Classification of dendrites with a countable set of end points

Sophia Zafiridou

Department of Mathematics, University of Patras, 26500 Patras, Greece

MSC:
54C25
54D35
54F50
54G12

Keywords:
Dendrites
End points
α-Derivative
Universal space

1. Introduction

In this article all spaces under consideration are metrizable and separable, and all ordinals are countable.

The order of point \( p \) in a space \( X \), written ord\((p, X)\), is the least cardinal or ordinal number \( \kappa \) such that \( p \) has an arbitrarily small neighborhood in \( X \) with boundary of cardinality \( \leq \kappa \) [6, §51, I, p. 274]. Hence, ord\((p, X)\) = \( \omega \) if \( p \) has an arbitrarily small neighborhood in \( X \) with a finite boundary but ord\((p, X)\) \neq n for every natural number \( n \).

Points of order one are called end points and the sets of end points of a space \( X \) is denoted by \( E(X) \). Points of order \( \geq 3 \) are called ramification points and the sets of ramification points of a space \( X \) is denoted by \( R(X) \).

A continuum is a non-empty compact and connected metric space. A dendrite is a locally connected continuum without simple closed curves. Any dendrite has a basis of open sets with finite boundaries and, therefore, is hereditarily locally connected [6, §51, VI, Theorem 4 and IV, Theorem 2]. Hence every subcontinuum of a dendrite is a dendrite.

Let \( X \) be a dendrite and \( x \in X \). Then the set of components of \( X \setminus \{x\} \) either is finite or form a null sequence [10, (1.1) on p. 88 and (2.6) on p. 92]. The order of \( x \) in \( X \) coincides with the number of components of \( X \setminus \{x\} \) whenever either of this is finite [10, p. 88]. Hence, the order of \( x \) in \( X \) is \( \omega \) iff the set of components of \( X \setminus \{x\} \) form a null sequence.

A space \( Z \) is universal in a class \( \mathcal{F} \) of spaces provided that \( Z \in \mathcal{F} \) and for each \( X \in \mathcal{F} \) there exists an embedding \( h : X \to Z \).

We recall some results concerning the existence of universal elements in the families of dendrites.

1. There exists a universal element in the family of all dendrites [11]. Simple construction of universal dendrite in the plane is given in [2].
2. There exists a universal element in the family of all dendrites with a closed set of end points [1].
3. In the family of all dendrites with a closed countable set of end points there is no universal element [12, p. 1937].
4. In the family of all dendrites with a countable set of end points there is no universal element [4].
Some results concerning the existence of universal elements or containing dendrites with a countable sets of end points for some families of dendrites with a countable sets of end points are in [12,13].

In this paper we prove that:

\( (\alpha) \) For each ordinal \( \alpha \) in the family of dendrites \( X \) such that the \( \alpha \)-derivative of the set of end points of \( X \) is empty there is no universal element.

\( (\beta) \) For each natural number \( n > 0 \) in the family of dendrites with ramification degree \( \leq n \) there exists a universal element (see Section 4 for the definition of ramification degree).

2. General notions and facts

Given a subset \( M \) of a space \( X \), we use the symbols \( \text{cl}_X(M) \) and \( M^d \) to denote the closure of \( M \) and the set of limit points of \( M \) in \( X \), respectively. We denote the diameter of \( M \) by \( \text{diam}(M) \), and the cardinality of \( M \) by \( \text{card}(M) \).

A set \( M \) is called \textit{dense in itself} if it contains no relatively isolated points, equivalently, if \( M \subseteq M^d \).

For each integer \( n \geq 2 \) by \textit{n-od with center} \( v \) and \textit{arms} \( A_i \), \( i = 1, \ldots, n \), is meant a union of \( n \) arcs \( A_i \) emanating from a point \( v \) and disjoint out of \( v \). Similarly, by \textit{\( \omega \)-od with a center} \( v \) and \textit{arms} \( A_i \), \( i = 1, 2, \ldots \) is meant a countable union \( \bigcup_{i=1}^{\infty} A_i \) of arcs \( A_i \) emanating from a point \( v \) and disjoint out of \( v \), such that \( \lim_{n \to \infty} \text{diam}(A_i) = 0 \).

The \textit{first point map} \( r : X \to Y \) for dendrite \( Y \) contained in a dendrite \( X \) is defined by letting \( r(x) = x \) if \( x \in Y \) and by letting \( r(x) \) to be a unique point \( r(x) \in Y \) such that \( r(x) \) is a point of any arc in \( X \) from \( x \) to any point of \( Y \) if \( x \notin Y \) (see [7, 10.26, p. 176]).

Let \( X \) be a dendrite. Given two points \( a \) and \( b \) of \( X \) we denote by \( ab \) the unique arc from \( a \) to \( b \) in \( X \) and by \( (ab) \) the set \( ab \setminus [a,b] \). We denote by \( R^d(X) \) (respectively, by \( E^d(X) \)) the set of limit points of \( R(X) \) (respectively, of \( E(X) \)) in \( X \) and by \( R_o \) the set of points of order \( \omega \).

The following facts will be used in the sequel ([3, Propositions 4.13–4.14], and [8, Lemma 4]).

**Fact 2.1.** If \( X \) and \( Y \) are dendrites and \( X \subseteq Y \), then

1. \( R(X) \subseteq R(Y) \);
2. \( E^d(X) \subseteq E^d(Y) \);
3. \( \text{card}(E(X)) \leq \text{card}(E(Y)) \).

**Fact 2.2.** For any dendrite \( X \) the set \( R(X) \) is countable and

$$ E^d(X) = R^d(X) \cup R_\omega(X). $$

A subset \( Y \) of dendrite \( X \) is called \textit{branch of } \( X \) at the point \( x \in X \) if \( Y \) is a component of \( X \setminus \{x\} \) [8].

