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A Generalized Multiple Scales Approach to a Class of Linear Differential Equations

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A technique for approximating uniformly the solutions for a class of ordinary linear differential equations with variable coefficients is developed. The coefficients are taken to contain a small or large parameter in a simple way. In particular, the coefficients vary on a single scale and are small and rapidly varying or large and slowly varying. The method employed is the following ("extension"). The ordinary differential equation is replaced by a set of partial differential equations that determine the unknown function in terms of a set of independent "scales." The partial differential equations, in conjunction with the requirement of uniformity of the approximation in an interval, help us establish the functional dependence of the scales in terms of the original independent variable ("scale functions").

With the use of two scales, we obtain an approximation to the amplitude and phase of each of the independent solutions of nth-order equations that improves perturbative and frozen approximations. In particular, "whipping tail" effects are eliminated. Under appropriate conditions, for second-order equations, the Liouville-Green (or WKBJ) approximation is readily recovered as a special case of our method. Several examples are given. It is essential, for the success of the approximation, that the scale functions be nonlinear as well as, in general, complex. Thus, the present approach generalizes earlier "time scale" analyses in several respects.

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1. Introduction

This paper is concerned with a technique for approximating the solutions for a class of ordinary linear differential equations with variable coefficients.

While the linear equation of the first order can be solved explicitly in the form of a quadrature over the coefficient, higher-order equations cannot be handled in this way.[1] For example, the second-order equations of Bessel and Mathieu yield transcendental functions which cannot be expressed in terms of quadratures over the coefficients of the defining equation. Approximate solutions in this form are, however, very desirable; for example, when it is of interest to study the effect of arbitrary variations in the coefficients.

In this paper we assume that a small (or large) parameter appears in the coefficients. We then develop a method, based on the concept of extension, for obtaining asymptotic approximations in the form of simply calculable functions of the coefficients. We will show that for second-order equations, the Liouville-Green (or WKBJ) approximation is recovered as a special case of our method. The technique is then extended to higher-order equations. Our main results are (3.3.10) and (3.3.18).

Direct expansion in powers of the parameter often leads to a serious misrepresentation of the true function for a certain range of the independent variable. This occurrence is termed a nonuniformity in the perturbation expansion. Techniques have been devised to overcome this difficulty and render the approximations uniformly valid, as, for example, an expansion [1] of the independent variable (developed by Lighthill), the method of matched asymptotic expansions [2], and the method of extension [3]. We shall mainly follow the method of extension and study a class of ordinary linear differential equations with variable coefficients. We shall see that the failure of the direct perturbation expansion has as its *raison d'etre* an inappropriate scale on which the function is observed. The natural scales can be interpreted as "clocks" which permit us to give a uniform description of the phenomenon and are determined by knowing the precise nature of the breakdown of the direct expansion.

We recall what is meant by a uniformly valid approximation. We will denote by ϵ a "small" parameter (i.e., $|\epsilon| \ll 1$) and the one-dimensional independent variable will be called the time. Given a function $y(t, \epsilon)$ of arbitrary shape (Fig. 1), $y_0(t, \epsilon)$ is said to be a uniformly valid approximation to $y(t, \epsilon)$ to order ϵ, in a specific interval, if and only if for all t in that interval:

$$y(t, \epsilon) = y_0(t, \epsilon) + O(\epsilon). \tag{1.1}$$

More generally we could have

$$y(t, \epsilon) = y_0(t, \epsilon) + o(1).$$

[1] We do not consider time-ordered exponentials of integrals over matrices "explicit".

