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Armendariz Rings and Reduced Rings

Nam Kyun Kim*

and

Yang Lee[†]

Department of Mathematics Education, Pusan National University, Pusan 609-735, Korea

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1. INTRODUCTION

Throughout this paper, all rings are associative with identity. Given a ring R, the polynomial ring over R is denoted by $R[x]$. This paper concerns the relationships between Armendariz rings and reduced rings, being motivated by the results in [1, 2, 7]. The study of Armendariz rings, which is related to polynomial rings, was initiated by Armendariz [2] and Rege and Chhawchharia [7]. A ring R is called *Armendariz* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. (The converse is obviously true.) A ring is called *reduced* if it has no nonzero nilpotent elements. Reduced rings are Armendariz by [2, Lemma 1] and subrings of Armendariz rings are also Armendariz obviously. We emphasize the connections among Armendariz rings, reduced rings, and classical quotient rings. Moreover several examples and counterexamples are included for answers to questions that occur naturally in the process of this paper.

*E-mail: nkkim@nms.kyunghee.ac.kr.
[†]E-mail: ylee@hyowon.pusan.ac.kr.

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2. ARMENDARIZ RINGS

First we consider some examples and counterexamples for Armendariz rings. Rege and Chhawchharia [7] showed that every n-by-n full matrix ring over any ring is not Armendariz, where $n \geq 2$. We have a similar result in the following.

EXAMPLE 1. Let R be a ring. We claim that n-by-n upper triangular matrix rings over R are not Armendariz, where $n \geq 2$. It is enough to show that the 2-by-2 upper triangular matrix ring over R is not Armendariz because each subring of an Armendariz ring is also Armendariz. Let S be the 2-by-2 upper triangular matrix ring over R, and let $f(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)x$ and $g(x) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)x$ be polynomials in $S[x]$. Then $f(x)g(x) = 0$, but $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right) \neq 0$. So S is not Armendariz and consequently every n-by-n upper triangular matrix ring over R is not Armendariz.

But we may find subrings of the 3-by-3 upper triangular matrix rings which may be Armendariz as in the following.

PROPOSITION 2. *Let R be a reduced ring. Then*

$$S = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in R \right\}$$

is an Armendariz ring.

Proof. We employ the method in the proof of [7, Proposition 2.5]. First notice that for

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & a_1 & d_1 \\ 0 & 0 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 & c_2 \\ 0 & a_2 & d_2 \\ 0 & 0 & a_2 \end{pmatrix} \in S$$

we can denote their addition and multiplication by

$$(a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2)$$

and

$$\begin{aligned}(a_1, b_1, c_1, d_1)&(a_2, b_2, c_2, d_2) \\ &= (a_1a_2, a_1b_2 + b_1a_2, a_1c_2 + b_1d_2 + c_1a_2, a_1d_2 + d_1a_2),\end{aligned}$$

respectively. So every polynomial in $S[x]$ can be expressed in the form $(p_0(x), p_1(x), p_2(x), p_3(x))$ for some $p_i(x)$'s in $R[x]$.

Let $f(x) = (f_0(x), f_1(x), f_2(x), f_3(x))$ and $g(x) = (g_0(x), g_1(x), g_2(x), g_3(x))$ be elements of $S[x]$. Assume that $f(x)g(x) = 0$. Then $f(x)g(x) = (f_0(x)g_0(x), f_0(x)g_1(x) + f_1(x)g_0(x), f_0(x)g_2(x) + f_1(x)g_3(x) + f_2(x)g_0(x), f_0(x)g_3(x) + f_3(x)g_0(x)) = 0$. So we have the following system of equations:

