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On Stationarity Conditions for a Certain Periodic Random Process*

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The statistical analysis of a periodic signal $x(t)$ accompanied by additive noise $n(t)$ [assumed stationary and independent of $x(t)$] is of importance in the study of communication theory and modulation processes.[1] Thus, we consider $y(t) = x(t) + n(t)$, where $x(t)$ is a known periodic function, and $n(t)$ is as described above. The usual approach treats $y(t)$ — and therefore $x(t)$ — as stationary, so that the second order properties of $y(t)$ can be most easily calculated. But except in certain trivial cases, $x(t)$ is nonstationary, and the results obtained via a stationarity assumption are erroneous.[2] This difficulty is avoided by using $x(t + \alpha)$ in place of $x(t)$, where α is a random variable uniformly distributed over the period of $x(t)$, and independent of $x(t)$ and $n(t)$.[3,4]

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[1] See, e.g., RICE, S. O. Mathematical analysis of random noise. *Bell System Tech. J.* (1944–5). Reprinted in WAX, N. "Selected Papers on Noise and Stochastic Processes," art 3.10. Dover, New York, 1954.

[2] A check of a number of texts and papers indicates that this erroneous assumption is frequently made.

[3] A random variable X is said to be uniformly distributed over a set E of finite Lebesgue measure $m(E) > 0$ if for any measurable set F we have Prob $(X \in F) = m(E \cap F)/m(E)$. By the Radon-Nikodym theorem this corresponds to a probability density positive constant on E, and zero on the complement of E.

[4] An equivalent assumption is made by Rice (see footnote 1). Some of the more rigorous engineering texts introduce $x(t + \tau)$ explicitly, with τ uniformly distributed. See DAVENPORT, W.B., AND ROOT, W.L., "An Introduction to the Theory of Random Signals and Noise," section 8–6. McGraw-Hill, New York, 1958.

At best, intuitive reasoning has been used in the literature to argue that a uniformly distributed α should be added to the argument of $x(t)$ to insure stationarity. In this paper, we shall show that the above procedure is actually sufficient to guarantee stationarity. More generally, those probability distributions of α for which $x(t+\alpha)$ is stationary can be completely characterized in terms of certain properties of $x(t)$ itself. Most important among these is the description of the σ-field which is induced on the real line by $x(t)$.

Throughout this work we shall assume $x(t)$ Borel measurable, non-random, and periodic with periodicity 1. The random variable α is taken to be finite valued, so that $x(t+\alpha)$ remains unchanged if we take $\tau = \alpha (\text{mod } 1)$ in place of α. Finally, we shall read $t+\tau$ as "$(t+\tau)$ (mod 1)."

Let $A_j = \{\tau | x(t_j + \tau) < a_j\}$, and take $A = \bigcap\limits_1^n A_j$. Evidently, A is a Borel set on the unit interval for arbitrary n, and any a_j, t_j, $j = 1, 2, \ldots, n$. Thus, $x(t+\tau)$ is defined as a random process; indeed, the probabilities $P(\tau \in A)$ [which we shall write $P(A)$] are the multivariate distribution functions which describe the process $x(t+\tau)$.

The indicator function $I_A(\cdot)$ is defined as usual, with the understanding that its argument be taken (mod 1). By considering the minimum σ-field generated by sets such as A, we arrive at $S(A)$, which is a subfield of the Borel sets on the unit interval.[5]

The definition of stationarity is most conveniently expressed in terms of indicators; $x(t+\tau)$ is stationary if and only if the expectation

$$E[I_B(t+\tau) - I_B(t-h+\tau)] = 0 \tag{1}$$

for every $B \in S(A)$, every t, and every h.

For the first result, we characterize the probability distributions of τ which render $x(t+\tau)$ stationary in terms of the Fourier expansion of indicators. Accordingly, we define for every set $B_\delta \in S(A)$ the Fourier coefficients $c_{n\delta} = \int_0^1 I_{B_\delta}(t)\, e^{-i2\pi nt}\, dt$ of the corresponding indicator. We then call $N_\delta = \{n | c_{n\delta} \neq 0\}$, and take $N = \bigcup\limits_\delta N_\delta$, the union extending over all δ such that $B_\delta \in S(A)$.