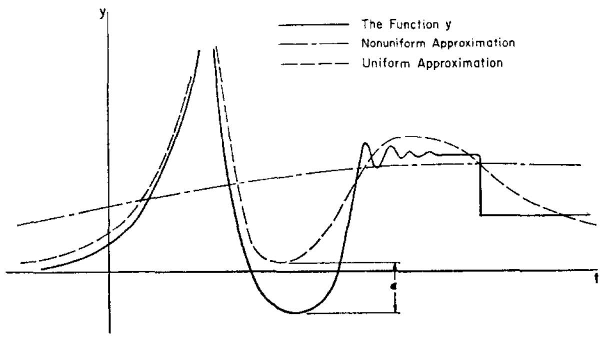

FIG. 1. Approximations to y.

That is, the relation (1.1) holds if and only if the error between the function and its approximation is uniformly small within the domain of interest. Precise definitions are given by Erdelyi [4] and Wasow [5].

2. The Concept of Extension

The origin of the concept can be traced to the work of Poincaré on the secular expansion in celestial mechanics, and to the works of Krylov, Bogoliubov, and Mitropolsky [6] who allowed a slow variation in the constants arising in lowest-order perturbation theory. The technique of multiple scales was applied to certain nonlinear differential equations by Cole and Kevorkian [7]. Some problems in celestial mechanics have also been treated in this manner by Kevorkian [8]. Frieman [9] and Sandri [3] have developed it in the context of the theory of irreversible processes [10]. These applications employ linear time scales. Also, one of us has considered a general technique (method of extension) of uniformization and has discussed the relations of this method to the others mentioned [3].

The fundamental idea of the method of extension is to enlarge the domain of the independent variable to a space of higher dimension. Thinking of the independent variable as time, we introduce a set of new independent "clocks". A complete reparameterization of the lowest-order term in the perturbation expansion can thus be achieved. The clock variables, in general, will not be restricted to be real. The "clocks" are so chosen as to eliminate the nonuniformities of direct perturbation theory. In the extended domain, uniform approximations to the unknown function may then be obtained. The simplest extension introduces a set of linear scales.

There are many problems, however, for which linear scales are inadequate. It is the aim of this paper to demonstrate the need for, and the usefulness of, nonlinear scales. This will be accomplished by studying a class of linear ordinary differential equations with variable coefficients. In general, the scales may turn out to be complex quantities. Thus, the present paper generalizes earlier time scales analyses in several respects.

We illustrate the idea of extension with a very simple example. This example is treated in Section 3.1 from the point of view of differential equations. Consider a slow exponential decay

$$y = \exp(-\epsilon t). \tag{2.1}$$

Direct expansion of y in powers of ϵ yields

$$y = 1 - \epsilon t + \frac{\epsilon^2 t^2}{2!} + \cdots . \tag{2.2}$$

A finite term representation of this exponential series fails for $t \gtrsim 1/\epsilon$. A physical picture comes to mind if we take y to represent an observable quantity such as a displacement from a reference position or a temperature difference between two bodies. An observer who measures y and records it using a clock with the units of $\tau_0 = t$ will have to wait for a long time (the longer, the smaller ϵ is) before he can observe a perceptible change in y and will have considerable difficulty in ascertaining the exponential nature of the quantity. Instead, if our observer were to use the slow variable $\tau_1 = \epsilon t$, i.e., a "super" clock which measures time in giant units of t/ϵ, the nature of the phenomenon would transpire clearly since then our function can be written simply as $y = \exp(-\tau_1)$. The method of extension aims at facilitating such a useful change of variable. Its purpose is to enable us to perform readings on appropriate scales by employing a sufficient number of independent "observers". Thus, in a phenomenon exhibiting a mixed behavior in time, the slow and fast motions are to be extracted individually. Figure 2 shows a schematic of the concept.

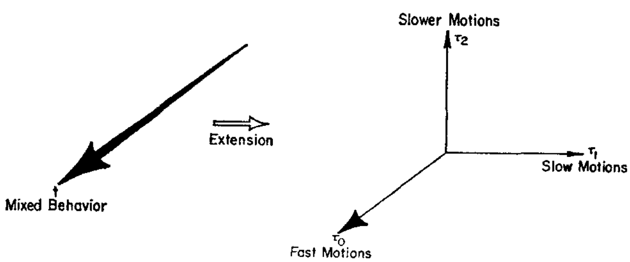

FIG. 2. Extension of the Domain.