$$f_0(x)g_0(x) = 0; \tag{0}$$

$$f_0(x)g_1(x) + f_1(x)g_0(x) = 0; \tag{1}$$

$$f_0(x)g_2(x) + f_1(x)g_3(x) + f_2(x)g_0(x) = 0; \tag{2}$$

$$f_0(x)g_3(x) + f_3(x)g_0(x) = 0. \tag{3}$$

Use the fact that $R[x]$ is reduced. From Eq. (0), we see that $g_0(x)f_0(x) = 0$. If we multiply Eq. (1) on the right side by $f_0(x)$, then $f_0(x)g_1(x)f_0(x) + f_1(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_1(x) = 0$ and hence $f_1(x)g_0(x) = 0$. Also if we multiply Eq. (3) on the right side by $f_0(x)$, then $f_0(x)g_3(x)f_0(x) + f_3(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_3(x) = 0$ and hence $f_3(x)g_0(x) = 0$. Now if we multiply Eq. (2) on the right side by $f_0(x)$, then $f_0(x)g_2(x)f_0(x) + f_1(x)g_3(x)f_0(x) + f_2(x)g_0(x)f_0(x) = 0$. So $f_0(x)g_2(x) = 0$ and hence Eq. (2) becomes

$$f_1(x)g_3(x) + f_2(x)g_0(x) = 0. \tag{3'}$$

If we multiply Eq. (3′) on the right side by $f_1(x)$, then we have $f_1(x)g_3(x) = 0$ and so $f_2(x)g_0(x) = 0$.

Now let

$$f(x) = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} x^i \quad \text{and} \quad g(x) = \sum_{j=0}^{m} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} x^j,$$

where $f_0(x) = \sum_{i=0}^{n} a_i x^i$, $f_1(x) = \sum_{i=0}^{n} b_i x^i$, $f_2(x) = \sum_{i=0}^{n} c_i x^i$, $f_3(x) = \sum_{i=0}^{n} d_i x^i$, $g_0(x) = \sum_{j=0}^{m} a'_j x^j$, $g_1(x) = \sum_{j=0}^{m} b'_j x^j$, $g_2(x) = \sum_{j=0}^{m} c'_j x^j$, and $g_3(x) = \sum_{j=0}^{m} d'_j x^j$. Then we obtain that $a_i a'_j = 0$, $a_i b'_j = 0$, $b_i a'_j = 0$, $a_i c'_j = 0$, $b_i d'_j = 0$, $c_i a'_j = 0$, $a_i d'_j = 0$, and $d_i a'_j = 0$ for all i, j by the preceding results, the condition that R is reduced, and [2, Lemma 1]. Consequently

$$\begin{pmatrix} a_i & b_i & c_i \\ 0 & a_i & d_i \\ 0 & 0 & a_i \end{pmatrix} \begin{pmatrix} a'_j & b'_j & c'_j \\ 0 & a'_j & d'_j \\ 0 & 0 & a'_j \end{pmatrix} = 0$$

for all i, j and therefore S is an Armendariz ring.

Let S be a reduced ring and let

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in S \right\}.$$

Based on Proposition 2, one may suspect that R_n may be also an Armendariz ring for $n \geq 4$. But the following example erases the possibility.

EXAMPLE 3. Let S be a ring and let

$$R_4 = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & a_{14} \\ 0 & a & a_{23} & a_{24} \\ 0 & 0 & a & a_{34} \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in S \right\}.$$

Let

$$f(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$$

and

$$g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} x$$

be polynomials in $R_4[x]$. Then $f(x)g(x) = 0$, but

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$$

So R_4 is not Armendariz. Similarly, for the case of $n \geq 5$, we have the same result.

Given a ring R and a bimodule ${}_R M_R$, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

COROLLARY 4 [7, Proposition 2.5]. *Let R be a reduced ring. Then the trivial extension $T(R, R)$ is an Armendariz ring.*

Proof. Notice that $T(R, R)$ is isomorphic to

$$U = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b \in R \right\}$$

and that each subring of an Armendariz ring is also Armendariz. Thus $T(R, R)$ is an Armendariz ring by Proposition 2.

From Corollary 4, one may suspect that if R is Armendariz then the trivial extension $T(R, R)$ is Armendariz. But the following example eliminates the possibility.

EXAMPLE 5. Let T be a reduced ring. Then $R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in T\}$ is an Armendariz ring by Corollary 4. Let S $= \{\left(\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix}\right) \mid A, B \in R\}$ and let

$$f(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$$

and

$$g(x) = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} x$$

be polynomials in $S[x]$. Then $f(x)g(x) = 0$, but

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \neq 0.$$

Thus S is not Armendariz.

