e + G_l^e \cdot f)(1-x)^s - (G_l^e + (1-x)G_l^e \cdot f)\}.$$

It is clear that $G_l^e \cdot f = (G \cdot f)_l^e$. Thus, from (6),

$$\begin{aligned} G_l^e + G_l^e \cdot f = \frac{(-1)^l}{x}\{&(G + G \cdot e + G \cdot f + G \cdot f \cdot e)(1-x)^l \\ &- (G + G \cdot f + (1-x)(G \cdot e + G \cdot f \cdot e))\}, \end{aligned}$$

and

$$\begin{aligned} G_l^e + (1-x)G_l^e \cdot f = \frac{(-1)^l}{x}\{&(G + G \cdot e + (1-x)(G \cdot f + G \cdot e))(1-x)^l \\ &- (G + (1-x)(G \cdot e + G \cdot f) \\ &+ (1-x)^2 G \cdot f \cdot e)\}. \end{aligned}$$

Hence, we have

$$\begin{aligned} G^{e,f}_{l,s} = \frac{(-1)^{s+l}}{x^2}\{&(G+G\cdot e+G\cdot f+G\cdot f\cdot e)(1-x)^{s+l} \\ &-(G+G\cdot f+(1-x)(G\cdot e+G\cdot f\cdot e))(1-x)^s \\ &-(G+G\cdot e+(1-x)(G\cdot f+G\cdot e))(1-x)^l \\ &+(G+(1-x)(G\cdot e+G\cdot f)+(1-x)^2 G\cdot f\cdot e)\}. \end{aligned}$$

The 'coefficient' of $(1-x)^{s+l}$ above is exactly $G-e-f$, as $G+G\cdot e=G-e$ and $G\cdot f+G\cdot f\cdot e=(G\cdot f)-e$, and clearly $(G\cdot f)-e=(G-e)\cdot f$, giving

$$\begin{aligned} G+G\cdot e+G\cdot f+G\cdot f\cdot e &= (G-e)+(G-e)\cdot f \\ &= (G-e)-f. \end{aligned}$$

This establishes the case $k=2$ from the previous case ($k=1$). □

We will see that Theorem 3.1 has some deep consequences in terms of the location of chromatic roots, especially when restricted to the natural subfamily of uniform subdivisions.

4. Uniform subdivisions and the limits of their chromatic roots

When subdividing each edge of E' the same number of times, Theorem 3.1 specializes to the following.

Theorem 4.1. *Let $E'=\{e_1,\ldots,e_k\}\subseteq E$, and $G^{E'}_l$ the graph obtained from G by subdividing each edge of E' into a path of length l. Then*

$$\begin{aligned} \pi(G^{E'}_l,x) = \frac{(-1)^{kl}}{x^k}\{&\pi(G-E',x)(1-x)^{kl} - f_{k-1}(x)(1-x)^{(k-1)l} \\ &+f_{k-2}(x)(1-x)^{(k-2)l} - \cdots + (-1)^{k-1}f_1(x)(1-x)^l + (-1)^k g_{E'}(x)\}, \end{aligned} \tag{7}$$

where $g_{E'}$ is given by (5), *and the f's* (*again*) *are polynomials that depend on G* (*and E'*) *but* not *on l.*

The key point is that expansion (7) expresses $\pi(G^{E'}_l,x)$ as a recursive family, and hence we can employ the power of the Beraha–Kahane–Weiss Theorem. In doing so, we find *all* of the limit points of the chromatic roots of the uniform subdivisions $\{G^{E'}_l : l\geqslant 1\}$ of G (without explicitly finding the chromatic roots of each of these graphs!).

We need yet one bit of technical notation; $\tilde{E}'$ will denote the edges of E' that are not bridges.