A *geometric interpretation* can be given as follows. Consider a three-dimensional space (Fig. 3) with orthogonal axes, τ_0, τ_1, and $\mathbf{y}$. Readings on "fast" and "slow" clocks are represented, respectively, by points along τ_0 and τ_1 coordinates and $\mathbf{y}$ is defined to be the function

$$\mathbf{y}(\tau_0, \tau_1) = \exp(-\tau_1). \tag{2.3}$$

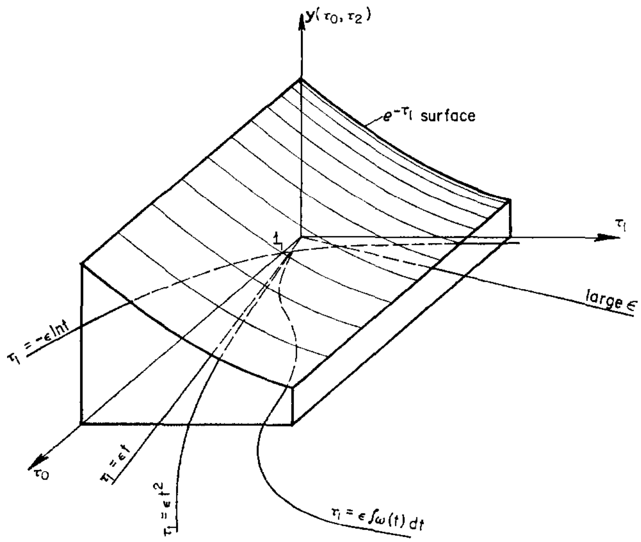

FIG. 3. Function surface in extended space.

Graphically, $\mathbf{y}(\tau_0, \tau_1)$ is represented by the cylindrical surface in Fig. 3 which is constant in τ_0, but decays exponentially in τ_1. To relate $\mathbf{y}(\tau_\theta, \tau_1)$ to $y(t)$, substitute $\tau_0 = t$ and $\tau_1 = \epsilon t$ into (2.3), then

$$\mathbf{y}(t, \epsilon t) = y(t). \tag{2.4}$$

The function $\mathbf{y}(\tau_0, \tau_1)$ is said to be an extension of $y(t)$. We are now prepared to give a *formal definition* of the concept of extension.

Given a function $y(t, \epsilon)$ where t is, in general, an n-dimensional vector, and a function $\mathbf{y}(\tau_0, \tau_1, ..., \tau_{N-1})$ of the N-independent variables $\tau_0, \tau_1, ..., \tau_{N-1}$ (each of which is an n-dimensional vector), $\mathbf{y}$ is said to be an extension of y if and only if there exists a set of $N \cdot n$ functions

$$\tau_k = \tau_k(t, \epsilon) \qquad k = 0, 1, 2, ..., N-1 \tag{2.5}$$

which, when inserted into $\mathbf{y}$, give:

$$\mathbf{y}(\tau_0(t, \epsilon), \tau_1(t, \epsilon), ..., \tau_{N-1}(t, \epsilon)) = y(t, \epsilon). \tag{2.6}$$

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The space of N-tuplets $\tau = \{\tau_0, \tau_1, ..., \tau_{N-1}\}$ is called the extension of the domain, and the locus defined by the equations $\tau_k = \tau_k(t, \epsilon)$ is called the "trajectory" in the extended domain. The result of substituting the trajectory in the extended function is the "restriction" of $\mathbf{y}$ and is denoted by $\mathbf{y}(\tau)|_t$. We shall denote the mappings described above by the following notation:

$$t \to \{\tau_0, \tau_1, ..., \tau_{N-1}\}, \qquad y \to \mathbf{y}.$$

It is evident that there are infinitely many extensions which correspond to a given function. In particular, if $\mathbf{y}$ is an extension of y, the result of multiplication by an arbitrary extension of the unit function and addition of an arbitrary extension of the zero function is also an extension. Simple examples of these extensions for the trajectory $\tau_0 = t$ and $\tau_1 = \epsilon t$ are:

$$1 \to \exp(\tau_1 - \epsilon\tau_0), \qquad 0 \to 1 - \exp(\tau_1 - \epsilon\tau_0). \tag{2.7}$$