By Anderson and Camillo [1, Theorem 2], polynomial rings over Armendariz rings are also Armendariz. So we may conjecture that skew polynomial rings over Armendariz rings are also Armendariz. Recall that for a ring R with an endomorphism α of R and an α-derivation δ of R, the *Ore extension* of R, denoted by $R[x; \alpha, \delta]$, is the ring obtained by giving the polynomial ring over R with the new multiplication $xr = \alpha(r) + \delta(r)$ for all $r \in R$. When $\delta = 0$, we write $R[x; \alpha]$ for $R[x; \alpha, 0]$ and call it a *skew polynomial ring* (also called an *Ore extension of endomorphism type*). However there exists an Armendariz ring R over which the skew polynomial ring is not an Armendariz ring as in the following.

EXAMPLE 6. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider the ring $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Let $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then R is a commutative reduced ring; hence R is Armendariz by [2, Lemma 1]. Now let $\alpha\colon R \to R$ be defined by $\alpha((a, b)) = (b, a)$. Then α is an automorphism of R. We claim that $R[x; \alpha]$ is not Armendariz. Let $f(y) = (1,0) + [(1,0)x]y$ and $g(y) = (0,1) + [(1,0)x]y$ be elements in $R[x; \alpha][y]$. Then $f(y)g(y) = 0$, but $(1,0)[(1,0)x] \neq 0$. Therefore $R[x; \alpha]$ is not an Armendariz ring.

Armendariz obtained some results on reduced rings in [2]. We here obtain the same results on Armendariz rings. A ring is called *abelian* if every idempotent of it is central.

LEMMA 7. *Armendariz rings are abelian.*

Proof. By the method in the proof of [1, Theorem 6].

Given a ring R, the formal power series ring over R is denoted by $R[[x]]$.

LEMMA 8. *Suppose that a ring R is abelian. Then we have the following*:

(1) *Every idempotent of $R[x]$ is in R and $R[x]$ is abelian.*

(2) *Every idempotent of $R[[x]]$ is in R and $R[[x]]$ is abelian.*

Proof. $R[x]$ is a subring of $R[[x]]$ and so it is enough to prove (2). For $f \in R[[x]]$, assume that $f^2 = f$, where $f = e_0 + e_1 x + \cdots + e_n x^n + \cdots$. Then we have the system of equations

$$e_0^2 = e_0; \tag{0}$$

$$e_0 e_1 + e_1 e_0 = e_1; \tag{1}$$

$$e_0 e_2 + e_1 e_1 + e_2 e_0 = e_2; \tag{2}$$

$$\ldots;$$

$$e_0 e_n + e_1 e_{n-1} + \cdots + e_n e_0 = e_n; \tag{n}$$

$$\ldots .$$

Observing that Eq. (0) yields that e_0 is an idempotent, so it is central. If we multiply Eq. (1) on the left side by e_0, then $e_0e_1 + e_0e_1e_0 = e_0e_1$. But $e_0e_1e_0 = e_0e_1$ because e_0 is central. So $e_0e_1 = 0$ and so $e_1 = 0$; hence Eq. (2) becomes $e_0e_2 + e_2e_0 = e_2$. If we multiply Eq. (2) on the left side by e_0, then $e_0e_2 + e_0e_2e_0 = e_0e_2$. But $e_0e_2e_0 = e_0e_2$. Hence $e_0e_2 = 0$ and so $e_2 = 0$. Now assume that k is a positive integer such that $e_i = 0$ for all $1 \leq i \leq k$. Then equation $(k+1)$ becomes $e_0e_{k+1} + e_{k+1}e_0 = e_{k+1}$, and so $e_{k+1} = 0$ by the same method. Therefore $f = e_0 \in R$ and also $R[[x]]$ is abelian.

By Kaplansky [3], a ring R is called *Baer* if the right annihilator of every nonempty subset of R is generated by an idempotent. Closely related to these rings are p.p.-rings; a ring R is called a right p.p.-ring if each principal right ideal of R is projective, or equivalently, if the right annihilator of each element of R is generated by an idempotent. A ring R is called a p.p.-ring if it is both a right and a left p.p.-ring. Baer rings are clearly right p.p.-rings. We denote the right annihilator over a ring R by $r_R(-)$. Abelian right p.p.-rings are reduced (for, letting R be an abelian right p.p.-ring and $r^2 = 0$ for $r \in R$, then there exists $e^2 = e \in R$ such that $r_R(r) = eR$; hence $r \in r_R(r)$ implies $r = er = re = 0$) and so they are also left p.p.-rings. Thus we may obtain the following two theorems with the help of [2, Theorems A and B]. However we prove them a little independently in this paper.