We are now able to state:

THEOREM 1: *A necessary condition for the stationarity of $x(t+\tau)$ is that $F(\tau)$ (the probability distribution function of τ) solves the trigonometric moment problem*

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[5] In theorem 4, we shall also have occasion to consider $\overline{S(A)}$, the completion of $S(A)$. There is no need to do this as long as our considerations are confined to Borel measurable $x(t)$.

$$\sigma_n = \int_0^1 e^{i2\pi n\tau}\, dF(\tau) \tag{2}$$

where the σ_n satisfy

$$\sigma_0 = 1, \tag{3}$$

$$\sigma_n = 0 \quad \textit{for all} \quad n \in N, \quad n \neq 0. \tag{4}$$

$$\textit{The } \sigma_n \textit{ are nonnegative definite.} \tag{5}$$

Conversely, let $F(\tau)$ be any absolutely continuous function satisfying (3), (4), *and* (5). *Then $F(\tau)$ is a probability distribution function, and $x(t + \tau)$ is stationary if τ is distributed according to $F(\tau)$.*

COROLLARY 1: If τ is uniformly distributed over the unit interval then $x(t + \tau)$ is stationary.

COROLLARY 2: Suppose every $n \in N$. Then $x(t + \tau)$ is stationary if and only if τ is uniformly distributed over the unit interval.

COROLLARY 3: Let M_δ be the completion in $L_2(0, 1)$ of the linear manifold generated by translations of $I_{B_\delta}(t)$, where such translations are defined by $T^a I_{B_\delta}(t) = I_{B_\delta}(t - a)$, $0 \leqslant a < 1$. Suppose there exists a $B_\delta \in S(A)$ such that $M_\delta = L_2(0, 1)$. Then $x(t + \tau)$ is stationary if and only if τ is uniformly distributed over the unit interval.

COROLLARY 4: Suppose that $x(t) \in L_2(0, 1)$, and that the subspace in $L_2(0, 1)$ generated by translations $T^a x(t) = x(t - a)$ is complete in $L_2(0, 1)$. Then $x(t + \tau)$ is stationary if and only if τ is uniformly distributed on the unit interval.

COROLLARY 5: If the complement of N (designated $\bar{N}$) contains integers other than $n = 0$, there exist a nondenumerable set of absolutely continuous distribution functions (for τ) which render $x(t + \tau)$ stationary.

PROOF OF THEOREM: For any $B_\delta \in S(A)$, let $z(t, \tau, h) = I_{B_\delta}(t + \tau) - I_{B_\delta}(t - h + \tau)$ and

$$z_n(t, \tau, h) = \sum_{-n}^{n} c_{k\delta}\, u_k(h)\, e^{i2\pi kt}\, e^{i2\pi k\tau} \tag{6}$$

where $c_{k\delta}$ are the Fourier coefficients of $I_{B_\delta}(t)$, and $u_k(h) = 1 - e^{-i2\pi kh}$ with h to be chosen later. Because any indicator is in $L_2(0, 1)$, we have $\text{l.i.m.}_{n\to\infty} z_n(t, \tau, h) = z(t, \tau, h)$ for all τ and h. There follows *a fortiori*

$$\lim_{n\to\infty} \int_0^\lambda z_n \, dt = \int_0^\lambda z \, dt \tag{7}$$

for an arbitrary $\lambda \in (0, 1)$.

Since (7) certainly holds for all (finite valued) τ, we may take the expectation on both sides. On the right side, $z(t, \tau, h)$ is bounded, jointly measurable, and therefore certainly integrable on the product space of t and τ; hence, Fubini's theorem is applicable. On the left side, we observe that $\int_0^\lambda z_n \, dt$ converges to $\int_0^\lambda z \, dt$ for every (finite valued) τ; further, successive application of the Schwarz and Bessel inequality shows that $|\int_0^\lambda z_n \, dt| \leqslant 1$ for all (finite valued) τ. Hence, Lebesgue's bounded convergence theorem is applicable. Putting these results together enables us to obtain

$$\lim_{n\to\infty} \left\{ E \int_0^\lambda z_n \, dt \right\} = \int_0^\lambda E[z] \, dt. \tag{8}$$