**Lemma 2.1.** If \( X \) is a dendrite and \( e \in E(X) \cap E^d(X) \), then \( (ex) \cap R(X) \neq \emptyset \) for each \( x \in X \setminus \{e\} \) [8].

**Proof.** Let \( Y \) be a branch of \( X \) at the point \( x \) such that \( e \in Y \). Since \( Y \) is open and \( e \in E^d(X) \), then there exists \( e^* \in Y \cap E(X) \setminus \{e\} \). Each connected subset of dendrite is arcwise connected and \( Y \) is connected (see [7, p. 169]), therefore \( ee^* \subseteq Y \). Since \( X \) is hereditarily unicoherent (see [7, p. 180]), it follows that \( ex \cap ee^* \) is connected and closed subset of half-open arc \( xe \setminus \{x\} \). Since \( \text{ord}(e,X) = 1 \), then \( ex \cap ee^* \) is an arc. Hence, there is a point \( b \in (ex) \) such that \( ex \cap ee^* = eb \). Clearly \( be \cup bx \cup be^* \) is a triod with a center \( b \). Thus \( b \in R(X) \), which completes the proof. \( \Box \)

**Lemma 2.2.** Let \( X \) be a dendrite and \( Y = Y(x) \) be a branch of \( X \) at a point \( x \in X \) such that the set \( E(Y) \) is dense in itself. Then there exists a point \( b \in Y \) at which some branches \( Y_0(b) \) and \( Y_1(b) \) begin such that \( Y_0(b) \subset Y, Y_1(b) \subset Y, Y_0(b) \cap Y_1(b) = \emptyset, \) and each of the sets \( E(Y_0(b)) \) and \( E(Y_1(b)) \) is dense in itself.

**Proof.** Let \( e \in E(Y) \). Since \( E(Y) \subseteq E(X) \) and \( E(Y) \) is dense in itself, it follows that \( e \in E^d(Y) \subseteq E^d(X) \) and \( e \notin R_\omega(X) \). Then, from relation (1) above, \( e \in R^d(X) \). Therefore, \( (ex) \cap R(X) \neq \emptyset \) by Lemma 2.1. Let \( b \in (ex) \cap R(X) \) and \( Y_0(b), Y_1(b) \) are components of \( X \setminus \{b\} \) that do not contain the point \( x \). Clearly, \( Y_0(b) \) and \( Y_1(b) \) satisfy the requirements of lemma. \( \Box \)

**Theorem 2.1.** If the set of end points of dendrite \( X \) is dense in itself, then the set of end points of \( X \) is uncountable.

**Proof.** It suffices to show that \( X \) contains the Gehman dendroid (see [8] or [9] for the definition of Gehman dendroid). Since the set of end points of Gehman dendroid is uncountable, from Fact 2.1(3) it will follow that the set \( E(X) \) is uncountable.
Let \( x \in E(X) \). Then the set \( V_Y(x) = X \setminus \{x\} \) is a branch of \( X \) at \( x \). Since \( E(X) \) is dense in itself, the set \( E(Y_Y(x)) \) is dense in itself. By Lemma 2.2, there exists a point \( b_0 \in Y_Y(x) \) at which some branches \( V_0(b_0) \) and \( Y_1(b_0) \) begin such that \( V_0(b_0) \subset Y_Y(x) \), \( Y_1(b_0) \subset V_Y(x) \), \( V_0(b_0) \cap Y_1(b_0) = \emptyset \), and each of the sets \( E(Y_0(b_0)) \) and \( E(Y_1(b_0)) \) is dense in itself.

The rest of the proof that end in construction of the Gehman dendroid in \( X \) is similar to the analogous construction in the proof of Theorem 1 in [8] (see the "only if" part). Instead of Lemma 7 of [8], here is used Lemma 2.2. \( \square \)

### 3. Scattered spaces

A space \( M \) is said to be scattered provided that every non-empty subspace of \( M \) has a relatively isolated point. For every ordinal \( \alpha \), the \( \alpha \)-derivative of space \( M \) is defined by induction as follows [5, §24, IV]:

\[
M^{(0)} = M,
\]

\[
M^{(\alpha + 1)} = \{ \text{all limit points of } M^{(\alpha)} \} \text{ in } M^{(\alpha)},
\]

\[
M^{(\alpha)} = \bigcap_{\beta < \alpha} M^{(\beta)} \text{ for a limit ordinal } \alpha.
\]

If \( M^{(\alpha)} = \emptyset \) for some ordinal \( \alpha \), then the least such ordinal is called type of \( M \) and is denoted by \( \text{type}(M) \).

The facts below are well known and are easily obtained.

**Fact 3.1.** A space \( M \) is scattered if and only if there exists an ordinal \( \alpha \) such that \( \text{type}(M) = \alpha \).

**Fact 3.2.** Any scattered space is countable [5, §23, V].

**Fact 3.3.** If a space \( \tilde{M} \) is scattered and \( M \subseteq \tilde{M} \), then \( \text{type}(M) \leq \text{type}(\tilde{M}) \).

**Proposition 3.1.** If \( \text{type}(M) = \beta \) for a space \( M \), then \( \text{type}(M \setminus M^{(\alpha)}) = \alpha \) for any \( 0 < \alpha \leq \beta \).