THEOREM 9. *Let R be an Armendariz ring. Then R is a p.p.-ring if and only if $R[x]$ is a p.p.-ring.*

Proof. Assume that R is a p.p.-ring. Let $p = a_0 + a_1x + \cdots + a_mx^m \in R[x]$. There exists $e_i^2 = e_i \in R$ such that $r_R(a_i) = e_iR$, for $i = 0, 1, \dots, m$. Let $e = e_0e_1 \cdots e_m$. Then by Lemma 7, $e^2 = e \in R$ and $eR = \bigcap_{i=0}^{m} r_R(a_i)$. So $pe = a_0e + a_1ex + \cdots + a_mex^m = 0$. Hence $eR[x] \subseteq r_{R[x]}(p)$. Let $q = b_0 + b_1x + \cdots + b_nx^n \in r_{R[x]}(p)$. Since $pq = 0$ and R is Armendariz, $a_ib_j = 0$ for all $0 \leq i \leq m$, $0 \leq j \leq n$. Then $b_j \in e_0e_1 \cdots e_mR = eR$ for all $j = 0, 1, \dots, n$. Hence $q \in eR[x]$. Consequently $eR[x] = r_{R[x]}(p)$ and thus $R[x]$ is a p.p.-ring.

Conversely, assume that $R[x]$ is a p.p.-ring. Let $a \in R$. By Lemma 8, there exists an idempotent $e \in R$ such that $r_{R[x]}(a) = eR[x]$. Hence $r_R(a) = r_{R[x]}(a) \cap R = eR$ and therefore R is a p.p.-ring.

THEOREM 10. *Let R be an Armendariz ring. Then R is a Baer ring if and only if $R[x]$ is a Baer ring.*

Proof. Assume that R is Baer. Let A be a nonempty subset of $R[x]$ and let A^* be the set of all coefficients of elements of A. Then A^* is a nonempty subset of R and so $r_R(A^*) = eR$ for some idempotent $e \in R$.

Since $e \in r_{R[x]}(A)$, we get $eR[x] \subseteq r_{R[x]}(A)$. Now, let $g = b_0 + b_1 x + \cdots + b_t x^t \in r_{R[x]}(A)$. Then $Ag = 0$ and hence $fg = 0$ for any $f \in A$. Thus $b_0, b_1, \ldots, b_t \in r_R(A^*) = eR$ since R is Armendariz. Hence there exist $c_0, c_1, \ldots, c_t \in R$ such that $g = ec_0 + ec_1 x + \cdots + ec_t x^t = e(c_0 + c_1 x + \cdots + c_t x^t) \in eR[x]$. Therefore $R[x]$ is Baer.

Conversely, assume that $R[x]$ is a Baer ring. Let B be a nonempty subset of R. Then $r_{R[x]}(B) = eR[x]$ for some idempotent $e \in R$ by Lemma 8. Hence $r_R(B) = eR$ and therefore R is a Baer ring.

In the following text we obtain similar results for the formal power series rings.

PROPOSITION 11. *Suppose that a ring R is abelian. Then we have the following*:

(1) *If $R[[x]]$ is a p.p.-ring, then R is a p.p.-ring.*

(2) *If $R[[x]]$ is a Baer ring, then R is a Baer ring.*

Proof. By the same methods in the proofs of Theorems 9 and 10.

COROLLARY 12. *Suppose that a ring R is an Armendariz ring. Then we have the following*:

(1) *If $R[[x]]$ is a p.p.-ring, then R is a p.p.-ring.*

(2) *If $R[[x]]$ is a Baer ring, then R is a Baer ring.*

Proof. Combining Lemma 7 and Proposition 11.

Remark. The converse of Corollary 12(1) is not true in general by the following argument. Take the ring R in [5, Example 1(1)]. Notice that R is a Boolean ring and hence it is a p.p.-ring. R is also an Armendariz ring because it is reduced. However $R[[x]]$ is not a p.p.-ring by the argument in [4, Example 4].