Two types of freedom are available: the choice of the trajectory and the choice of the extension of y itself. Both are utilized in obtaining a $\mathbf{y}$ with a simpler and smoother dependence on the parameter than that offered by y. Such a dependence should clearly facilitate the determination of uniformly valid approximations in the domain of interest. It is clear that "in general" the concept of extension can be applied to the range as well as to the domain of the function y.

The derivatives, and indeed entire differential expressions, can be treated as functions on t and can be extended with the above definition. Derivatives of y are, of course, *functionals* on y but *functions* on t. Consider, for example,

$$\phi(t) \equiv \frac{dy}{dt} + \epsilon\omega(t)\, y. \tag{2.8}$$

An extension of ϕ corresponding to the trajectory $\tau_0 = t$, $\tau_1 = \epsilon t$ is

$$\mathbf{\Phi} \equiv \frac{\partial \mathbf{y}}{\partial \tau_0} + \epsilon \frac{\partial \mathbf{y}}{\partial \tau_1} + \epsilon\omega(\tau_0)\, \mathbf{y}. \tag{2.9}$$

Extensions similar to the one given in (2.9) are readily constructed for a general trajectory. Note that we have extended

$$\omega(t) \to \omega(\tau_0). \tag{2.10}$$

This particularly simple choice will be maintained throughout the rest of this paper. Clearly, the extension (2.10) can be used only if the variable coefficient depends on a single time scale and does not contain ϵ independently. Quite generally, for a linear differential equation for $y(t)$ with nonconstant coefficients $\omega_i(t)$

$$L([\omega_i], y) = 0 \tag{2.11}$$

the freedom available in extension corresponds to the choice of the trajectory *and* to the extension of ω_i. We shall study below the nonlinear trajectory $\tau_0 = t$, $\tau_1 = \epsilon k(t)$ (where k is to be determined) in conjunction with the extension (2.10).

3. Linear Differential Equations

We will now apply these ideas to some linear differential equations. We use the first-order equation to illustrate the mechanics of our approximation scheme. The class of equations discussed below can be characterized qualitatively by the following restrictions on the coefficients:

(i) all coefficients vary on a single scale,

(ii) the coefficients are either large and slowly-varying or small and rapidly-varying.

3.1. *First-order Equations*

Consider, first, the linear equation of the first order with *constant coefficient*,

$$\frac{dy}{dt} + \epsilon y = 0. \tag{3.1.1}$$

Direct Taylor expansion in powers of ϵ yields, with $y(0) = 1$,

$$y = \sum_{n=0}^{\infty} \epsilon^n y_n(t), \qquad y_n = (-1)^n \frac{t^n}{n!}$$

which corresponds to (2.2). As shown in the discussion following (2.2), this expansion is not uniformly valid. Our method readily uniformizes this simple case. Take the extension $t \to \{\tau_0, \tau_1\}$ with linear trajectory

$$\tau_0 = t, \qquad \tau_1 = \epsilon t \tag{3.1.2}$$

The time derivative operator is extended as

$$\frac{d}{dt} \to \frac{\partial}{\partial \tau_0} + \epsilon \frac{\partial}{\partial \tau_1}.$$