Theorem 4.2. *If E' is a subset of E containing at least one edge that is not a bridge of G, then the limits of the chromatic roots of the family $\{G_l^{E'}\}$ are exactly*

(i) *the circle $|z-1|=1$,*
(ii) *the roots of $\pi(G-\tilde{E}',x)$ outside $|z-1|=1$, and*
(iii) *the roots of $g_{\tilde{E}'}(x)$ inside $|z-1|=1$.*

Proof. Let us first examine the case where E' contains *no* bridges, in which case $\tilde{E}'=E'$. Clearly, $\pi(G-E',x)$ is not identically zero. And neither is $g_{E'}(x)$, for suppose $g_{E'}(x)\equiv 0$. Then, from (7), we would have that $(1-x)^l$ divides $G_l^{E'}$. However, it is well known (cf. [17]) that the multiplicity of 1 as a chromatic root of a graph is the number of blocks in the graph. Since E' contains no bridges, for each l the graph $G_l^{E'}$ has the same number of blocks as G, so that the multiplicity of 1 as a chromatic root of $G_l^{E'}$ cannot possibly go to infinity with l, a contradiction.

Now ignoring the factor $(-1)^{kl}/x^k$ in (7) and rewriting $(-1)^k g_{E'}(x)$ as $(-1)^k g_{E'}(x)1^l$, we can apply Corollary 2.4, and therefore (i) and (ii) of Theorem 2.3, to get the limits of the chromatic roots of $\{G_l^{E'}\}$.

Applying (i) of Theorem 2.3, note that we immediately get the circle $|z-1|=1$ as limits, by setting all of the $|\lambda_i(z)|$ equal, i.e.,

$$|1-z|^k=|1-z|^{k-1}=\cdots=|1-z|=1.$$

And this is the only situation where (i) of Theorem 2.3 gives any limits, for setting any fewer than all of the $|\lambda_i(z)|$ equal here will, upon applying (i), amount to finding values z such that $|z-1|=1$ and $|z-1|>1$, which is impossible.

Moving on to (ii) of Theorem 2.3, one application gives the roots z of $\pi(G-E',x)$ such that

$$|1-z|^k>|1-z|^i \quad \text{for all } i=0,1,\ldots,k-1,$$

that is, the roots z of $\pi(G-E',x)$ such that $|1-z|>1$. Another application of (ii) gives the roots z of $g_{E'}$ such that

$$1>|1-z|^i \quad \text{for all } i=1,2,\ldots,k,$$

that is, the roots z of $g_{E'}$ such that $|1-z|<1$.

Finally, applying (ii) to any j such that $0<j<k$ would amount to finding roots z of f_j such that $|1-z|<1$ and $|1-z|>1$, which is clearly impossible.

Now for the case where E' does contain some bridges, note that subdividing a bridge $e\in E'$ is merely the replacement of a bridge by a path, from which it follows (quite easily, using the complete cutset theorem) that $\pi(G_l^{E'},x)=(x-1)^{l-1}\pi(G_l^{E'-e},x)$. Repeating this argument recursively over all the bridges in E', we have that the limits of the chromatic roots of $\{G_l^{E'}\}$ are exactly those of $\{G_l^{\tilde{E}'}\}$, from which the result follows from the previous case. This completes the proof. □

When $E'=E$, it is clear that $G-\tilde{E}'$ is a forest, therefore having no chromatic roots outside $|z-1|=1$. Also, we write $\tilde{g}(x)$ instead of $g_{\tilde{E}}(x)$.

Corollary 4.3. *Let G_l be the graph obtained from G by subdividing each edge into a path of length exactly l. Then, if G is not a forest, the limits of the chromatic roots of the family G_l are exactly*:

(i) *the circle* $|z-1|=1$,
(ii) *the roots of* $\tilde{g}(x)$ *inside* $|z-1|=1$.

Based on direct calculation for various small graphs, it appears that (ii) of Corollary 4.3 simply never happens, except of course for the point $z=1$ which is clearly a root of $\tilde{g}(x)$. In fact, we conjecture that if z is a root of $\tilde{g}(x)$, then $|z-1|$ is either 0 or 1.

5. Application I: chromatic roots with negative real part

Our first application of our investigation of subdivisions concerns chromatic roots with negative real part.

Very little is known about the chromatic roots lying in the left-half plane. It was conjectured [10] in 1980 that in fact there are none at all. However, Read and Royle [13] showed recently by direct calculation with cubic graphs that they do exist. Independently, in [14,15,8] the existence of *infinitely* many chromatic roots with negative real part was demonstrated. In particular, in [8] it was shown that the graph $\Theta_{a,a,a}$ has a chromatic root with negative real part for each $a \geqslant 8$, and that the moduli of these roots get arbitrarily small (the symbol $\Theta_{a,b,c}$ refers to the graph consisting of two vertices (called *terminals*) joined by three internally disjoint paths of lengths a, b, and c, and is called a *generalized theta graph*).

