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Abstract Versions of L'Hôpital's Rule for Holomorphic Functions in the Framework of Complex B-Modules

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Abstract versions of L'Hôpital's rule are proved for the "ratio" $f(z)(g(z))^{-1}$, where $f : S \to X$, $g : S \to A$ are vector-valued holomorphic functions defined in a region of the complex plane containing S, A being a complex unital Banach algebra, and X a complex Banach module over A. Both cases, (i) $(g(z))^{-1} \xrightarrow{A} 0$, and (ii) $f(z) \xrightarrow{X} 0$, $g(z) \xrightarrow{A} 0$, as $z \xrightarrow{S} \alpha$, α being either finite or infinite, are considered when $f'(z)(g'(z))^{-1}$ has a finite limit. Applications are given to the asymptotics of linear second-order differential equations in Banach algebras.

1. INTRODUCTION

The past 15 years have witnessed a renewed interest in the generalization of the well-known L'Hôpital theorem. Almost all contributions, however, were concerned with the weakening of the classical hypotheses, remaining in any case within the realm of *real* functions of *one real* variable. The only exception in this context appeared in [1], where the case of complex-valued functions of one real variable was addressed. More recently, in [7, 8] extensions of the rule have been proposed in the framework of Banach modules, *still* for functions of a *real* variable. The reader is referred to [7] for a fairly complete list of references of the aforementioned contributions.

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In this paper, abstract versions of L'Hôpital's theorem are presented for the "ratio" $f(z)(g(z))^{-1}$ where $f: S \to X$, $g: S \to A$ are *vector-valued holomorphic* functions defined in a region of the *complex* plane containing S, A being a complex unital Banach algebra and X a complex Banach module over A. When $f'(z)(g'(z))^{-1}$ has a finite limit as $z \xrightarrow{S} \alpha$, with α either finite or infinite, sufficient conditions are given to ensure that $f(z)(g(z))^{-1}$ has the same limit. Both indeterminate cases, (i) $(g(z))^{-1} \xrightarrow{A} 0$, and (ii) $f(z) \xrightarrow{X} 0$, $g(z) \xrightarrow{A} 0$, as $z \xrightarrow{S} \alpha$, are considered (cf. Theorems 2.1, 2.4).

In Section 3, some applications to the asymptotics of linear second-order differential equations in commutative Banach algebras, as well as in the matrix algebra $M_n(\mathbf{C})$, are shown. We stress, finally, that not even the simple scalar case ($X = A = \mathbf{C}$) has been considered before, apart from the most trivial occurrences.

2. The Main Theorems

In what follows, the properties of holomorphic functions taking values in a complex Banach space will be used; for the basic theory, we refer to [4, Ch. 3]. Moreover, we shall term, for short, a function "holomorphic in S", $S \subseteq \mathbf{C}$, when it is holomorphic in a region (an open connected set) containing S. When necessary, the integrals used throughout the paper are intended in the sense of Bochner. Here we state the first result of the paper.

Theorem 2.1. *Let X be a complex right Banach module over the complex unital Banach algebra A, S a subset of the complex plane, and α a finite or infinite limit-point of S, $\alpha \in \partial S$.*

Let $f: S \to X$, $g: S \to A$ be holomorphic, with $g(z)$, $g'(z) \in \mathrm{Inv}(A)$ $\forall z \in S$, and $\|(g(z))^{-1}\|_A \to 0$, as $z \xrightarrow{S} \alpha$. Moreover, suppose that

(i)
$$\lim_{z \xrightarrow{S} \alpha} f'(z)(g'(z))^{-1} = L \in X; \tag{1}$$

(ii) *there exists $\xi \in S$ such that, for any $z \in S$, there is a simple piecewise regular path in S connecting ξ and z, say $\mathscr{L}_{\xi,z}$, parametrized by $\eta: [t_\xi, t_z] \to S$, $\eta(t_\xi) = \xi$, $\eta(t_z) = z$, such that, if $u \in \mathscr{L}_{\xi,z}$, then $\mathscr{L}_{\xi,u}$ coincides with $\mathscr{L}_{\xi,z}$ from ξ to u, and*

$$\limsup_{z \xrightarrow{S} \alpha} \int_{t_\xi}^{t_z} \|g'(\eta(t))(g(z))^{-1}\|_A \, |\eta'(t)| \, dt < \infty; \tag{2}$$