We now verify the necessity condition of the theorem. If $x(t + \tau)$ is indeed stationary, $E[z] = 0$ for all t and h follows from (1) and the definition of $z(t, \tau, h)$. Hence, the left side of (8) must be zero for all λ and h. Evaluating this side by carrying out the indicated operations yields

$$\sum_{\substack{-\infty \\ k\neq 0}}^{\infty} \frac{c_{k\delta}}{2\pi ik} u_k(h)\sigma_k(e^{i2\pi k\lambda} - 1) = 0 \tag{9}$$

with the σ_k as defined by (2). We shall show that (9) implies that

$$c_{k\delta} \, u_k(h)\sigma_k = 0 \qquad \text{all } k, \text{ all } h. \tag{10}$$

Indeed, if a Fourier series $\Sigma b_n e^{i2\pi nt}$ has $\Sigma |b_n|^2 < \infty$, there exists a $f(t) \in L_2(0, 1)$ having the b_n as Fourier coefficients. For any Fourier series, convergent or not, term by term integration is valid over any interval [1]. An application of this fact is that

$$\int_0^\lambda f(t) \, dt = \lambda b_0 + \sum_{n\neq 0} \frac{b_n}{i2\pi n} (e^{i2\pi n\lambda} - 1). \tag{11}$$

Suppose now that the right side of (11) is zero for every λ. Since $f(t)$ is integrable, a standard measure theoretic result leads to the conclusion that $f(t) = 0$ a.e. Consequently, $b_n = 0$ for all n.

In our problem, the role of b_n is played by $c_{n\delta}\, u_n(h)\sigma_n$. For any h, $|u_n(h)| \leqslant 2$. Since $F(\tau)$ is a probability distribution, $|\sigma_n| \leqslant 1$. Finally, $I_{B_\delta}(t) \in L_2(0, 1)$, so that $\Sigma |c_{n\delta}|^2 < \infty$. Thus we may make use of the preceding paragraph, and (10) follows.

If h is any irrational number $u_n(h) \neq 0$ unless $n = 0$. It is clear, therefore, that (10) is satisfied only if $\sigma_n = 0$ whenever $n \in N_\delta$. The above argument may be repeated for any $B_\delta \in S(A)$, so that we conclude $\sigma_n = 0$ for every $n \in \bigcup\limits_{\delta} N_\delta = N$. This is precisely the condition (4). As for (3) and (5), we need only observe that these are automatic because $F(\tau)$ is a probability distribution function [2].

We turn now to the proof of the second part of the theorem. In the first place, (3) and (5) suffice to assure that $F(\tau)$ is a probability distribution function [2]. Since (4) is satisfied by hypothesis, (10) holds for any δ such that $B_\delta \in S(A)$, and for all h. Therefore, $E[z_n] = 0$ for all δ, t, and h.

For each n, $z_n(t, \tau, h)$ is bounded and jointly measurable. This permits an interchange of integrations, i.e., $E\int_0^\lambda z_n\, dt = \int_0^\lambda E[z_n]\, dt = 0$. In other words, (8) implies that

$$\int_0^\lambda E[z]\, dt = \int_0^\lambda \{E[I_{B_\delta}(t + \tau) - I_{B_\delta}(t - h + \tau)]\}\, dt = 0 \tag{12}$$

for any $\lambda \in (0, 1)$. Let us call $E[z(t, \tau, h)] = g(t)$ (we suppress the h). Then (12) is equivalent to $g(t) = 0$ a.e. To complete the proof we must show that $g(t) \equiv 0$; we accomplish this by exhibiting the continuity of $g(t)$ as a consequence of the absolute continuity of $F(\tau)$.

The derivative $f(\tau) = F'(\tau)$ exists a.e., and it follows from the absolute continuity of $F(\tau)$ that $F(\tau) = \int_0^\tau f(\zeta)\, d\zeta$. Then for any number a we have the easily verified inequality[6]

$$|g(t) - g(t - a)| \leqslant \int_0^1 |f(\tau) - f(\tau + a)|\, d\tau. \tag{13}$$

Since $f(\tau) \in L_1(0, 1)$, the right side of (13) becomes zero as $a \to 0$. The latter is a familiar result which may also be proved by Lusin's theorem and the absolute continuity of the integral; we omit the proof here.