**Proof.** Fix \( 0 < \alpha \leq \beta \). Since \( (M \setminus M^{(\alpha)})^{(\alpha)} \subseteq (M \setminus M^{(\alpha)}) \cap M^{(\alpha)} \), then \( (M \setminus M^{(\alpha)})^{(\alpha)} = \emptyset \). It remains to show that \( (M \setminus M^{(\alpha)})^{(\delta)} \neq \emptyset \) for any \( \delta < \alpha \).

Fix \( \delta < \alpha \). Since \( \text{type}(M) \geq \alpha \), it follows that \( M^{(\delta)} \setminus M^{(\alpha)} \neq \emptyset \). Note that

\[
M^{(\delta)} \setminus M^{(\alpha)} = M^{(\delta)} \cap (M \setminus M^{(\alpha)}).
\]

Using the fact that \( M \setminus M^{(\alpha)} \) is open, one can prove by induction on \( \delta \) that

\[
M^{(\delta)} \cap (M \setminus M^{(\alpha)}) \subseteq (M \setminus M^{(\alpha)})^{(\delta)}.
\]

Consequently, \( (M \setminus M^{(\alpha)})^{(\delta)} \neq \emptyset \), which completes the proof. \( \square \)

**Construction of a space \( E_n \) such that \( (E_n)^{(\alpha)} \) is a single point**

The sets \( E_n \) are defined by induction as subsets of the Cantor ternary set \( C \) [3, p. 21].

Set \( E_1 = \{0\} \cup \{1/3^n \colon n \in \{0, 1, 2, \ldots\}\} \). Then \( (E_1)^{(1)} = \emptyset \). Suppose that the sets \( E_\alpha \) have been defined for all \( 1 \leq \alpha < \alpha_0 \).

We associate to \( \alpha_0 \) a sequence \( \langle \alpha_n \rangle_{n=1}^{\infty} \) of ordinals less than \( \alpha_0 \) as follows:

(i) if \( \alpha_0 = \alpha + 1 \), then \( \alpha_n = \alpha \) for all \( n \),

(ii) if \( \alpha_0 \) is a limit ordinal, then \( \alpha_{n+1} = \text{lim}_{n} \alpha_n \).

For each \( n \in \{1, 2, \ldots\} \), we locate in \( C \cap [2/3^n, 3/3^n] \) a copy \( E^n_{\alpha_n} \) of \( E_{\alpha_n} \) diminished \( 3^n \) times in such a way that \( (E^n_{\alpha_n})^{(\alpha_n)} = [2/3^n, 3/3^n] \), and define \( E_{\alpha_0} = \bigcup_{n=1}^{\infty} E^n_{\alpha_n} \). Clearly, \( (E_{\alpha_0})^{(\alpha)} = \emptyset \).

### 4. Ramification degree

Given a continuum \( X \) and \( Y \subseteq X \), we denote by \( \text{irr}(Y) \) the subcontinuum of \( X \) which is irreducible about \( Y \). Any continuum contains an irreducible subcontinuum about any of its closed subsets [10, (11.2), p. 17].

Let \( X \) be a dendrite. For every countable ordinal \( \alpha \) we define inductively the closed subset \( X_{(\alpha)} \) of \( X \) as follows

\[
X_{(0)} = X; \quad X_{(\alpha+1)} = \begin{cases} \text{irr} (\text{cl}_X X_{(\alpha)} (R(X_{(\alpha)}))), & \text{if } R(X_{(\alpha)}) \neq \emptyset, \\ \emptyset, & \text{if } R(X_{(\alpha)}) = \emptyset; \end{cases} \quad X_{(\alpha)} = \bigcap_{\beta < \alpha} X_{(\beta)} \text{ for a limit ordinal } \alpha.
\]
Obviously each $X_{(\alpha)}$ either is a subcontinuum of $X$ or is empty.
If $\alpha < \beta$, then $X_{(\alpha)} \supseteq X_{(\beta)}$. The well ordered family of decreasing closed sets
\[ X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{(\alpha)} \supseteq \cdots, \quad \alpha < \omega_1 \]
is countable [5, §24, II, Theorem 2]. Therefore there exists an ordinal $\alpha_0$ such that $X_{(\alpha_0)} = X_{(\alpha)}$ for each $\alpha_0 < \alpha$.

**Definition 4.1.** We say that a dendrite $X$ has a ramification degree if there exists an ordinal $\alpha$ such that $X_{(\alpha)} = \emptyset$. In this case the ordinal
\[ \text{rdeg}(X) = \min\{\alpha : X_{(\alpha)} = \emptyset\} \]
is called the ramification degree of $X$.

Observe that if $\alpha$ is a limit ordinal and $X_{(\beta)} \neq \emptyset$ for each $\beta < \alpha$, then, since each $X_{(\beta)}$ is compact, $X_{(\alpha)} \neq \emptyset$. So we conclude that

**Fact 4.1.** The ramification degree of dendrite may be only isolated ordinal.

**Theorem 4.1.** If a dendrite $X$ has ramification degree, then each subdendrite $Y$ of $X$ has ramification degree and $\text{rdeg}(Y) \leq \text{rdeg}(X)$.

**Proof.** From Fact 4.1 there exists an ordinal $\beta$ such that $\text{rdeg}(X) = \beta + 1$. It suffices to prove that $Y_{(\alpha)} \subseteq X_{(\alpha)}$ for each ordinal $\alpha$. Then $Y_{(\beta+1)} = Y_{(\beta+1)} \subseteq Y_{(\beta)} = \emptyset$, which means $\text{rdeg}(Y) \leq \beta + 1$.

We proceed by induction. Clearly $Y_0 = Y \subseteq X = X_0$. Let $\alpha_0$ be an ordinal and suppose that $Y_{(\alpha)} \subseteq X_{(\alpha)}$ for each $\alpha < \alpha_0$.