Of course, theta graphs are very specific graphs. But it turns out they are the key ingredient in characterizing the graphs in general which have a *subdivision* having a chromatic root with negative real part. It is from Theorem 4.2 that we are able to make this connection, and is perhaps a bit surprising that indeed most graphs have a subdivision having a chromatic root with negative real part. We say that G has a Θ-subgraph if G has a subgraph isomorphic to some generalized theta graph. The *co-rank* (or *cycle rank*) of a graph $H(V,E)$ is $|E|-|V|+c$, where c is the number of components of H.

Theorem 5.1. *The following are equivalent.*

(i) *G has a subdivision having a chromatic root with negative real part*,
(ii) *G has a block of co-rank at least* 2,
(iii) *G has a Θ-subgraph.*

Proof. We prove the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). It is easy to see that connected graphs of co-rank 0 are trees, and of co-rank 1 are cycles. Thus, if no block of G has co-rank larger than 1, then the blocks of G are simply bridges and cycles. Hence, the blocks of any subdivision of G are

K_2's and cycles as well. And since neither K_2 nor any cycle has a chromatic root with negative real part, neither do the subdivisions of G.

(ii) $\Rightarrow$ (iii). Without loss of generality, assume that G is 2-connected and has co-rank at least 2. Let e be an edge of G. Clearly, e is not a bridge, and therefore lies on a cycle, say C_1, of G. Now G must have an edge, say e', other than those of C_1 (or else $G = C_1$ and G has co-rank 1). Since G is 2-connected, e and e' both belong to a cycle $C_2(\neq C_1)$ and $e \in C_1 \cap C_2$. Now, C_1 together with a component of $C_1 \oplus C_2$ (the symmetric difference) is a θ-subgraph of G, as required.

(iii) $\Rightarrow$ (i). It easy to see (from the complete cutset theorem) that it is enough to show the result holds for 2-connected graphs. So we assume G is 2-connected. Start with a Θ-subgraph of G, and subdivide it into $\Theta_{a,a,a}$ for some fixed $a \geqslant 8$, obtaining a subdivision H of G containing as a subgraph $H_0 = \Theta_{a,a,a}$. Now, let E' be the edges of H that do not lie on H_0. Then $H - E'$ consists exactly of H_0 and possibly some isolated vertices, a graph which we know [8] has a chromatic root with negative real part. Thus if E' is empty, then we are done. And if not, then consider the family $\{H_l^{E'}\}$. Since E' has no bridges and $H - E'$ has a chromatic root z with negative real part, then by (ii) of Theorem 4.2, z will be a limit of the chromatic roots of $\{H_l^{E'}\}$. The $H_l^{E'}$'s are indeed subdivisions of G, and it follows that for l sufficiently large, $H_l^{E'}$ has a chromatic root with negative real part. □

From this theorem, we immediately deduce the following.

Corollary 5.2. *Every non-empty graph has a series–parallel extension having a chromatic root with negative real part.*

We can sometimes make use of subdivisions to generate, from a single chromatic root with negative real part, an infinite cluster of such chromatic roots. For suppose e is an edge of G which is not a bridge, and that $G - e$ has a chromatic root z with negative real part. Then, from (ii) of Theorem 4.2, z is a limit of the chromatic roots of the family $\{G_l^e\}$. If, in addition, z is *not* a chromatic root of $G \cdot e$, then, from (6), z will not be a chromatic root of any G_l^e, and so in fact must be a *limit point* of the chromatic roots of the family $\{G_l^e\}$. Together with Theorem 2.2, we find a sequence $\{z_l\}$ in $\mathbb{C}\backslash\{z\}$ such that $\pi(G_l^e, z_l) = 0$ and $z_l \to z$.