Now, we extend the dependent variable as $y(t) \to \mathbf{y}(\tau_0, \tau_1)$ and obtain, equating powers of ϵ:

$$\frac{\partial \mathbf{y}}{\partial \tau_0} = 0 \tag{3.1.3}$$

$$\frac{\partial \mathbf{y}}{\partial \tau_1} + \mathbf{y} = 0. \tag{3.1.4}$$

From (3.1.3) and (3.1.4) $\mathbf{y}(\tau_0, \tau_1) = A(\tau_1) = c \exp(-\tau_1)$. We now restrict the extended function along the trajectory (3.1.2) and obtain

$$y(t) = c \exp(-\epsilon t) \tag{3.1.5}$$

which is the exact solution of (3.1.1) with $y(0) = c$, and independent of ϵ. It is clear that the treatment is independent of whether ϵ is small or large. This situation will hold for the case with variable coefficient also. For higher-order equations, however, it will be important to distinguish the two cases.

We now consider equations with a variable coefficient. We first show that the extension with linear scales (3.1.2) is inadequate. We will then use non-linear time scales. Consider, in fact, the linear equation:

$$\frac{dy}{dt} + \epsilon\omega(t)\, y = 0 \tag{3.1.6}$$

with a coefficient that depends on a single time scale. Taylor expansion (i.e., the direct perturbation expansion) expresses y in terms of powers of $\int \omega\, dt$, while the correct result is the exponential function of $\int w\, dt$ given by (3.1.14). Using the trajectory (3.1.2) and with the extension (2.10) for ω, we have, equating powers of ϵ,

$$\frac{\partial \mathbf{y}}{\partial \tau_0} = 0 \tag{3.1.7}$$

$$\frac{\partial \mathbf{y}}{\partial \tau_1} + \omega(\tau_0)\, \mathbf{y} = 0 \tag{3.1.8}$$

i.e., $\mathbf{y}(\tau_0, \tau_1) = A(\tau_1)$ and $A'(\tau_1) + \omega(\tau_0)\, A(\tau_1) = 0$, which leads to a contradiction unless ω is a constant. A uniform approximation to y can therefore not be obtained with the linear scales of (3.1.2).

Consider now the extension $t \to \{\tau_0, \tau_1\}$ with the nonlinear trajectory

$$\tau_0 = t \qquad \tau_1 = \epsilon k(t) \tag{3.1.9}$$

where $k(t)$ is as yet an undetermined "clock" (or "scale") function. The derivative operator now is extended as

$$\frac{d}{dt} \to \frac{\partial}{\partial \tau_0} + \epsilon \dot{k} \frac{\partial}{\partial \tau_1} \tag{3.1.10}$$

leading to the equations:

$$\frac{\partial \mathbf{y}}{\partial \tau_0} = 0 \tag{3.1.11}$$

$$\dot{k}(\tau_0) \frac{\partial \mathbf{y}}{\partial \tau_1} + \omega(\tau_0)\, \mathbf{y} = 0. \tag{3.1.12}$$

We obtain, on integrating, $\mathbf{y}(\tau_0, \tau_1) = A(\tau_1)$ and

$$\frac{A'}{A}(\tau_1) = -\frac{\omega}{k}(\tau_0) = s \tag{3.1.13}$$

where s is a constant. Thus, $\mathbf{y} = c \exp(s\tau_1)$, where c is a constant and $\tau_1 = -(\epsilon/s) \int \omega \, d\tau_0$. Clearly, s can be set equal to unity without loss of generality. Upon restriction, the solution is given by

$$y(t) = c \exp\left(-\epsilon \int \omega(t)\, dt\right) \tag{3.1.14}$$

which is the exact solution of (3.1.6).

We note that our method "simplifies" the problem reducing the original variable coefficient case to the constant coefficient case plus an explicit quadrature (see Eqs. (3.1.13)). In other words, the method yields a suitable change of either the dependent or of the independent variable.

It can be verified that the trajectory represents the characteristics of the partial differential equation obtained by the extension of the given ordinary differential equation. The partial differential equations (3.1.11) and (3.1.12) are compatible with each other as can be readily shown by cross differentiation. The compatibility conditions will be obtained only approximately for higher-order equations. The compatibility conditions play a particularly important role in the matrix formulation of the problem (see, e.g., Ref. 5). This approach will be discussed elsewhere.