(iii) *$\|f'(z)(g'(z))^{-1}\|$ is bounded in S, and $\|g(z)\|$ is bounded in $S \cap \{z : |z-\alpha| = \delta\}$, $\forall \delta < \delta_0$, for some $\delta_0 > 0$, if α is finite, or in $S \cap \{z : |z| = R\}$, $\forall R > R_0$, for some $R_0 > 0$, if $\alpha = \infty$. Then*

$$\lim_{z \xrightarrow{S} \alpha} f(z)(g(z))^{-1} = L. \tag{3}$$

Clearly, *left-modules* can replace right-modules here as well as in Theorem 2.4 below, with obvious modifications.

Proof. For simplicity, we shall prove the theorem in the case $\alpha = \infty$. Taking $\varepsilon > 0$, in view of (i) there exists $R_\varepsilon > 0$ such that $\|f'(z)(g'(z))^{-1} - L\|_X < \varepsilon$ for $z \in S$, $|z| \geqslant R_\varepsilon$. By the holomorphy of f and g we can write [4, Ch. 3, Section 2]

$$f(z)(g(z))^{-1} - L = \int_{\mathscr{L}_{\xi,z}} [f'(\eta) - Lg'(\eta)]\, d\eta (g(z))^{-1} + [f(\xi) - Lg(\xi)](g(z))^{-1}, \tag{4}$$

where the second summand is an infinitesimal as $z \xrightarrow{S} \infty$. Let us consider the point $\xi_\varepsilon = \eta(t_\varepsilon)$, where $t_\varepsilon = \max\{t \in [t_\xi, t_z], |\eta(t)| = R_\varepsilon\}$, and split the first summand in (4) as

$$\int_{\mathscr{L}_{\xi,z}} [f'(\eta) - Lg'(\eta)]\, d\eta (g(z))^{-1} = \int_{\mathscr{L}_{\xi,\xi_\varepsilon}} [f'(\eta) - Lg'(\eta)]\, d\eta (g(z))^{-1} + \int_{\mathscr{L}_{\xi_\varepsilon,z}} [f'(\eta) - Lg'(\eta)]\, d\eta (g(z))^{-1}. \tag{5}$$

By the definition itself of Banach module, there exists $M > 0$ such that $\|m\beta\|_X \leqslant M\, \|m\|_X\, \|\beta\|_A$, $\forall m \in X$, $\forall \beta \in A$ (cf. [2, Ch. 1, Section 9, Def. 12]), and observing that (2) is equivalent to

$$\exists K > 0, \exists \rho > 0\colon \int_{t_\xi}^{t_z} \|g'(\eta(t))(g(z))^{-1}\|_A\, |\eta'(t)|\, dt \leqslant K, \forall z \in S, |z| > \rho, \tag{6}$$

we get for the second summand in (5) the estimate

$$\left\| \int_{\mathscr{L}_{\xi_\varepsilon,z}} [f'(\eta) - Lg'(\eta)]\, d\eta (g(z))^{-1} \right\|_X \leqslant \int_{t_\varepsilon}^{t_z} \|[f'(\eta(t))(g'(\eta(t)))^{-1} - L]\, g'(\eta(t))(g(z))^{-1}\|_X\, |\eta'(t)|\, dt \leqslant M\varepsilon \int_{t_\varepsilon}^{t_z} \|g'(\eta(t))(g(z))^{-1}\|_A\, |\eta'(t)|\, dt \leqslant MK\varepsilon, \tag{7}$$

for $z \in S$, $|z| > \rho \vee R_\varepsilon$.

As for the first summand in (5), we observe that by (iii) there exists $N>0$ such that $\|f'(z)(g'(z))^{-1}-L\|_X \leqslant N$ for all $z \in S$, and hence

$$\begin{aligned}
&\left\| \int_{\mathscr{L}_{\xi,\xi_\varepsilon}} [f'(\eta)-Lg'(\eta)]\,d\eta(g(z))^{-1} \right\|_X \\
&\quad \leqslant MN \int_{t_\xi}^{t_\varepsilon} \|g'(\eta(t))(g(z))^{-1}\|_A \,|\eta'(t)|\,dt \\
&\quad \leqslant MN\,\|g(\xi_\varepsilon)\|_A\,\|(g(z))^{-1}\|_A \int_{t_\xi}^{t_\varepsilon} \|g'(\eta(t))(g(\xi_\varepsilon))^{-1}\|_A\,|\eta'(t)|\,dt \\
&\quad \leqslant MNK\,\|g(\xi_\varepsilon)\|_A\,\|(g(z))^{-1}\|_A,
\end{aligned} \tag{8}$$