If $\alpha_0$ is an isolated ordinal, then $Y_{(\alpha)} = \cap_{\alpha < \alpha_0} Y_{(\alpha)} \subseteq \cap_{\alpha < \alpha_0} X_{(\alpha)} = X_{(\alpha_0)}$.

Let $\alpha_0$ be an isolated ordinal. By assumption, $Y_{(\alpha_0-1)} \subseteq X_{(\alpha_0-1)}$. If either $Y_{(\alpha_0-1)} = \emptyset$ or $R(Y_{(\alpha_0-1)}) = \emptyset$, then $Y_{(\alpha_0)} = \emptyset \subseteq X_{(\alpha_0)}$. If $R(Y_{(\alpha_0-1)}) \neq \emptyset$, then $R(Y_{(\alpha_0-1)}) \subseteq R(X_{(\alpha_0-1)})$.

Thus
\[ Y_{(\alpha)} = \text{irr}(\text{cl}_{X_{(\alpha)}} R(Y_{(\alpha)})) \subseteq \text{irr}(\text{cl}_{X(\alpha)} R(X_{(\alpha)})) = X_{(\alpha_0)}. \]

The proof of the theorem is complete. \( \square \)

Given a dendrite $X$ we denote by $E^{(\alpha)}(X)$ the $\alpha$-derivative of the set of end points of $X$.

**Theorem 4.2.** Any dendrite $X$ with a scattered set of end points has a ramification degree. Moreover, $\text{type}(E(X)) \leq \text{rdeg}(X)$.

**Proof.** Suppose on the contrary that $X$ has no ramification degree. Then there exists an ordinal $\alpha$ such that $X_{(\alpha)} = X_{(\alpha+1)} = \text{irr}(\text{cl}_{X_{(\alpha)}} R(X_{(\alpha)}))$. Thus $E(X_{(\alpha)}) \subseteq E(X_{(\alpha+1)}) \subseteq E(X_{(\alpha)})$. Theorem 2.1 now implies that the set $E(X_{(\alpha)})$ is uncountable. Then the set $E(X)$ is uncountable from Fact 2.1(3), contrary to Fact 3.2.

Let $\text{type}(E(X)) = \alpha_0$. It suffices to show that $E^{(\alpha)}(X) \subseteq X_{(\alpha)}$ for each ordinal $\alpha$. Since $E^{(\alpha)}(X) \neq \emptyset$ for each $\alpha < \alpha_0$, it will follow that $X_{(\alpha)} \neq \emptyset$ for each $\alpha < \alpha_0$, so $\text{rdeg}(X) \geq \alpha_0$.

The proof will be done by induction on $\alpha$. Clearly $E^{(0)}(X) = E(X) \subseteq X = X_0$. Suppose that $E^{(\alpha)}(X) \subseteq X_{(\beta)}$ for each $0 \leq \beta < \alpha$.

If $\alpha$ is a limit ordinal, then $E^{(\alpha)}(X) = \cap_{\beta < \alpha} E^{(\beta)}(X) \subseteq \cap_{\beta < \alpha} X_{(\beta)} = X_{(\alpha)}$.

Let $\alpha$ be an isolated ordinal. Then $E^{(\alpha-1)}(X) \subseteq X_{(\alpha-1)} \cap E(X)$. Hence, each point of $E^{(\alpha-1)}(X)$ is of order 1 in $X_{(\alpha-1)}$. Thus $E^{(\alpha-1)}(X) \subseteq E(X_{(\alpha-1)})$. Therefore, $E^{(\alpha)}(X) \subseteq E^{(1)}(X_{(\alpha-1)})$.

If $\beta \in E^{(1)}(X_{(\alpha-1)})$, then $\beta \in \text{cl}_{X_{(\alpha-1)}} R(X_{(\alpha-1)}) \subseteq X_{(\alpha)}$. Thus $E^{(\alpha)}(X) \subseteq E^{(1)}(X_{(\alpha-1)}) \subseteq X_{(\alpha)}$, which completes the proof. \( \square \)

**Dendrite $G_{\beta}$ with type** $\text{type}(E(G_{\beta})) = \text{rdeg}(G_{\beta}) = \beta + 1$

Let $G$ denote the Gehman dendrite of the plane with an orthogonal coordinate system. Then $E(G) = C \times \{0\}$. For any ordinal $\beta > 0$ we will define the dendrites $G_{\beta} \subseteq G$ [3, p. 21].

Let $G_\beta$ be a segment with end points $(0, 0)$ and $(1/2, 1/2)$. Then $\text{type}(E(G_\beta)) = \text{rdeg}(E(G_\beta)) = 1$.

For $1 < \beta < \omega_1$ the dendrite $G_{\beta}$ is a subcontinuum of $G$ irreducible about the set $E_{\beta} \times \{0\}$, where $E_{\beta}$ is the set constructed in Section 3. Since $E(G_{\beta}) = E_{\beta}$, then $E_{\beta}(G_{\beta}) = E_{\beta}(0, 0)$. Thus $\text{type}(E(G_{\beta})) = \beta + 1$.

If $\beta$ is an isolated ordinal, then $(G_{\beta})_{(\beta)}$ is a segment. If $\beta$ is a limit ordinal, then $(G_{\beta})_{(\beta)} = (0, 0)$. In either case we have $\text{rdeg}(G_{\beta}) = \beta + 1$. 


5. Construction of dendrites $B^n$ and $K^n$

In this section for any natural number $n \geq 1$ we will construct in the plane dendrites $B^{n-1}$ and $K^n$, each with a countable set of end points and such that

(i) $B^{n-1}_{(n-1)}$ is a single point,
(ii) $K^n_{(n-1)}$ is a segment.