Let R be a reduced ring. Then the trivial extension $T = T(R, R)$ is Armendariz by Corollary 4; notice that the prime radical $P(T)$ of T is $\left(\begin{smallmatrix} 0 & r \\ 0 & 0 \end{smallmatrix}\right)$ with $r \in R$ (hence it is Armendariz by applying the definition of Armendariz rings to rings without identity) and that $T/P(T) \cong R$ is reduced (hence it is also Armendariz). So one may suspect that if a ring R is an abelian ring such that $R/P(R)$ and $P(R)$ are Armendariz, then R is Armendariz, where $P(R)$ is the prime radical of R. However, the following example erases the possibility.

EXAMPLE 13. Let $\mathbb{Z}$ be the ring of integers and let

$$R = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a - b \equiv c \equiv 0 \pmod 2 \right\}.$$

Then $P(R) = \{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \equiv 0 \pmod 2 \}$ and so $P(R)$ is Armendariz. Also the only idempotents of R are $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So R is abelian.

Next note that $R/P(R)$ is reduced and so it is Armendariz. In fact,

$$\begin{aligned} R/P(R) &= \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a - b \equiv 0 \pmod 2 \right\} \\ &\cong \{ (a, b) \mid a - b \equiv 0 \pmod 2 \}. \end{aligned}$$

If $(a, b)^2 = (a^2, b^2) = (0, 0)$, then $a = 0$ and $b = 0$.

Now we claim that R is not Armendariz. Let $f(x) = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x$ and $g(x) = \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} x$. Then $f(x)g(x) = 0$, but $\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$. Therefore R is not an Armendariz ring.

Moreover we conjecture that R is an Armendariz ring if for any nonzero proper ideal I of R, R/I and I are Armendariz. However, we also have a counterexample to this situation as in the following.

EXAMPLE 14. Let F be a field and consider the ring

$$R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}.$$

Then by Example 1, R is not Armendariz.

Now we show that R/I and I are Armendariz for any nonzero ideal I of R. Note that the only nonzero proper ideals of R are $\begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$, and $\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. First, let $I = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$. Then $R/I \cong F$ and so R/I is Armendariz. So we claim that I is Armendariz. For $f(x), g(x) \in I[x]$, suppose that $f(x)g(x) = 0$, where $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$ and $g(x) = \beta_0 + \beta_1 x + \cdots + \beta_m x^m$. Let $\alpha_i = \begin{pmatrix} a_i & b_i \\ 0 & 0 \end{pmatrix}$ and $\beta_j = \begin{pmatrix} c_j & d_j \\ 0 & 0 \end{pmatrix}$, where $0 \le i \le n$ and $0 \le j \le m$. Assume that $\alpha_0 \neq 0$ and $\beta_0 \neq 0$. Then $a_0 c_0 = 0 = a_0 d_0$. If $a_0 \neq 0$, then $c_0 = 0$ and $d_0 = 0$, which is a contradiction. So $a_0 = 0$ and hence $b_0 \neq 0$. This implies that $\alpha_0 \beta_j = 0$ for all j, $0 \le j \le m$. Hence the coefficient of x in $f(x)g(x)$ is $\alpha_1 \beta_0 = 0$. Then $a_1 c_0 = 0 = a_1 d_0$. If $a_1 \neq 0$, then $c_0 = 0$ and $d_0 = 0$, which is also a contradiction. So $\alpha_1 \beta_j = 0$ for all j, $0 \le j \le m$. Continuing this process, we have $\alpha_i \beta_j = 0$ for all $i, j, 0 \le i \le n$ $0 \le j \le m$. Therefore I is Armendariz.

Next let $J = \left(\begin{smallmatrix} 0 & F \\ 0 & F \end{smallmatrix}\right)$. Then $R/K \cong F$ and so R/J is Armendariz. By the same method, we have that J is Armendariz.

Finally, let $K = \left(\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right)$. Then $R/K \cong F \oplus F$ and so R/K is Armendariz. Also $K^2 = 0$ and so K is Armendariz.