for $|z|>R_\varepsilon$, provided that $|\xi_\varepsilon| = R_\varepsilon$ has been chosen greater then ρ in order to apply (ii). The proof is complete since, by (iii), in (8) $\|g(\xi_\varepsilon)\|_A$ is bounded in $S \cap \{z : |z| = R_\varepsilon\}$, and thus the right-hand side of (8) is infinitesimal as $z \to \infty$ in $S \cap \{z : |z| > R_\varepsilon\}$. Q.E.D.

Remark 2.2. When condition (2) in (ii) can be replaced by the stronger one

$$\limsup_{z \xrightarrow{S} \alpha} \int_{t_\xi}^{t_z} \|g'(\eta(t))\|_A\,|\eta'(t)|\,dt\,\|(g(z))^{-1}\|_A < \infty, \tag{9}$$

then the latter can be given a simple *geometric interpretation* if the path $\mathscr{L}_{\xi,z}$ coincides with an arc of a *fixed* curve, say $\gamma_{\xi,\alpha}$, joining ξ to α in S, for all $z \in \gamma_{\xi,\alpha}$. In fact, if $l_{\xi,z}$ denotes the length of the curve (in A) $g(\mathscr{L}_{\xi,z})$, then

$$\frac{l_{\xi,z}}{\|g(z)-g(\xi)\|_A} \leqslant l_{\xi,z}\,\|(g(z)-g(\xi))^{-1}\|_A \leqslant \kappa l_{\xi,z}\,\|(g(z))^{-1}\|_A,$$

for some constant $\kappa > 0$ and z in a suitable neighborhood of α in S. Therefore, identifying $l_{\xi,z}$ in (9), it becomes clear that the ratio of the length of $g(\mathscr{L}_{\xi,z})$ to the corresponding chord is asymptotically finite as $z \to \alpha$ *on* $\gamma_{\xi,\alpha}$, in this case. Obviously, (2) and (9) coincide when $A = \mathbf{C}$, e.g..

If S is *closed*, necessarily $\alpha = \infty$. In fact, α is a singular point for $g(z)$ since $\|g(z)\| \geqslant 1/\|(g(z))^{-1}\| \to +\infty$ as $z \xrightarrow{S} \alpha$. It follows that condition (iii) above is automatically satisfied, $S \cap \{z : |z| = R\}$ being a compact subset of the holomorphy region.

Remark 2.3. Let α be an *isolated* (finite) singularity for g and $\|(g(z))^{-1}\| \to 0$ as $z \to \alpha$. Then it is necessarily a pole for g (since $\|g\| \to +\infty$ as $z \to \alpha$, cf. [4]). If α is also a pole for f, then Theorem 2.1 can be applied *whenever* the coefficient g_{-m} in the Laurent expansion of $g(z)$,

$g(z) = g_{-m}(z-\alpha)^{-m} + g_{-m+1}(z-\alpha)^{-m+1} + \dots$, is *invertible* in A. One can choose for S a suitable open punctured disc centered in α; as path $\mathscr{L}_{\xi,z}$ joining ξ (fixed in S) to z, one can follow the circular arc centered in α with radius $|\xi - \alpha|$ from ξ to the intersection with the straight line through α and z, and then the portion of such a line up to z. This result could also be seen directly manipulating the Laurent series representing f and g (with g_{-m} invertible) in the neighborhood of *any* pole of them. Note, however, that *when* g_{-m} is invertible, if α is a pole, then necessarily $\|(g(z))^{-1}\| \to 0$ as $z \to \alpha$.

It is worthwhile to observe that Theorem 2.1 can be applied when α is a pole for g and g_{-m} is invertible, *whatsoever* the behavior of f around α might be (provided that S contains paths $\mathscr{L}_{\xi,z}$ as above, e.g.).