Given two points $a$ and $b$ of the plane we denote by $\overline{ab}$ the straight line segment joining $a$ and $b$.

For $n = 1$ we define $B^0 = [q]$ and $K^1 = \overline{qe}$, where $q$ and $e$ are two points of the plane.

Assume that we have defined the dendrites $B^{n-1}$ and $K^n$ for some $n \geq 1$.

Let $\{K^n_i(q(i))\}_{i=1}^\infty$ be a family of dendrites such that

(i) $K^n_i(q(i))$ is a copy of $K^n$ and $(K^n_i(q(i)))_{i=1}^{n-1} = \overline{qe}$;
(ii) $K^n_i(q(i)) \cap K^n_j(q(j)) = \{q(i)\}$ for any $i_1, i_2 = 1, 2, \ldots$;
(iii) $\lim_{i \to \infty} \overline{\text{diam}}(K^n_i(q(i))) = 0$.

We define $B^n = \bigcup_{i=1}^\infty K^n_i(q(i))$.

Now we define the dendrite $K^{n+1}$. Take a dense countable set $\{q_1, q_2, \ldots, q_j, \ldots\}$ of $\overline{qe} \setminus \{q, e\}$. Let $\{B^n(q(i))\}_{i=1}^\infty$ be a family of copies of $B^n$ such that

(i) $B^n(q(i))$ is a copy of $B^n$ and $B^n_{i-1}(q(i)) = \{q(i)\}$;
(ii) $B^n(q(i)) \cap \overline{qe} = \{q(i)\}$;
(iii) $B^n(q(i)) \cap B^n(q(j)) = \emptyset$ if $j_1 \neq j_2$;
(iv) $\lim_{i \to \infty} \overline{\text{diam}}(B^n(q(i))) = 0$.

We define $K^{n+1} = \overline{qe} \cup (\bigcup_{i=1}^\infty B^n(q(i)))$.

Remark 5.1. We give another description of the construction of dendrites $K^n$. As above, $K^1$ is a segment $\overline{qe}$. To define $K^2$ we take a countable dense subset $Q_0 = \{q_1, \ldots, q_j, \ldots\}$ of $\overline{qe} \setminus \{q, e\}$ and join to each $q_i$ an $\omega$-od $B^1(q(i)) = \bigcup_{i=1}^\infty \overline{qe_i}$ consisting of segments in such a way that the conditions (i)-(iv) above are satisfied. We call each $\overline{qe_i}$ segment of first degree. We repeat the above procedure taking a countable dense subset $Q_i$ in the interior of each segment $\overline{qe_i}$ of first degree and joining an $\omega$-od to each point of $Q_i$. The resulting dendrite is $K^3$ and so on.

One can easily verify the following facts.

Fact 5.1. For each $1 \leq m \leq n$ the subdendrite $K^n_{(n-m)}$ of $K^n$ is homeomorphic to $K^m$ and $E(K^n_{(n-m)}) \subseteq E(K^n)$.

Fact 5.2. There exists an embedding $h : K^n \to B^n$, $n = 1, 2, \ldots$, such that $h(E(K^n)) \subseteq E(B^n)$.

From (i) and (ii) above it follows that

Fact 5.3. $\text{rdeg}(K^n) = \text{rdeg}(B^{n-1}) = n$, for any $n = 1, 2, \ldots$.

6. Classification of dendrites by type of the set of end points

Theorem 6.1. For any ordinals $\alpha$ and $\beta$ such that $0 < \alpha \leq \beta$ there exists a dendrite $D(\alpha, \beta)$ such that $\text{type}(E(D(\alpha, \beta))) = \alpha$ and $\text{rdeg}(D(\alpha, \beta)) = \beta + 1$.

Proof. Fix an ordinal $\beta > 0$ and consider the dendrite $G_\beta$ with a scattered set of end points defined in Section 4.

We denote $\alpha = (0, 0)$, $e = (-1, 0)$, and define $D(\alpha, \beta) = G_\beta \cup \overline{qe}$.

Since $R(D(\alpha, \beta)) \cap \overline{qe} = \emptyset$, it follows that $(D(\alpha, \beta))_{(1)} = (G_\beta)_{(1)}$. Therefore $(D(\alpha, \beta))_{(\beta)} = (G_\beta)_{(\beta)}$. Thus $\text{rdeg}(D(\alpha, \beta)) = \text{rdeg}(G_\beta) = \beta + 1$. 

Since $E(D(\beta, \beta)) = (E(G_\beta) \setminus \{x\}) \cup \{e\}$, $E^{(\varnothing)}(G_\beta) = \{x\}$, and $e$ is an isolated end point of $D(\beta, \beta)$, it follows that

$$E^{(\gamma)}(D(\beta, \beta)) = E^{(\gamma)}(G_\beta) \setminus \{x\} \neq \emptyset, \quad \text{for each } \gamma < \beta$$

and $E^{(\beta)}(D(\beta, \beta)) = \emptyset$. Thus $type(E(D(\beta, \beta))) = \beta$.

Let $\alpha$ and $\beta$ be ordinals such that $0 < \alpha < \beta$. Consider the dendrite $G_\beta$ defined in Section 4. Then $E^{(\varnothing)}(G_\beta) = E^{(\alpha)}$ is some countable infinite set $\{x_1, x_2, \ldots\}$. Consider a family of disjoint arcs $\{x_ie_i\}_{i=1}^{\infty}$ such that $\lim_{i \to \infty} \text{diam}(x_ie_i) = 0$ and $x_i, x_i \cap G_\beta = \{x_i\}$ for each $i$. Set $D(\alpha, \beta) = G_\beta \cup \bigcup_{i=1}^{\infty} x_ie_i$.