[6] Wherever a probability density function appears its argument is to be taken (mod 1).

PROOF OF COROLLARY 1: If τ is uniformly distributed, probability measure is absolutely continuous with respect to Lebesgue measure (see footnote 3). From the same footnote, it follows that $F(\tau) = \tau$ if τ is uniformly distributed. Substituting this $F(\tau)$ in (2) gives $\sigma_0 = 1$, $\sigma_n = 0$ when $n \neq 0$. These values of σ_n satisfy the conditions (3), (4), and (5); in particular, (4) is satisfied for any N whatsoever. The desired result is then a direct consequence of the second part of Theorem 1.

A proof of this corollary can also be given independent of Theorem 1. The proof rests on the fact that the Lebesgue measure [equivalent to probability measured when $F(\tau) = \tau$] of any Borel set is invariant under translation.

PROOF OF COROLLARY 2: "If" is true in any case from Corollary 1. To show "only if," observe that if every $n \in N$, (3) and (4) require that $\sigma_0 = 1$, $\sigma_n = 0$ for all $n \neq 0$. Since the trigonometric moment problem has a unique solution (when all σ_n are specified) up to an arbitrary additive constant, that solution must be $F(\tau) = \tau$ [3].

PROOF OF COROLLARY 3: We show that the hypothesis of this corollary implies that every $n \in N$; then Corollary 2 is applicable. That every $n \in N$ follows in turn from the "if" part of the following *lemma*: *Let $f(t) \in L_2(0, 1)$, and let M be the completion in $L_2(0, 1)$ of the linear manifold generated by translations of $f(t)$, where such a translation is defined by $T^a f(t) = f(t - a)$. Let c_n be the Fourier coefficients of $f(t)$ (as previously defined). Then $c_n \neq 0$ for every n if and only if $M = L_2(0, 1)$.*[7]

The proof of the lemma follows. The condition is necessary; if $c_n = 0$ then $g(t) = e^{i2\pi nt}$ is orthogonal to $f(t - a)$ for every a. For sufficiency, assume $c_n \neq 0$ (all n), and suppose there exists a nontrivial $g(t) \in L_2(0, 1)$ which is orthogonal to $f(t - a)$ for every a. Now by Parseval's relation

$$\sum c_n g_n^* e^{i2\pi na} = \int_0^1 f(t - a) g^*(t)\, dt = 0 \qquad \text{all } a \tag{14}$$

where the g_n are the Fourier coefficients of $g(t)$, and * denotes complex conjugacy. It follows from the uniqueness property of Fourier series that $c_n g_n^* = 0$ for every n, so that indeed $g_n = 0$ (for every n). Hence $g(t)$ can be orthogonal to the $f(t - a)$ only if it is zero a.e.

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[7] This result is undoubtedly well known, although the author has been unable to find it in the literature. A Fourier transform analogue is given by BOCHNER, S. AND CHANDRASEKHAR, K., Fourier transforms. *Ann. Math. Studies* **19**, 148–149 (1949).

Proof of Corollary 4: In this proof only, let $z(t, \tau, h) = x(t + \tau) - x(t - h + \tau)$. If $x(t + \tau)$ is stationary, we can demonstrate that $E[x]$ exists, and that indeed $E[z] = 0$ for every t and h.

For every (finite valued) τ, $\int_0^1 |x(t + \tau)| \, dt = M < \infty$. Then $E\int_0^1 |x(t + \tau)| \, dt = \int_0^1 E|x(t + \tau)| \, dt = M$ by Fubini's theorem, so that $E[|x(t+\tau)|] < \infty$ for a.e. t. But stationarity implies $P(T^h A_j) = P(A_j)$ for all A_j and h, where A_j is as previously defined, and $T^h A_j = \{\tau | x(t_j - h + \tau) < a_j\}$. Therefore, $E[|x(t + \tau)|] = E[|x(t - h + \tau)|]$ for all h, and hence $E[|x(t + \tau)|] = M$, the constant mentioned above. Since all necessary integrals exist, we may repeat the argument with $x(t + \tau)$ replacing $|x(t + \tau)|$. Then also $E[x(t + \tau)] = E[x(t - h + \tau)]$, i.e., $E[z]$ exists and is zero.[8]