The second result of this paper is given by

THEOREM 2.4. *Let X be a complex right Banach module over the complex unital Banach algebra A, S a subset of the complex plane, and α a finite or infinite limit-point of S.*

Let $f: S \to X$, $g: S \to A$ be holomorphic, with $g(z)$, $g'(z) \in \mathrm{Inv}(a)$ $\forall z \in S$, and $f(z) \xrightarrow{X} 0$, $g(z) \xrightarrow{A} 0$ as $z \xrightarrow{S} \alpha$. Moreover suppose that

(i) $$\lim_{z \xrightarrow{S} \alpha} f'(z)(g'(z))^{-1} = L \in X; \tag{10}$$

(ii) *for every $z \in S$ in a neighborhood $N(\alpha)$ of α there is a simple piecewise regular path in S, say $\mathscr{L}_{\alpha,z}$, connecting α to z, parametrized by $\eta: [t_\alpha, t_z] \to S \cup \{\alpha\}$, $\eta(t_\alpha) = \alpha$, $\eta(t_z) = z$, such that $|\eta(t) - \alpha| \leqslant |z - \alpha|$ for all $t \in [t_\alpha, t_z]$ when α is finite, or $|\eta(t)| \geqslant |z|$ when $\alpha = \infty$, and*

$$\limsup_{z \xrightarrow{S} \alpha} \int_{t_\alpha}^{t_z} \|g'(\eta(t))(g(z))^{-1}\|_A \, |\eta'(t)| \, dt < \infty. \tag{11}$$

Then

$$\lim_{z \xrightarrow{S} \alpha} f(z)(g(z))^{-1} = L. \tag{12}$$

Note that "piecewise regularity" of $\eta(t)$ is intended in a slightly generalized sense when $\alpha = \infty$.

Proof. As in (4), but recalling that now $f(\alpha) = 0$ and $g(\alpha) = 0$, we have

$$\begin{aligned} f(z)(g(z))^{-1} - L &= \int_{\mathscr{L}_{\alpha,z}} [f'(\eta) - Lg'(\eta)] \, d\eta (g(z))^{-1} \\ &= \int_{t_\alpha}^{t_z} [f'(\eta(t)) - Lg'(\eta(t))] \, \eta'(t) \, dt (g(z))^{-1}, \end{aligned} \tag{13}$$

for $z \in N(\alpha)$, since $f(\eta(t))$ and $g(\eta(t))$ are absolutely continuous in $[t_\alpha, t_z]$, being $f(\eta(t))$, $g(\eta(t)) \in AC_{loc}([t_\alpha, t_z])$ and $g'(\eta(t))\,\eta'(t) \in L([t_\alpha, t_z])$; $f'(\eta(t))\,\eta'(t) \in L([t_\alpha, t_z])$ then follows from (i). When $t_\alpha = -\infty$, by saying that these functions are $AC([t_\alpha, t_z])$ we mean that they are the integral between t_α and t of their derivatives (as they vanish in t_α); cf. [7] for the analogous observations in the *real* domain. Therefore

$$\|f(z)(g(z))^{-1} - L\|_X \leqslant M \int_{t_\alpha}^{t_z} \|f'(\eta(t))[g'(\eta(t))]^{-1} - L\|_X \times \|g'(\eta(t))(g(z))^{-1}\|_A \, |\eta'(t)| \, dt.$$

By the geometric property of the path $\eta(t)$ as in (ii), $\eta(t)$ belongs to some neighborhood of α whenever z does, and, consequently, $\|f'(\eta(t))[g'(\eta(t))]^{-1} - L\|_X < \varepsilon$. Finally, by (11),

$$\|f(z)(g(z))^{-1} - L\|_X \leqslant MK\varepsilon$$

for some $K > 0$ and z in a suitable neighborhood of α. Q.E.D.