The clock function (scale function) $k(t)$ can be highly nonlinear. Some simple examples are shown in Fig. 3. Further, with first-order equations, a uniformly valid (in fact, exact) solution can be obtained in any interval (although not necessarily through a clock). This is not possible for higher-order equations.

The analysis is extended without difficulty to the case in which $\epsilon\omega$ in (3.1.6) is preplaced by $\sum_n \phi_n(\epsilon)\, \omega_n(t)$, where ϕ_n is an asymptotic sequence [4], by means of the trajectory $\tau_n = \phi_n(\epsilon)\, k_n(t)$.

The necessity of using nonlinear scales is appreciated by treating (3.1.1) after the change of variable $t' = 1/t$.

For both small and large ϵ the extension (3.1.10) is appropriate. This will be seen not to be the case for higher-order equations.

3.2. *Second-Order Equations.*

Consider the second-order equation in "canonical" form; that is, the first derivative has been eliminated. We shall assume, as in Section 3.1, that the frequency function depends on a single time scale. We can then write

$$y'' + \epsilon\omega(t)\, y = 0. \tag{3.2.1}$$

We shall study this equation both for small and large values of the parameter ϵ. We will see that in both cases the partial differential equation that will be left unsatisfied contains derivatives with respect to the slowest time scale. It can be readily verified, in view of the choice (2.10), that the class of second-order equations considered below corresponds to either rapidly-varying, low frequency (Case (1)) or slowly-varying, high frequency (Case (2)). The derivative will be extended differently in the two cases.

Case (1)—*Coefficient with Small Parameter:* $0 < |\epsilon| \ll 1$. The direct perturbation expansion may be nonuniform for values of t which depend on the nature of $\omega(t)$. To study the low frequency with rapid variation we invoke the extension (3.1.10) and (2.10) and are led to the set of equations:

$$\frac{\partial^2 \mathbf{y}}{\partial {\tau_0}^2} = 0 \tag{3.2.2}$$

$$\ddot{k} \frac{\partial \mathbf{y}}{\partial \tau_1} + 2\dot{k} \frac{\partial^2 \mathbf{y}}{\partial \tau_0 \, \partial \tau_1} + \omega(\tau_0)\, \mathbf{y} = 0 \tag{3.2.3}$$

$$\dot{k}^2 \frac{\partial^2 \mathbf{y}}{\partial {\tau_1}^2} = 0. \tag{3.2.4}$$

We have chosen $\partial \mathbf{y}/\partial \epsilon = 0$ which is adequate for first-order theory. Higher-order theory requires that either $\mathbf{y}$ be expanded or that additional scales be used. From (3.2.2) we find

$$\mathbf{y}(\tau_0, \tau_1) = A(\tau_1)\, \tau_0 + B(\tau_1). \tag{3.2.5}$$

Two solutions are generated by using (3.2.3) and (3.2.5). These give rise to two clock functions that can be written as:

$$k_1 = \frac{1}{\tau_0} \int {\tau_0}^2 \omega \, d\tau_0 - \int \tau_0 \omega \, d\tau_0 \tag{3.2.6}$$

$$k_2 = \int \tau_0 \omega \, d\tau_0 - \tau_0 \int \omega \, d\tau_0 \, . \tag{3.2.7}$$

Upon restriction along the trajectory (3.1.9), the approximate solutions to first order can be written as:

$$\tilde{y}_1 = c_1 t \exp \left\{ \epsilon \left(\frac{1}{t} \int t^2 \omega \, dt - \int t\omega \, dt \right) \right\} \tag{3.2.8}$$

$$\tilde{y}_2 = c_2 \exp \left\{ + \epsilon \left(\int t\omega \, dt - t \int \omega \, dt \right) \right\} . \tag{3.2.9}$$

The constants of integration associated with (3.2.8) and (3.2.9) require careful consideration in applications.