Obviously $D(\alpha, \beta)$ is a dendrite and $E(D(\alpha, \beta)) = \{e_1, e_2, \ldots\} \cup (E_\beta \setminus E^{(\alpha)})$. Since each $e_i$ is isolated in $E(D(\alpha, \beta))$, then $E^{(1)}(D(\alpha, \beta)) = (E_\beta \setminus E^{(\alpha)}(1))$. That type $E(D(\alpha, \beta)) = type(E(\beta) \setminus E^{(\alpha)}) = \alpha$ follows from Proposition 3.1.

Observe that $R(D(\alpha, \beta)) = R(G_\beta)$ and $R^d(D(\alpha, \beta)) = R^d(G_\beta)$. Thus $D(\alpha, \beta)$ is dendrite with type $\beta$ and, therefore, $rdeg(D(\alpha, \beta)) = rdeg(G_\beta) = \beta + 1$, which complete the proof. \(\square\)

**Theorem 6.2.** In the family of all dendrites $X$ with $type(E(X)) \leq \alpha$ there is no universal element.

**Proof.** Suppose, on the contrary, that $Y$ is a universal element in the family of all dendrites $X$ with $type(E(X)) \leq \alpha$. Then $type(E(Y)) = \alpha_Y \leq \alpha$.

From Theorem 4.2 we have $rdeg(Y) = \beta_Y \geq \alpha_Y$. Consequently, by Theorem 4.1, for any dendrite $X \subseteq Y$ we have $rdeg(X) \leq \beta_Y$.

It follows that if $X$ is dendrite with $type(E(X)) \leq \alpha$, then $rdeg(X) \leq \beta_Y$. However, for a dendrite $D(\alpha_Y, \beta_Y)$ defined in Theorem 6.1 we have

$$type(E(D(\alpha_Y, \beta_Y))) = \alpha_Y \leq \alpha \quad \text{and} \quad rdeg(D(\alpha_Y, \beta_Y)) = \beta_Y + 1 > \beta_Y,$$

which is a contradiction. \(\square\)

**7. Classification of dendrites by ramification degree**

**Theorem 7.1.** For any limit ordinal $\alpha$ in the family of all dendrites $X$ with $rdeg(X) \leq \alpha$ there is no universal element.

**Proof.** Let $\alpha$ be a limit ordinal. Suppose, on the contrary, that $Z$ is a universal element in the family of all dendrites $X$ with $rdeg(X) \leq \alpha$. Since the ramification degree of a dendrite is never a limit ordinal, then $rdeg(Z) = \beta < \alpha$. But for the dendrite $G_\beta$ we have $\alpha > rdeg(G_\beta) = \beta + 1 > rdeg(Z)$, which is a contradiction. \(\square\)

**Remark 7.1.** Let $X$ be a dendrite such that $X_{(m+1)} \neq \emptyset$ for some natural number $m \geq 0$. Since $X_{(m+1)} = \text{irr}(cl_{X_{(m)}}(R(X_{(m)})))$, it follows that

$$R(X_{(m)}) \subseteq X_{(m+1)} \subset X_{(m)}.$$

Hence, the components of $X_{(m)} \setminus X_{(m+1)}$ are of the form $re \setminus \{r\}$, where $re$ is free arc of $X_{(m)}$, $r \in X_{(m+1)}$, and $re \cap X_{(m+1)} = \{r\}$. Let $f_m : X_{(m)} \to X_{(m+1)}$ be the first point map. From now on we shall denote

$$\tilde{R}(X_{(m)}) = \{f_m(x) : x \in X_{(m)} \setminus X_{(m+1)}\}. \quad (3)$$

From the above we conclude that

$$X_{(m)} = X_{(m+1)} \cup \left(\bigcup \{f_m^{-1}(r) : r \in \tilde{R}(X_{(m)})\}\right), \quad (4)$$

where each $f_m^{-1}(r)$ is either an arc having $r$ as one of its end points, or an $n$-od with a center $r$, or an $\omega$-od with a center $r$, and $X_{(m+1)} \cap f_m^{-1}(r) = \{r\}$.

Clearly, $\tilde{R}(X_{(m)}) \subseteq X_{(m+1)}$. From (3) it follows that if $r \in \tilde{R}(X_{(m)})$, then

$$\text{ord}(r, X_{(m+1)}) < \text{ord}(r, X_{(m)}) \leq \text{ord}(r, X).$$

In fact, $\text{ord}(r, X) \geq 3$ for all $r \in \tilde{R}(X_{(m)}) \setminus E(X_{(m+1)})$ and $\text{ord}(r, X) \geq 2$ for all $r \in \tilde{R}(X_{(m)}) \cap E(X_{(m+1)})$.

Hence

$$\tilde{R}(X_{(m)}) \subseteq (R(X) \cap X_{(m+1)}) \cup (E(X_{(m+1)}) \setminus E(X)). \quad (5)$$

**Theorem 7.2.** For any natural number $n \geq 1$ and for any dendrite $X$ such that $rdeg(X) = n$ and $X_{(n-1)}$ is an arc there exists an embedding $h : X \to K^n$ such that $h(E(X)) \subseteq E(K^n)$.
**Proof.** For \( n = 1 \) the dendrite \( X \) is an arc and, therefore, \( X \) is embeddable in a segment \( K^1 \).

Let \( X \) be a dendrite such that \( \text{rdeg}(X) = n > 1 \), and \( X_{(n−1)} \) is an arc \( q_X \in X \).