Remark 2.5. The integral appearing in the key-condition (11) is finite, e.g., when $g'(z)$ has a finite limit as $z \xrightarrow{S} \alpha$. This is unnecessary, however. Taking $g(z) = \sqrt{z}$, where the square root is positive for z positive, $\alpha = 0$, and $S = \{z \in \mathbf{C} : z \neq 0, |\arg(z)| < \pi\}$, we obtain that $g'(z) = 1/2\sqrt{z} \to \infty$ as $z \xrightarrow{S} 0$, but the integral above is finite and more (11) is satisfied. In fact, choosing as paths $\mathscr{L}_{\alpha,z}$ the rays joining $\alpha = 0$ to z, we obtain

$$\frac{1}{|g(z)|} \int_{t_0}^{t_z} |g'(\eta(t))\,\eta'(t)| \, dt = \frac{1}{\sqrt{|z|}} \int_0^{|z|} \frac{1}{2\sqrt{t}} \, dt \equiv 1.$$

Remark 2.6. If α is an *analiticity* point for g, and $(S \cup \{\alpha\}) \cap B(\alpha, \delta)$ is, for some $\delta > 0$, a *star domain* with respect to α, then condition (11) is satisfied provided that the first *nonzero* derivative of g in α is *invertible* in A. Details are similar to those in the case of the pole (cf. Remark 2.3), and are left to the reader.

If α is an analiticity point for both f and g, one can see *directly* from their Taylor expansions that L'Hôpital's theorem holds whenever the first nonzero derivative of g in α is invertible. In this occurrence, the case $\lambda = \infty$ can be included.

Remark 2.7. As in Remark 2.2 suppose that condition (11) in Theorem 2.4 can be replaced by the stronger one

$$\limsup_{z \xrightarrow{S} \alpha} \int_{t_\alpha}^{t_z} \|g'(\eta(t))\|_A \, |\eta'(t)| \, dt \, \|(g(z))^{-1}\|_A < \infty. \tag{14}$$

Then, if the path $\mathscr{L}_{\alpha,z}$ coincides with an arc of a fixed curve in S, ending in α, say γ, for all $z \in \gamma$, a *geometric interpretation* can be given. In fact, (14) shows that the length of the curve (in A), $g(\mathscr{L}_{\alpha,z})$, is of order of the length of the chord $\|g(z)\|_A$.

When S itself *is a curve*, Theorems 2.1 and 2.4 can be obtained as special cases from the analogous (abstract) cases *on* **R** (cf. [7, 8]). Indeed, we have

$$\lim_{z \xrightarrow{S} \alpha} f(z)(g(z))^{-1} = \lim_{t \to t_\alpha} f(\eta(t))[g(\eta(t))]^{-1}$$

$$= \lim_{t \to t_\alpha} f'(\eta(t))[g'(\eta(t))]^{-1} = \lim_{z \xrightarrow{S} \alpha} f'(z)(g'(z))^{-1},$$

since holomorphy of f and g and piecewise regularity of η imply the required absolute continuity properties [7, 8].

Exploiting this reduction, we can give a simple counterexample in connection with Theorem 2.4. Let be $f(z) = z^2$, $g(z) = z^2 e^{i/z}$, $S = (0, +\infty)$, $\alpha = 0$. Then, taking $\eta(t) = t$, clearly $f'(t)/g'(t) \to 0$ as $t \to 0^+$, while $f(t)/g(t)$ has no limit (cf. [1, 7]). On the other hand,

$$\limsup_{t \to 0^+} \int_0^t \left| \frac{g'(s)}{g(t)} \right| ds = \lim_{t \to 0^+} \frac{1}{t^2} \int_0^t |2s - i| \, ds = +\infty.$$

One can observe that, taking $S = \{z \in \mathbf{C} : \operatorname{Im} z \geq 0,\ z \neq 0\}$, again $f'(z)/g'(z) \to 0$ while $f(z)/g(z)$ has no limit, as $z \xrightarrow{S} 0$. Then Theorem 2.4 implies that the limsup *must* be $+\infty$ *whichever* family of piecewise regular paths is chosen, all the other assumptions being satisfied. Condition (11) (as well as (2) in Theorem 2.1) is not necessary, however. In fact, the case $f = g = z^2 e^{i/z}$ shows that the thesis of L'Hôpital's rule trivially holds.

In closing, we present the following versions of the complex L'Hôpital's rules, which *cannot* be obtained from the theorems above.

THEOREM 2.8. *Suppose that all hypotheses of Theorem* 2.1 [2.4] *hold, for the pair of holomorphic functions* $f : S \to \mathbf{C}$, $g : S \to A$. *Then*

$$\lim_{z \xrightarrow{S} \alpha} f'(z)(g'(z))^{-1} = L \in A$$

implies that

$$\lim_{z \xrightarrow{S} \alpha} f(z)(g(z))^{-1} = L.$$

The proofs are similar to those of Theorems 2.1, 2.4, and thus are left to the reader.