Fix any $a \geqslant 8$, for instance, and let $F = \Theta_{a,a,a} + uv$, where u and v are the terminals of $\Theta_{a,a,a}$. Then $F - uv = \Theta_{a,a,a}$, which we know [8] has a chromatic root z with negative real part, while $F \cdot uv$ is just three cycles intersecting on a single vertex, a graph having no chromatic roots with negative real part whatsoever, as the chromatic roots of cycles lie on the disk $|z-1| = 1$. Thus, $\{F_l^{uv}\}$ is a family of graphs producing infinitely many chromatic roots with negative real part.

6. Application II: bounding the chromatic roots of large subdivisions

Our second application of our investigation of chromatic roots of subdivisions centers on the location of the roots of *large* subdivisions of a graph. A few plots of the chromatic roots of subdivisions of several small graph will indicate that subdividing edges tends to draw the chromatic roots closer to the disk $|z-1| \leqslant 1$. It was shown in [7] that *co-rank* is an upper bound for $|z-1|$ where z is any chromatic root of the graph. The roots of any subdivision of G, therefore, are bounded by 1 plus the co-rank of G. However, more is true; in [6] it was proven that for any $\varepsilon > 0$, there is *some* subdivision of G having all its chromatic roots in $|z-1| < 1+\varepsilon$. This belies the empirical evidence that *all* large subdivisions have its chromatic roots in the salient disc. Our results here are indeed strong enough to prove this fact.

Theorem 6.1. *For any $\varepsilon > 0$, there is an $L = L(G, \varepsilon)$ such that, if we subdivide each edge of G into a path of length at least L, then all chromatic roots of the resulting graph G' lie in $|z-1| < 1+\varepsilon$.*

Proof. Let $\varepsilon > 0$ be given. Suppose $E = \{e_1, \dots, e_m\}$ are the edges of G, and that we subdivide edge e_i into a path of length l_i, $i = 1, \dots, m$. We obtain a graph $G^{e_1,\dots,e_m}_{l_1,\dots,l_m}$, whose chromatic polynomial, by Theorem 3.1, is $\pi(G^{e_1,\dots,e_m}_{l_1,\dots,l_m}, x) = ((-1)^{\sum_{i=1}^{k} l_i}/x^m)\mathscr{F}_{l_1,\dots,l_m}(x)$, where

$$
\begin{aligned}
\mathscr{F}_{l_1,\dots,l_m}(x) = {} & \pi(G-E,x)(1-x)^{\sum_{i=1}^{m} l_i} \\
& - \sum_{1 \leqslant i_1 < \cdots < i_{m-1} \leqslant k} f_{i_1,\dots,i_{m-1}}(x)(1-x)^{\sum_{j=1}^{m-1} l_{i_j}} \\
& + \sum_{1 \leqslant i_1 < \cdots < i_{m-2} \leqslant k} f_{i_1,\dots,i_{m-2}}(x)(1-x)^{\sum_{j=1}^{m-2} l_{i_j}} - \cdots \\
& + (-1)^{m-1} \sum_{1 \leqslant i_1 \leqslant m} f_{i_1}(x)(1-x)^{l_{i_1}} + (-1)^m g_E(x)
\end{aligned}
$$

is the expression (in braces) in Theorem 3.1 (with k replaced by m).

As we remarked earlier, the roots of $\pi(G^{e_1,\dots,e_m}_{l_1,\dots,l_m}, x)$, and therefore of $\mathscr{F}_{l_1,\dots,l_m}$, are bounded by the co-rank μ of G. Let $C = C(G, \varepsilon) > 0$ be a bound for the maximum modulus of the f's and g_E on $1+\varepsilon \leqslant |1-z| \leqslant \mu$. Choose $L > 0$ large enough that $(\varepsilon^n(1+\varepsilon)^L)/C > 2^m - 1$. Suppose that $l_1, \dots, l_m$ are all larger than L, and, without loss of generality, $l_1 \leqslant l_2 \leqslant \cdots \leqslant l_m$. Let z be such that $1+\varepsilon \leqslant |1-z| \leqslant \mu$; we will show that $|\mathscr{F}_{l_1,\dots,l_m}(z)| > 0$.