Clearly

\[
X = X_{(0)} \supseteq \cdots \supseteq X_{(n−2)} \supseteq X_{(n−1)} = q_X X.
\]

From the construction of \( K^n \) we have also

\[
K^n = K^n_{(0)} \supseteq K^n_{(1)} \supseteq \cdots \supseteq K^n_{(n−2)} \supseteq K^n_{(n−1)} = \overline{qX}.
\]

Let

\[
f_m : X_{(m)} \to X_{(m+1)}, \quad g_m : K^n_{(m)} \to K^n_{(m+1)}, \quad m = 0, \ldots, n−2,
\]

be the first point maps.

Let \( m \in \{0, \ldots, n−2\} \). From (4) we have

\[
X_{(m)} = X_{(m+1)} \cup \left( \bigcup \{ f_m^{-1}(r) : r \in \bar{R}(X_{(m)}) \} \right).
\]

where each \( f_m^{-1}(r) \) is either an arc having \( r \) as one of its end points, or an \( n \)-od with a center \( r \), or an \( \omega \)-od with a center \( r \), and \( X_{(m+1)} \cap f_m^{-1}(r) = \{r\} \). Also

\[
K^n_{(m)} = K^n_{(m+1)} \cup \left( \bigcup \{ g_m^{-1}(q) : q \in \bar{R}(K^n_{(m)}) \} \right).
\]

The sets \( g_m^{-1}(q) \) in (6) have the following properties:

(i) \( g_m^{-1}(q) \) is an \( \omega \)-od with center \( q \), whose arms are free arcs of \( K^n_{(m)} \), and \( K^n_{(m+1)} \cap g_m^{-1}(q) = \{q\} \);

(ii) \( g_m^{-1}(q) \cap R(K^n) \) is dense in \( g_m^{-1}(q) \) for \( 1 \leq m \leq n−2 \);

(iii) \( E(g_m^{-1}(q)) \subseteq E(K^n) \).

For each \( q \in \bar{R}(K^n_{(m)}) \) we represent the \( \omega \)-od \( g_m^{-1}(q) \) as a finite union of \( \omega \)-ods \( B_i(q) \) having only the point \( q \) in common:

\[
g_m^{-1}(q) = B_0(q) \cup \cdots \cup B_{n−2}(q), \quad B_0(q) \cap \cdots \cap B_{n−2}(q) = \{q\}.
\]

We can rewrite (6) as follows

\[
K^n_{(m)} = K^n_{(m+1)} \cup \left( \bigcup \{ B_0(q) \cup \cdots \cup B_{n−2}(q) : q \in \bar{R}(K^n_{(m)}) \} \right).
\]

From the construction of \( K^n \) we have

\[
\bar{R}(K^n_{(m)}) = R(K^n) \cap (K^n_{(m+1)} \setminus K^n_{(m+2)}).
\]

By the above \( \bar{R}(K^n_{(i)}) \cap \bar{R}(K^n_{(i−1)}) = \emptyset \) for each \( i = 1, \ldots, n−2 \).

We set \( \bar{K}^{n}_{m+1} = \overline{qX} \) and for \( m = n−2, \ldots, 0 \) we define

\[
\bar{K}^n_m = \bar{K}^n_{m+1} \cup \left( \bigcup \left\{ B_m(q) : q \in \bigcup_{i=n−2}^m (R(K^n_{(i)}) \cap \bar{K}^n_{i+1}) \right\} \right).
\]

Since, from (8), \( \bigcup_{i=n−2}^m \bar{R}(K^n_{(i)}) = R(K^n) \cap K^n_{(m+1)} \), we take

\[
\bar{K}^n_m = \bar{K}^n_{m+1} \cup \left( \bigcup \{ B_m(q) : q \in R(K^n) \cap \bar{K}^n_{m+1} \} \right).
\]

Note that:

(iv) \( \bar{K}^n_m \subseteq K^n_{(m)} \),

(v) \( E(\bar{K}^n_m) \subseteq E(K^n_{(m)}) \), and

(vi) \( B_m(q) \cap \bar{K}^n_{m+1} = \{q\} \) for each \( q \in R(K^n) \cap \bar{K}^n_{m+1} \).

For each \( m = 0, \ldots, n−1 \) (starting with \( m = n−1 \)) we shall define an embedding \( h_m : X_{(m)} \to \bar{K}^n_{(m)} \) in such a way that

(1) \( h_m(E(X_{(m)}) \cap E(X)) \subseteq E(\bar{K}^n_{(m)}) \),

(2) \( h_m(E(X_{(m)}) \setminus E(X)) \subseteq R(K^n) \cap \bar{K}^n_{(m+1)} \),

(3) \( h_m(R(X) \cap X_{(m)}) \subseteq R(K^n) \cap \bar{K}^n_{m+1} \).
and for $m = 0, \ldots, n - 2$

$$(4)_m h_{m|X_{(m+1)}} = h_{m+1}. $$

Then $h_0$ will be the required embedding.

First we define the embedding $h_{n-1} : X_{(n-1)} = q_X e_X \to q_{\bar{e}} = K^n_{(n-1)}$.

From the construction of $K^n$ the set $R(K^n) \cap \bar{q}\bar{e} = \{q_1, q_2, \ldots \}$ is dense in $\bar{q}\bar{e} \setminus \{q, e\}$. Take two points $q_j, q_{j'} \in \{q_1, q_2, \ldots \} \cup \{q, e\}$ that satisfy the following conditions

(a) if $q_j, e_X \in E(X)$, then $q_j = q$ and $q_{j'} = e$;
(b) if $q_j \in E(X)$ and $e_X \notin E(X)$, then $q_j = q$ and $q_{j'} \in \{q_1, q_2, \ldots \}$;
(c) if $q_j \notin E(X)$ and $e_X \in E(X)$, then $q_j \in \{q_1, q_2, \ldots \}$ and $q_{j'} = e$;
(d) if $q_j \notin E(X)$ and $e_X \notin E(X)$, then $q_j, q_{j'} \in \{q_1, q_2, \ldots \}$.