Now let $z_n(t, \tau, h) = \Sigma_{-n}^{+n} c_k \, u_k(h) \, e^{i2\pi kt} \, e^{i2\pi k\tau}$, where the c_k are the Fourier coefficients of $x(t)$. Then $\int_0^1 |z_n| \, dt$ is uniformly bounded with respect to (finite valued) τ and h (use the Bessel and Schwarz inequalities). Furthermore, the preceding considerations have already shown that $E\int_0^1 |z| \, dt \leqslant 2M$. These bounds permit us to repeat the arguments of Theorem 1 verbatim, with the result that $c_k \sigma_k = 0$ for each $k \neq 0$. By the lemma in Corollary 3, all $c_k \neq 0$, so that $\sigma_k = 0$ for all $k \neq 0$. This, together with $\sigma_0 = 1$, leaves $F(\tau) = \tau$ as the unique solution (up to some arbitrary constant) of the trigonometric moment problem which yields the required σ_k.

Proof of Corollary 5: Because each $I_{B_\delta}(t)$ is real, $n \in \bar{N}$ implies $-n \in \bar{N}$, and conversely. Therefore, if $\bar{N} \cap \{n \neq 0\}$ is nonempty (as assumed in the hypothesis), there exists an $n \geqslant 1$ belonging to $\bar{N}$.

Let s be any real number, $1 < s < \infty$, and consider

$$f_s(\tau) = 1 + (s - 1) \sum_{\substack{n \geqslant 1 \\ n \in \bar{N}}} s^{-n} \cos 2\pi n\tau. \tag{15}$$

It is clear that $f_s(\tau)$ is a probability density function on $[0, 1)$, so that (3) and (5) are satisfied. Direct verification of (4) is an easy calculation. Therefore, the $\{f_s(\tau)\}$, $1 < s < \infty$, are a nondenumerable set of densities each satisfying the sufficiency conditions of Theorem 1, and implying the stationarity of $x(t + \tau)$.

The characterization of stationarity properties of $x(t + \tau)$ provided by theorem one and its corollaries is indirect at best; in specific cases verification of stationarity is rendered difficult if not impossible. The

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[8] In the engineering literature, $E[z] = 0$ is sometimes (erroneously) taken as a definition for, rather than a consequence of, stationarity.

next two theorems are devoted to types of $x(t)$ for which stationarity properties can be readily determined. The first of these (Theorem 2) makes use of the ideas of Theorem 1, while the second (Theorem 3) employs measure theoretic concepts which emphasize the relationship of stationarity conditions to the σ-field induced by $x(t)$.

THEOREM 2: *Let there exist an $a \neq 0$ such that $x(t + a) = x(t)$ for every t. Then we have:*

(i) *If a is irrational, and $x(t)$ is left (or right) continuous at some point t_0, then $x(t + \tau)$ is stationary irrespective of the probability distribution of τ.*

(ii) *If a is rational, call $a = p/q$, where p and q are relatively prime integers. Then $x(t + \tau)$ is stationary if τ is uniformly distributed over any interval[9] of length k/q, $k = 1, 2, \ldots, q$.*

PROOF: For (i) we show that $x(t)$ is identically a constant, i.e., that $x(t_0) = x(t_0 - \tau)$ for arbitrary τ. If this is true, $I_B(t + \tau) = I_B(t + \tau - h)$ identically for any $B \in S(A)$, and every t, τ, and h. Thus, (1) is satisfied for any distribution of τ.

To demonstrate the constancy of $x(t)$, we first let $t = t_0 - \tau$, and obtain $x(t_0 - \tau + na) = x(t_0 - \tau)$ by induction for any integer n. Therefore,[10] $x(t_0 - \tau + \tau_n) = x(t_0 - \tau)$, where $\{\tau_n\}$ is a denumerably dense set in $[0, 1)$. Let $\tau_n \searrow \tau$ if $x(t)$ is right continuous at t_0 (otherwise, take $\tau_n \nearrow \tau$); the desired result follows.