3. Examples and Applications

Several applications of the previous theorems can be given for the purpose of illustration. Some of them are of independent interest.

Example 3.1. Let $f: S \to X$, X being a complex Banach space, $g: S \to \mathbf{C}$, both holomorphic. Suppose that S is any fixed annular sector with vertex at the origin. Then, if $g' \sim z^p$, $p > 0$, and $f'/z^p \to \lambda$ as $z \xrightarrow{S} \infty$, z^p representing a given holomorphic branch in S, we obtain $f/g \to \lambda$ as $z \xrightarrow{S} \infty$.

To prove this, observe first that the result holds for $g'(z) \equiv z^p$ in S. In fact, suppose that S' is an annular subsector of S (with the same angle) where (iii) of Theorem 2.1 holds, and ξ is any fixed point in S'. Then, choosing as path $\mathscr{L}_{\xi,z}$ a circular arc with center in 0 and radius $|\xi|$ from ξ to the intersection with the straight half-line through 0 and z, and then the part of such a line up to z, we obtain

$$\begin{aligned}\limsup_{z \xrightarrow{S} \infty} &(p+1)\,|z|^{-p-1} \int_{t_\xi}^{t_z} |\eta(t)|^p\, |\eta'(t)|\, dt \\ &= \limsup_{z \xrightarrow{S} \infty} (p+1)\,|z|^{-p-1} \\ &\quad \times \left[|\xi|^{p+1}\, |\mathrm{Arg}\, z - \mathrm{Arg}\, \xi| + \frac{|z|^{p+1} + |\xi|^{p+1}}{p+1} \right] = 1,\end{aligned}$$

and hence $f/g \to \lambda$ in this case (by Theorem 2.1). If $g' \sim z^p$, applying the previous result to the ratio $g'(z)/z^p$ we obtain that $g(z) \sim z^{p+1}/(p+1)$. Therefore, $f'/g' = (f'/z^p)(z^p/g') \to \lambda$ which shows that, again, $f/g \to \lambda$ since such a g verifies the limsup condition. In fact, $|g|$ can be bounded from below by $|z|^{p+1} c_0/(p+1)$, and $|g'|$ from above by $|z|^p c_1$, for some constants c_0, c_1, in a suitable neighborhood of ∞, $N(\infty)$, and ξ can be chosen in $N(\infty) \cap S'$, where S' is chosen as above.

Example 3.2 (*applications to abstract differential equations*).

(a) Consider the linear abstract differential equation

$$Y'' + Q(z)Y = 0, \qquad z \in S, \tag{15}$$

where $Q \in H(S; A)$, S being an open annular sector with vertex at the origin, A a complex *commutative* Banach algebra with unity, E; for generalities on this subject, the reader is referred to [5]. Then, *if* there is a solution to (15), $Y_1(z)$, with $Y_1(z) \sim E$ and $zY_1'(z) \to 0$ as $z \xrightarrow{S} \infty$, *then* there is a solution $Y_2(z)$, with $Y_2(z) \sim zE$ and $Y_2'(z) \sim E$ as $z \xrightarrow{S} \infty$, *and conversely*.

In fact, representing $Y_2(z)$ in terms of $Y_1(z)$ as

$$Y_2(z) = Y_1(z) \int_{z_0}^{z} (Y_1(t))^{-2}\, dt, \tag{16}$$

where $z_0 \in S$ belongs to a suitable neighborhood of ∞ (where $\exists (Y_1(z))^{-1}$), and $z \in S$, $|z| > |z_0|$, we obtain

$$\frac{Y_2(z)}{z} \sim \frac{1}{z} \int_{z_0}^{z} (Y_1(t))^{-2}\, dt, \qquad \text{as} \quad z \xrightarrow{S} \infty. \tag{17}$$

Then, by L'Hôpital's rule,

$$\frac{Y_2(z)}{z} \sim (Y_1(z))^{-2} \to E, \qquad \text{as} \quad z \xrightarrow{S} \infty \tag{18}$$