To that end, note that $\pi(G-E, z) = z^n$, whose modulus is at least ε^n. Set $y = |1-z|$. Then, by the triangle inequality,

$$
\begin{aligned}
|\mathscr{F}_{l_1,\dots,l_m}(z)| \geqslant {} & |z|^n \, y^{\sum_{i=1}^{m} l_i} - \sum_{1 \leqslant i_1 < \cdots < i_{m-1} \leqslant m} |f_{i_1,\dots,i_{k-1}}(z)| \, y^{\sum_{j=1}^{m-1} l_{i_j}} \\
& - \sum_{1 \leqslant i_1 < \cdots < i_{m-2} \leqslant m} |f_{i_1,\dots,i_{k-2}}(z)| \, y^{\sum_{j=1}^{m-2} l_{i_j}} - \cdots
\end{aligned}
$$

$$- \sum_{1 \leqslant i_1 < i_2 \leqslant m} |f_{i_1,i_2}(z)|\, y^{l_{i_1}+l_{i_2}} - \sum_{1 \leqslant i_1 < m} |f_{i_1}(z)|\, y^{l_{i_1}} - |g_E(z)|$$

$$\geqslant \varepsilon^n\, y^{\sum_{i=1}^{m} l_i} - \sum_{1 \leqslant i_1 < \cdots < i_{m-1} \leqslant m} C\, y^{\sum_{j=1}^{m-1} l_{i_j}}$$

$$- \sum_{1 \leqslant i_1 < \cdots < i_{m-2} \leqslant m} C\, y^{\sum_{j=1}^{m-2} l_{i_j}} - \cdots - \sum_{1 \leqslant i_1 < i_2 \leqslant m} C\, y^{l_{i_1}+l_{i_2}}$$

$$- \sum_{1 \leqslant i_1 < m} C\, y^{l_{i_1}} - C.$$

Note that on the right-hand side of the above there are exactly $2^m - 1$ terms in y (without combining any terms of like degree). We rewrite the expression as

$$C\left(\frac{\varepsilon^n}{C} y^{n_1} - y^{n_2} - \cdots - y^{n_p} - 1\right),$$

where $p = 2^m - 1$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_p$. In particular, $n_1 = \sum_{i=1}^{m} l_i$ and $n_2 = \sum_{i=2}^{m} l_i$. Then

$$\begin{aligned}
\frac{|\mathscr{F}_{l_1,\ldots,l_m}(z)|}{C} &\geqslant \frac{\varepsilon^n}{C} y^{n_1} - y^{n_2} - \cdots - y^{n_p} - 1 \\
&= y^{n_p}\left(\frac{\varepsilon^n}{C} y^{n_1-n_p} - y^{n_2-n_p} - \cdots - y^{n_{p-1}-n_p} - 1\right) - 1 \\
&= y^{n_p}\left(y^{n_{p-1}-n_p}\left(\frac{\varepsilon^n}{C} y^{n_1-n_{p-1}} - y^{n_2-n_{p-1}} - \cdots \right.\right. \\
&\qquad \left.\left. - y^{n_{p-2}-n_{p-1}} - 1\right) - 1\right) - 1 \\
&\vdots \\
&= y^{n_p}\left(y^{n_{p-1}-n_p}\left(y^{n_{p-2}-n_{p-1}} \cdots y^{n_3-n_4}(y^{n_2-n_3}(\frac{\varepsilon^n}{C} y^{n_1-n_2} - 1)\right.\right. \\
&\qquad \left.\left. -1)\cdots - 1) - 1\right) - 1.
\end{aligned}$$

Now, $n_1 - n_2 = l_1 \geqslant L$ and $y \geqslant 1 + \varepsilon > 1$. Hence,

$$\begin{aligned}
\frac{|\mathscr{F}_{l_1,\ldots,l_m}(z)|}{C} &> \frac{\varepsilon^n}{C}(1+\varepsilon)^L \underbrace{-1 - 1 \cdots - 1 - 1 - 1}_{p \text{ such}} \\
&= \underbrace{\frac{\varepsilon^n}{C}(1+\varepsilon)^L}_{>p} - p \\
&> 0,
\end{aligned}$$

which completes the proof. □

Direct calculations with generalized theta graphs suggests that this may be only half of the story. More specifically, we conjecture that the region $|z-1|<1+\varepsilon$ in Theorem 6.1 can be replaced by $\{z\in\mathbb{C}: 1-\varepsilon<|z-1|<1+\varepsilon\}\cup\{1\}$.