In the proof of (ii), the periodicity of $x(t)$ permits us to assume that $p < q$. Again using induction, we get $x(t + np/q) = x(t)$. It is easily shown that some $0 < n < q$ gives $(np/q) (\text{mod } 1) = 1/q$; hence $x(t)$ is actually periodic with period $1/q$.

Taking $A_j = \{\tau | x(t_j + \tau) < a_j\}$ as before, we note that $\tau \in A_j$ if $(\tau + k/q) \in A_j$ for each $k = 1, 2, \ldots, q$. This relationship is preserved under set operations, so that for every $B_\delta \in S(A)$, $\tau \in B_\delta$ if $(\tau + k/q) \in B_\delta$. Therefore, all $I_{B_\delta}(t)$ are periodic with period $1/q$. Consequently, the Fourier coefficients of any $I_{B_\delta}(t)$ are given by

$$c_{m\delta} = \left(\sum_{l=0}^{q-1} e^{-(i2\pi lm)/q} \right) \int_0^{1/q} I_{B_\delta}(t)\, e^{-i2\pi mt}\, dt \tag{16}$$

in which the summation is zero whenever m is not an integral multiple of q. In the notation of Theorem 1, $N \subset \{m | m = jq,\ j \text{ any integer}\}$.

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[9] An interval, as defined here, may also take the form $[b, 1) \cup [0, c)$ where $0 < c < b < 1$.

[10] The proof of this claim is an almost exact copy of the verification of theorem 16.C, HALMOS, P.R., "Measure Theory." Van Nostrand, Princeton, New Jersey. 1950.

To complete the proof through use of Theorem 1 we need only show that (4) is satisfied if τ has one of the probability density functions

$$f(\tau) = \begin{cases} q/k & \tau \in [c, c + k/q) \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

where c may be chosen arbitrary,[11] and k is any integer between 1 and q. For such an $f(\tau)$

$$\sigma_m = \frac{q\, e^{-i2\pi mc}}{i2\pi mk}\left(e^{(i2\pi mk)/q} - 1\right) \qquad m \neq 0 \tag{18}$$

which is zero whenever m is an integral multiple of q; i.e., $\sigma_m = 0$ for $m \in N$, $m \neq 0$.

THEOREM 3: *Let there exist a Borel set C such that $D = \{t | x(t) \in C\}$ is an interval whose Lebesgue measure $m(\cdot)$ is such that $0 < m(D) < 1$. Then $x(t + \tau)$ is stationary if and only if τ is uniformly distributed over the unit interval.*

COROLLARY: Under the conditions of the theorem the σ-field $S(A)$ [of sets $\{\tau | x(t_j + \tau) \in E\}$ generated by all $t_j \in [0, 1)$ and all Borel sets E] consists of all Borel sets on the unit interval.

PROOF OF THEOREM 3: In view of the preceding results, sufficiency is obvious. To prove necessity, let any suitable t_j be chosen, and define $A = \{\tau | x(t_j + \tau) \in C\}$. Then A has the same length as D, i.e., $m(A) = m(D)$. We may take $0 < m(A) \leqslant 1/2$ without loss of generality; if $1/2 < m(A) < 1$ we need only consider the complementary sets $\bar{A}$ and $\bar{C}$, and work with $\bar{A}$ in place of A.

Since A has positive measure, there exists a natural number n_0 such that $m(A) > 1/n_0$. Consider now the translation T^h defined by

$$T^h A = \{\tau | x(t_j + \tau - h) \in C\}. \tag{19}$$

Take $h = \pm [m(A) - 1/n]$, $n > n_0$. The $+$ or $-$ may be chosen to assure that $T^h A \cap A$ is a half-open interval. Indeed, $T^h A \cap A$ has length $1/n$, so that this set is either of the form $[a, a + 1/n)$ or $(a, a + 1/n]$, with the value of a depending on the particular choice of t_j.