Since the set $R(X) \cap q_X e_X$ is countable and the set $R(K^n) \cap q_{\bar{j}}, q_{\bar{j'}}$ is countable and dense in $\bar{q}_j \bar{q}_{j'}$, there exist a homeomorphism

$$h_{n-1} : q_X e_X \to q_{\bar{j}}, q_{\bar{j'}}$$

such that $h_{n-1}(q_X) = q_j$, $h_{n-1}(e_X) = q_{j'}$ and

$$h_{n-1}(R(X) \cap q_X e_X) \subseteq R(K^n) \cap q_{\bar{j}}, q_{\bar{j'}}. $$

Suppose that $0 \leq m \leq n - 2$ and $h_{n-1}, \ldots, h_{m+1}$ have been defined.

We shall define $h_m : X_{(m+1)} \to X_{(m+1)}$. For each $x \in X_{(m+1)}$ we put $h_m(x) = h_{m+1}(x)$. It remains to define $h_m$ on the sets $f_m^{-1}(r) \setminus \{r\}$, $r \in R(X_{(m)})$.

For each $r \in R(X_{(m)})$, we denote $h_{m+1}(r) = q_r$. From (5), (2)$_{m+1}$, and (3)$_{m+1}$ it follows that $q_r \in R(K^n) \cap \hat{K}_{m+1}$. Next from (vi) we conclude that $B_m(q_r) \cap \hat{K}_{m+1} = \{q_r\}$ and $B_m(q_r)$ is an $\omega$-od with center $q_r$.

The set $f_m^{-1}(r)$ is either an arc having $r$ as one of its end points, or an $\omega$-od with a center $r$, or an $\omega$-od with a center $r$. If $m < n - 2$, then $f_m^{-1}(r) \cap R(X)$ is countable and the set $B_m(q_r) \cap R(K^n)$ is dense in $B_m(q_r)$.

If $m = n - 2$, then $f_m^{-1}(r) \cap R(X) = \{r\}$ and $B_m(q_r) \cap R(K^n) = \{q_r\}$.

In either case there exists an (arm by arm) embedding $h'_m : f_m^{-1}(r) \to B_m(q_r)$ such that

(a) $h'_m(r) = q_r$;
(b) if $e \in E(f_m^{-1}(r)) \cap E(X)$, then $h'_m(e) \in E(B_m(q_r))$;
(c) if $e \in E(f_m^{-1}(r)) \setminus E(X)$, then $h'_m(e) \in B_m(q_r) \cap R(K^n)$;
(d) $h'_m(f_m^{-1}(r) \cap R(X)) \subseteq B_m(q_r) \cap R(K^n)$.

We define $h_m : X_{(m+1)} \to K_{(m)}$ by

$$h_m(x) = \begin{cases} h_{m+1}(x), & \text{if } x \in X_{(m+1)}, \\ h'_m(x), & \text{if } x \notin f_m^{-1}(r), \text{ } r \in R(X_{(m)}). \end{cases}$$

Then $h_m$ is an embedding that satisfies the properties (1)$_m$–(4)$_m$. $$\square$$

**Theorem 7.3.** If $X$ is a dendrite such that $\text{rdeg}(X) = n$, $n \in \{1, 2, \ldots \}$ and $X_{(n-1)}$ is a single point, then there exists an embedding $h : X \to K^n$ that $h(E(X)) \subseteq E(K^n)$.

**Proof.** The case $n = 1$ is trivial. Suppose that $n > 1$ and $X_{(n-1)} = \{q\}$. Then $X_{(n-1)}$ is either an $\omega$-od with a center $q$ or a $k$-od with a center $q$ for some natural number $k \geq 3$.

Let $q_1$ and $q_2$ be to different points of the plane. Join to $q_1$ a copy $Y$ of $X$ and to $q_2$ a copy $Z$ of $X$ in such a way that $Y_{(n-1)} = \{q_1\}$, $Z_{(n-1)} = \{q_2\}$, and $Y \cap Z = \emptyset$.

Obviously $\tilde{X} = Y \cup Z \cup q_1 \overline{q_2}$ is a dendrite and $\tilde{X}_{(n-1)} = \overline{q_1 \overline{q_2}}$. By Theorem 7.2 there is an embedding $h : \tilde{X} \to K^n$ such that $h(E(\tilde{X})) \subseteq E(K^n)$. Clearly, $h|_Z$ is the required embedding. $$\square$$

The corollary below follows from Theorems 7.2–7.3, and Fact 5.1.

**Corollary 7.1.** For any $n = 1, 2, \ldots$, the dendrite $K^n$ is universal in the family of all dendrites $X$ such that $\text{rdeg}(X) \leq n$. Moreover, for each dendrite $X$ with $\text{rdeg}(X) \leq n$ an embedding $h : X \to K^n$ can be defined in such a way that $h(E(X)) \subseteq E(K^n)$. 

8. Questions

1. Does a universal element exist in the family of dendrites $D$ with $\text{rdeg}(D) \leq \alpha$ for any fixed isolated ordinal $\alpha > \omega$.
2. Does a containing dendrite with a countable set of end points exist for the family of dendrites $D$ with $\text{rdeg}(D) \leq \alpha$ for any fixed ordinal $\alpha$.
3. Characterize the dendrites $D$ for which $\text{type}(E(D)) = \text{rdeg}(D)$.

References