(cf. Example 3.1 with $p = 1$). Moreover, from (16),

$$Y_2'(z) = Y_1'(z) \int_{z_0}^{z} (Y_1(t))^{-2}\, dt + (Y_1(z))^{-1}, \tag{19}$$

and hence, using again the rule, $Y_2'(z) \to E$. Conversely, if there is a solution $Y_2(z) \sim zE$ as $z \xrightarrow{S} \infty$, then from

$$Y_1(z) = Y_2(z) \int_{z}^{\infty} (Y_2(t))^{-2}\, dt, \tag{20}$$

valid for $z \in S$ with $|z|$ sufficiently large, applying Theorem 2.4 with $f(z) = \int_z^\infty (Y_2(t))^{-2}\, dt$, $g(z) = 1/z$, we obtain

$$Y_1(z) \sim z^2 (Y_2(z))^{-2} \to E, \; z \xrightarrow{S} \infty. \tag{21}$$

Clearly, f and g tend to 0 as $z \xrightarrow{S} \infty$, and the usual limsup term is equal to 1. All paths $\mathscr{L}_{\infty,z}$ to be used can be chosen as simple rays in S through z and 0. The statement concerning the derivative is verified as, from (20), obtains

$$zY_1'(z) = zY_2'(z) \int_{z}^{\infty} (Y_2(t))^{-2}\, dt - z(Y_2(z))^{-1} \to 0, \qquad \text{as} \quad z \xrightarrow{S} \infty,$$

where L'Hôpital's rule has been used again.

(b) Consider Eq. (15), with $Q \in H(S; A)$, S being an open annular sector with $|\operatorname{Arg} z| < \gamma < \pi/2$, and A a commutative unital C^*-algebra. Then, *if* there is a solution $Y_1(z)$ to (15) with $Y_1(z) \sim e^{-\sqrt{\beta} z}$,

$Y_1'(z) \sim -\sqrt{\beta}\, e^{-\sqrt{\beta} z}$ as $z \xrightarrow{S} \infty$, for some β *positive* in A, *then* there is a solution $Y_2(z)$, with $Y_2(z) \sim e^{\sqrt{\beta} z}$, $Y_2'(z) \sim \sqrt{\beta}\, e^{\sqrt{\beta} z}$ as $z \xrightarrow{S} \infty$. Here, $\sqrt{\beta}$ denotes the positive square root of β; cf. [6, Ch. 11]. The converse is also true.

In fact, from (16) obtains, for $z \xrightarrow{S} \infty$,

$$e^{-\sqrt{\beta} z} Y_2(z) \sim e^{-2\sqrt{\beta} z} \int_{z_0}^{z} (Y_1(t))^{-2}\, dt$$
$$\sim (2\sqrt{\beta}\, e^{2\sqrt{\beta} z})^{-1} (Y_1(z))^{-2} \to \frac{1}{2} (\sqrt{\beta})^{-1}, \tag{22}$$

where L'Hôpital's rule has been applied as in Theorem 2.1 with $f(z) = \int_{z_0}^{z} (Y_1(t))^{-2}\, dt$ and $g(z) = e^{2\sqrt{\beta} z}$. We stipulate that the sector S has been reduced (increasing its radius) if necessary, to guarantee both that Y_1 is invertible there (in view of its asymptotics), and to meet condition (iii) in Theorem 2.1 (in view of the fact that $f'(g')^{-1}$ has a finite limit). Therefore, Theorem 2.1 can be applied since $\|(g(z))^{-1}\|_A = \|e^{-2\sqrt{\beta} z}\|_A = \rho(e^{-2\sqrt{\beta} z}) = e^{-2 \min \sigma(\sqrt{\beta}) \operatorname{Re} z} \to 0$ as $z \xrightarrow{S} \infty$, $\rho(\cdot)$ and $\sigma(\cdot)$ denoting spectral radius and spectrum, respectively. Moreover, the limsup condition can be satisfied using the same paths as in Example 3.1. In fact, on the straight part we have

$$\begin{aligned} \|g'(\eta(t))(g(z))^{-1}\|_A &= \|2\sqrt{\beta} \exp\{2\sqrt{\beta}(e^{i \operatorname{Arg} z} t - z)\}\|_A \\ &\leqslant 2\, \|\sqrt{\beta}\|_A\, \rho(\exp\{2\sqrt{\beta}\, e^{i \operatorname{Arg} z}(t - |z|)\}) \\ &= 2\, \|\sqrt{\beta}\|_A \exp\{2 \min \sigma(\sqrt{\beta}) \cos(\operatorname{Arg} z)(t - |z|)\}, \end{aligned} \tag{23}$$

being $|\xi| \leqslant t \leqslant |z|$ there. Finally, rescaling $Y_2(z)$ appearing in (22) with the factor $2\sqrt{\beta}$, we obtain for Y_2 the asymptotic result predicted above, and an easy calculation (again based on rule) yields that for Y_2'.