Returning to the proof of the theorem, note that all we really needed of $G-E$ was the fact that it has no roots on $1<|z-1|<\mu$, for then we knew it is bounded away from zero on $1+\varepsilon\leqslant|z-1|\leqslant\mu$ for any fixed $\varepsilon>0$. So in fact any subset E' of E for which every root of $G-E'$ lies in $|z-1|\leqslant 1$ will do. Conversely, if E' is a subset of edges such that $G-E'$ has a root on $|z-1|>1$, then by (ii) of Theorem 4.2 and the remark immediately following the theorem, there will certainly be an $\varepsilon>0$ and an l such that $G_l^{E'}$ has a root on $|z-1|\geqslant 1+\varepsilon$. Hence, in the statement of Theorem 6.1, we can restrict to subdividing only edges in E' if and only if the graph $G-E'$ has all its chromatic roots in $|z-1|\leqslant 1$.

7. Concluding remarks

While we were interested here in large subdivisions, we ask whether subdividing each edge of G at least once is enough to guarantee that there are no *real* roots to the right of 2. This can be established for generalized theta graphs (cf. [9]), and what we can prove in general is the following.

Theorem 7.1. *If we subdivide each edge of G into a path of* even *length, then no real chromatic root of the resulting graph is* 2 *or more.*

Proof. For convenience, we will denote the chromatic polynomial $\pi(H,x)$ of a graph H by the symbol H itself. We argue by induction on the number m of edges in a graph. For $m=1$, the result is clear. Now, let $m\geqslant 2$ and suppose the result holds for all graphs of size at most $m-1$. Let G be a graph with m edges $e_1,\ldots,e_m$. Subdividing e_i into a path of even length l_i $(i=1,\ldots,m)$, we obtain a graph $G_{l_1,\ldots,l_m}^{e_1,\ldots,e_m}$, whose chromatic polynomial, by Eq. (6), is given by

$$
\begin{aligned}
G_{l_1,\ldots,l_m}^{e_1,\ldots,e_m} &= (G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}})_{l_m}^{e_m}\\
&= \frac{(-1)^{l_m}}{x}\Bigg(\underbrace{(G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}} + G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}}\cdot e_m)}_{G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}}-e_m}(1-x)^{l_m}\\
&\quad -\underbrace{(G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}} + (1-x)\,G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}}\cdot e_m)}_{(G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}}-e_m)-x\,G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}}\cdot e_m}\Bigg)\\
&= \frac{(-1)^{l_m}}{x}((G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}}-e_m)((1-x)^{l_m}-1)+x(G_{l_1,\ldots,l_{m-1}}^{e_1,\ldots,e_{m-1}}\cdot e_m)),
\end{aligned}
$$

and for $x \geqslant 2$, this is indeed positive, as $G^{e_1,\dots,e_{m-1}}_{l_1,\dots,l_{m-1}} - e_m = (G - e_m)^{e_1,\dots,e_{m-1}}_{l_1,\dots,l_{m-1}}$ and $G^{e_1,\dots,e_{m-1}}_{l_1,\dots,l_{m-1}} \cdot e_m = (G \cdot e_m)^{e_1,\dots,e_{m-1}}_{l_1,\dots,l_{m-1}}$ are both positive, by assumption, and $(1-x)^{l_m} - 1$ is positive since l_m is even. □

The number 2 here is best possible, for consider the graph $\Theta_{1,1,1}$. Among its even subdivisions (in the sense of Theorem 7.1) are the graphs $\Theta_{2l,2l,2l}$ ($l \geqslant 1$), whose chromatic polynomials are given (cf. [8]) by

$$\pi(\Theta_{2l,2l,2l}, x) = \frac{1-x}{x}((1-x)^{6l-1} - 3(x-1)^{2l} + 2 - x).$$

With this expression, we can verify that, for any given $\varepsilon > 0$, the graph $\Theta_{2l,2l,2l}$ will have a real chromatic root between $2 - \varepsilon$ and 2 for l sufficiently large.

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