Now observe that the sets $T^{j/n}(T^h A \cap A)$, $j = 1, 2, \ldots, n - 1$, partition the unit interval, and that, by stationarity, $P\{\tau \in T^{j/n}(T^h A \cap A)\}$

[11] If $(c + k/q) > 1$, this interval becomes $[c, 1) \cup [0, k/q)$.

is the same for all j. These facts, together with the additivity property of the probability measure, yield

$$nP\{\tau \in T^h A \cap A\} = \sum_{0}^{n-1} P\{\tau \in T^{j/n}(T^h A \cap A)\} \tag{20}$$

$$= P\{\tau \in \bigcup_{0}^{n-1} T^{j/n}(T^h A \cap A)\} = P\{\tau \in [0, 1)\} = 1.$$

Hence a second application of stationarity gives

$$P\left\{\tau \in \left[\frac{j}{n}, \frac{j+1}{n}\right)\right\} = P\{\tau \in T^{(j/n)-a}(T^h A \cap A)\} \tag{21}$$

$$= P\{\tau \in T^h A \cap A\} = 1/n.$$

Using additivity once more, we obtain for any rational $r \in [0, 1)$ the distribution function of τ as $F(r) = P\{\tau \in [0, r)\} = r$. This result is readily extended to arbitrary $\tau \in [0, 1)$. For since $F(\tau)$ is nondecreasing we have $r' \leqslant F(\tau) \leqslant r''$ whenever r', r'' are rationals such that $r' < \tau < r''$. If we now consider sequences $r' \nearrow \tau$ and $r'' \searrow \tau$ we see that $F(\tau) = \tau$, i.e., τ is uniformly distributed over the unit interval.

Proof of Corollary: It is clear from the proof of the theorem that the sets induced on the unit interval include all sets of the type $[a, a + 1/n)$, $n > n_0$, for any a. Denumerable set operations on such intervals yield in turn any interval of rational length, any interval of arbitrary length, and finally any Borel measurable set. Since $x(t)$ is itself a Borel measurable function, all such induced sets must be Borel measurable, so that the induced σ-field $S(A)$ does indeed coincide with the σ-field of Borel sets on the unit interval.

Theorem three is readily applied to forms of $x(t)$ that might occur in practice. For instance, if $x(t)$ is monotone, any interval in the range of $x(t)$ can be taken as the set C of the theorem. More generally, if $x(t)$ is right or left continuous, and has a unique supremum or infimum, the conditions of Theorem **3** can be satisfied. Another situation in which Theorem **3** is applicable is that in which $x(t)$ consists of a piecewise constant nonperiodic function; the construction of C in this case is left to the reader.

While Theorem **3** is often useful, the condition stated therein is not a necessary one. To construct an example where a C as described in the theorem fails to exist, consider $x(t) = t$ on the irrationals, and $x(t) = 0$ on the rationals. Even though we cannot find a proper C, $x(t)$

is such that stationarity results only if τ is uniformly distributed over the unit interval. To see this, define for any irrational $b \in (0, 1)$ and an arbitrary t the set $B \in S(A)$ by $B = \{\tau | x(t + \tau) < b\}$. The Fourier coefficients of $I_B(t)$ are then all nonzero, and reference to Corollary 2 of Theorem 1 shows τ must be uniformly distributed if $x(t + \tau)$ is to be stationary.

The above example rests upon the fact that two functions equal a.e. have the same Fourier expansion, so that if $x'(t)$ and $x''(t)$ induce σ-fields whose component sets differ only by sets of null measure, the necessity conditions on the stationarity of $x'(t + \tau)$ and $x''(t + \tau)$ will be the same. A more precise statement of this property is provided by

THEOREM 4: *Let $x(t)$ be Borel measurable and periodic on the unit interval, and let τ be distributed over the unit interval with a absolutely continuous distribution function $F(\tau)$. If $\hat{x}(t) = x(t)$ except on a set M (which may extend over the entire real line and need not be periodic) having Lebesgue measure $m(M) = 0$, then $\hat{x}(t + \tau)$ is a random process. Further, $\hat{x}(t + \tau)$ is stationary if and only if $F(\tau)$ is such that $x(t + \tau)$ is stationary.*

REMARK: Because $\hat{x}(t)$ may not be periodic, its argument is not read (mod 1).