The converse can be proved similarly, by Theorem 2.4. We only observe that the limsup condition is fulfilled being now

$$\begin{aligned} &\|g'(\eta(-t))(g(z))^{-1}\|_A \\ &\quad \leqslant 2\, \|\sqrt{\beta}\|_A \exp\{-2 \min \sigma(\sqrt{\beta}) \cos(\operatorname{Arg} z)(t - |z|)\}, \end{aligned}$$

where the paths $\mathscr{L}_{\infty, z}$ used are the straight half-lines through the origin and z, joining ∞ to z, thus $\eta(t) = -e^{i \operatorname{Arg} z} t$, $-\infty < t \leqslant -|z|$.

EXAMPLE 3.3 (*applications to matrix differential equations*).

Suppose that Eq. (15) is given, with $Q(z)$ *symmetric* $n \times n$ *holomorphic* matrix in an annular sector S as in Example 3.2(b). *If* there exists a solution $Y_1(z)$, with $Y_1(z) \sim e^{-\sqrt{B}z}$, $Y_1'(z) \sim -\sqrt{B}\, e^{-\sqrt{B}z}$ as $z \xrightarrow{S} \infty$, B being a *real symmetric positive definite* $n \times n$ matrix, and $Y_1(z)$ *commutes* with B (and thus with $\sqrt{B}$) for every $z \in S$, *then* there is a solution $Y_2(z)$, with $Y_2(z) \sim e^{\sqrt{B}z}$, $Y_2'(z) \sim \sqrt{B}\, e^{\sqrt{B}z}$ as $z \xrightarrow{S} \infty$. Such a solution also commutes with B. The converse statement can also be proved.

Note that the expression $Y_1(z) \sim e^{-\sqrt{B}z}$ is unambiguous, owing to the commutativity hypothesis. Since it is easily proved by its asymptotics that Y_1 is a "prepared" solution to (15), as $W(z) := (Y_1)^T Y_1' - (Y_1')^T Y_1 \to 0$, which implies that the matrix $W(z)$, being a constant, must be identically zero (cf. [3, 5]), the representation

$$Y_2(z) = Y_1(z) \int_{z_0}^{z} [(Y_1(t))^T Y_1(t)]^{-1}\, dt$$

holds, for z_0, z in a neighborhood of ∞. All proceeds then similarly to Example 3.2(b), and analogously regarding the converse statement.

This example is, to some extent, the noncommutative finite-dimensional counterpart of Example 3.2(b). The matrix case analogous to Example 3.2(a) is trivial, since L'Hôpital's rule given in Theorems 2.1, 2.4 can be applied componentwise.

EXAMPLE 3.4 (*extending derivatives of holomorphic functions up to boundary points*).

Suppose that $\phi \in H(\Omega; \mathscr{B}) \cap C^0(\Omega \cup \{\alpha\}; \mathscr{B})$, $\Omega \subseteq \mathbf{C}$ being a region, $\mathscr{B}$ a complex Banach space, $\alpha \in \partial\Omega$. If there exists $\lim_{z \xrightarrow{\Omega} \alpha} \phi'(z) = \lambda$, then

$$\lim_{z \xrightarrow{\Omega} \alpha} \frac{\phi(z) - \phi(\alpha)}{z - \alpha} = \lambda,$$

provided that $(\Omega \cup \{\alpha\}) \cap B(\alpha, \delta)$ is a *star-domain* with respect to α, for some $\delta > 0$. In fact, the limsup condition in (11), Theorem 2.4, is immediately satisfied with $g(z) = z - \alpha$ and taking for $\mathscr{L}_{\alpha,z}$ the segments joining each z in $(\Omega \cup \{\alpha\}) \cap B(\alpha, \delta)$ to α. This shows that λ is the derivative of ϕ "from inside" (cf. [7] for the real variable analogue).

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