Notice that for any nontrivial idempotent $e \in R$, if R is an Armendariz ring then so is eRe; but $\mathrm{Mat}_n(R)$ is not Armendariz and so "Armendariz" is not a Morita invariant property. But one may suspect that if eRe is an Armendariz ring for any nontrivial nonidentity idempotent e of R then R is an Armendariz ring. However, it is not true in general, by the following example.

EXAMPLE 15. Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and consider the ring $R = \left(\begin{smallmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{smallmatrix}\right)$. Then by Example 1, R is not Armendariz. Notice that the only nontrivial nonidentity idempotents of R are

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

and that $eRe \cong \mathbb{Z}_2$ is an Armendariz ring for any nontrivial nonidentity idempotent e in R.

3. ARMENDARIZ RINGS AND REDUCED RINGS

In [1], Anderson and Camillo assert that it does seem possible for R to be reduced but the classical quotient ring $Q(R)$ of R is not reduced. But we have the following affirmative result.

THEOREM 16. *Suppose that there exists the classical right quotient ring $Q(R)$ of a ring R. Then R is reduced if and only if $Q(R)$ is reduced.*

Proof. It is enough to show that if R is reduced then $Q(R)$ is reduced. Let q be a nonzero element of $Q(R)$ with $q^2 = 0$. Then there exist $a, b \in R$ with b regular such that $q = ab^{-1}$. So $ab^{-1}ab^{-1} = 0$. Clearly $b^{-1}a \in Q(R)$ and so there exist $c, d \in R$ with d regular such that $b^{-1}a = cd^{-1}$. Then $ac(bd)^{-1} = acd^{-1}b^{-1} = ab^{-1}ab^{-1} = 0$ and so $ac = 0$. Hence $(ca)^2 = 0$ and so $ca = 0$ since R is reduced. Now from $b^{-1}a = cd^{-1}$, we have $ad = bc$ in $Q(R)$. So $ada = bca = 0$. Thus $ad = 0$ and so $a = 0$ since d is regular, which is a contradiction. Therefore $Q(R)$ is reduced.

Anderson and Camillo also assert that for a semiprime left and right Noetherian ring R, R is Armendariz if and only if $Q(R)$ is reduced in the argument after [1, Theorem 6]. In the following corollary we add another condition.

COROLLARY 17. *Let R be a von Neumann regular ring and suppose that there exists the classical right quotient ring of R, $Q(R)$. Then the following statements are equivalent*:

(1) *R is Armendariz.*

(2) *R is reduced.*

(3) *$Q(R)$ is reduced.*

(4) *$Q(R)$ is Armendariz.*

Proof. (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) are straightforward. (2) $\Rightarrow$ (1): By [2, Lemma 1]. (1) $\Rightarrow$ (3): Assume that R is Armendariz. Then by Lemma 7, R is abelian von Neumann regular and so it is reduced; hence $Q(R)$ is reduced by Theorem 16. (3) $\Rightarrow$ (4): By [2, Lemma 1].

Anderson and Camillo [1, Theorem 7] proved that if R is a prime ring which is left and right Noetherian, then R is Armendariz if and only if R is reduced. We obtain this result under a weaker condition.

PROPOSITION 18. *Suppose that R is a semiprime right and left Goldie ring. Then R is Armendariz if and only if R is reduced.*

Proof. Since reduced rings are Armendariz by [2, Lemma 1], it is enough to show that if R is Armendariz then R is reduced. By hypothesis, R has the right and left classical quotient ring $Q(R)$, which is semisimple Artinian (up to isomorphism) by the Goldie theorem. Since we have right and left common denominators for finite sets of elements in $Q(R)$ by [6, Lemma 2.1.8], it follows that for $f(x), g(x) \in Q(R)[x]$ there exist regular elements a and b in R such that $af(x), g(x)b \in R[x]$. Assume that $f(x)g(x) = 0$. Recall that R is Armendariz and that a, b are regular; hence each coefficient of $f(x)$ annihilates each coefficient of $g(x)$. Thus $Q(R)$ is Armendariz. On the other hand, $Q(R)$ is von Neumann regular; hence $Q(R)$ is reduced by [1, Theorem 6] and so R is reduced.

COROLLARY 19. *Suppose that R is a semisimple Artinian ring. Then R is an Armendariz ring if and only if R is a finite direct sum of division rings.*

Proof. By Proposition 18.

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