PROOF: We complete $S(A)$ to $\overbrace{S(A)}$ as follows. Let $M_0 \in S(A)$ be such that $P[\tau \in M_0] = 0$, and consider the σ-field of sets $B \Delta M$, where $B \in S(A)$ and $M \subset M_0$. From the absolute continuity of $F(\tau)$, $m(M_0) = 0$ implies $P[\tau \in M_0] = 0$, and so $\overbrace{S(A)}$ contains all Lebesgue measurable sets. If the probability field is completed, the preimage of sets in $\overbrace{S(A)}$ belong to the completed probability σ-field. By extending the probability measure in the sense that $P[\tau \in B \Delta M] = P[\tau \in B]$, we obtain also[12] $\int_{B\Delta M} dF(\tau) = \int_B dF(\tau)$.

If $M = \{t | x(t) \neq \hat{x}(t)\}$, we note that for arbitrary t_j $T^{-t_j} M = \{\tau | x(t_j + \tau) \neq \hat{x}(t_j + \tau)\}$ is a translation of M, so that $m(T^{-t_j} M) = m(M) = 0$. Now take (for arbitrary a_j) $A_j = \{\tau | x(t_j + \tau) < a_j\}$ as before, and call $\hat{A}_j = \{\tau | \hat{x}(t_j + \tau) < a_j\}$. Then $\hat{A}_j = A_j \Delta M'$, where $M' \subset T^{-t_j} M$. M' belongs to $\overbrace{S(A)}$, and $m(M') = 0$; therefore, by the absolute continuity, $P[\tau \in M'] = 0$ also.

[12] For a general reference to measurability and completion of measure spaces see HALMOS, P.R., "Measure Theory," problem (11), p. 80, in particular. Van Nostrand, Princeton, New Jersey, 1950.

Evidently, $\hat{A}_j$ is measurable, as is $\hat{A} = \bigcap_1^n \hat{A}_j$ for any n, and arbitrary t_j, a_j, $j = 1, 2, \ldots, n$. Then $P(\hat{A})$ is defined, and in fact, $P(\hat{A})$ is a multivariate probability for $\hat{x}(t + \tau)$; thus $\hat{x}(t + \tau)$ is a random process.

Let A be defined as before, with the set of t_j and a_j the same as those for $\hat{A}$. We complete the proof of the theorem by showing that $P(A) = P(\hat{A})$, i.e., $x(t + \tau)$ and $\hat{x}(t + \tau)$ have the same distribution functions when a common probability distribution of τ is specified for both. If this is true, $P(T^h A) = P(T^h \hat{A})$, where the translation is as defined previously for any h. But stationarity of $x(t + \tau)$ is equivalent to $P(T^h A) = P(A)$, so that $P(T^h \hat{A}) = P(\hat{A})$ is implied thereby, and $\hat{x}(t + \tau)$ is stationary also. Since all relations are symmetrical, the stationarity of $\hat{x}(t + \tau)$ is turn implies that $x(t + \tau)$ is stationary.

The remaining proof is easy, for we have

$$|P(A) - P(\hat{A})| \leqslant \int |I_A(\tau) - I_{\hat{A}}(\tau)|\, dF(\tau) = \int_{A \Delta \hat{A}} dF(\tau) = P(A \Delta \hat{A}). \quad (22)$$

A calculation exhibits $A \Delta \hat{A}$ as at most a finite union of subsets of sets of null Lebesgue measure. Hence also $P(\tau \in A \Delta \hat{A}) = 0$.

We remark that the above result cannot be extended to an $F(\tau)$ with a jump component. As a counterexample, let $x(t) = 0$ identically, and $\hat{x}(t) = 0$ except at some one point. Then $x(t + \tau)$ is stationary for any distribution of τ, but a jump in $F(\tau)$ will render $\hat{x}(t + \tau)$ nonstationary. A considerably more complicated example is required to show that the above theorem need not hold for (nonabsolutely) continuous $F(\tau)$.

References

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3. Achieser, N.L., and Glasman, I.M., "Theorie der Linearen Operatoren im Hilbert Raum," pp. 144–148. Akademie Verlag, Berlin, 1954.