A sufficient condition for the approximation to break down is that the term in (3.2.4) which is neglected attains the same order of magnitude as any term in (3.2.2). This leads to the following criterion (Appendix B): the approximations (3.2.8) and (3.2.9) will fail when

$$\omega t^2 \sim \frac{1}{\epsilon}. \tag{3.2.10}$$

It can also be noticed that the constancy of the Wronskian of the approximating functions is destroyed when $\omega t^2 \sim 1/\epsilon$ and (3.2.3) and (3.2.4) are then no longer linearly independent. For example, for the Airy equation ($y'' + \epsilon t y = 0$) the failure of the approximations (3.2.8) and (3.2.9), ($t \gtrsim \epsilon^{-1/3}$) is correctly given by the above criterion.

One obtains the correct solution for the constant frequency case with the extension of the derivative (3.2.14) and (2.10). Application of the extension (3.1.10) yields an approximation that avoids the "whipping tail" defects of direct Taylor expansion but may not extend the interval over which the Taylor expansion is valid.

For the choice

$$\omega(t) = \frac{1}{t^2} \tag{3.2.11}$$

it is readily verified that our approximation formulae (3.2.8) and (3.2.9) substantially improve the Taylor expansion in ϵ. For $\epsilon \ll 1$ the WKB approximation is clearly at fault in this case [12].

Case (2)—*Coefficient with Large Parameter.* We will now change our point of view and consider (3.2.1) in the form

$$y'' + \lambda^2 \omega(t) y = 0 \tag{3.2.12}$$

where $|\lambda| \gg 1$, and by comparison with (3.2.1) $\lambda = \epsilon^{1/2}$. We now choose the trajectory

$$\tau_0 = t, \qquad \tau_1 = \lambda k(t) \tag{3.2.13}$$

with the derivative extended as

$$\frac{\partial}{\partial \tau_0} + \lambda \dot{k} \frac{\partial}{\partial \tau_1} = \frac{\partial}{\partial \tau_0} + \sqrt{\epsilon}\, \dot{k} \frac{\partial}{\partial \tau_1} \tag{3.2.14}$$

and have the set of equations:

$$\dot{k}^2(\tau_0) \frac{\partial^2 \mathbf{y}}{\partial {\tau_1}^2} + \omega(\tau_0)\, \mathbf{y} = 0 \tag{3.2.15}$$

$$\ddot{k} \frac{\partial \mathbf{y}}{\partial \tau_1} + 2\dot{k} \frac{\partial^2 \mathbf{y}}{\partial \tau_0\, \partial \tau_1} = 0 \tag{3.2.16}$$

$$\frac{\partial^2 \mathbf{y}}{\partial {\tau_0}^2} = 0. \tag{3.2.17}$$

We seek solutions of (3.2.15) in the form:

$$\mathbf{y}(\tau_0, \tau_1) = \alpha(\tau_0)\,\beta(\tau_1) = \alpha(\tau_0)\exp(\tau_1) \tag{3.2.18}$$

whence the clock k satisfies the equation:

$$\dot{k}^2 + \omega = 0 \tag{3.2.19}$$

Substitution into (3.2.16) yields $\alpha(\tau_0) = \omega^{-1/4}(\tau_0)$. Restriction along (3.2.13) yields the approximations

$$\hat{y}_1(t) = c_1\omega^{-1/4}\exp\left(i\lambda\int\omega^{1/2}\,dt\right) \tag{3.2.20}$$

$$\hat{y}_2(t) = c_2\omega^{-1/4}\exp\left(-i\lambda\int\omega^{1/2}\,dt\right) \tag{3.2.21}$$

which are recognized as the Liouville-Green (or WKBJ) formulae. Thus, with our method, the frequency and the amplitude variation are associated with the fast and slow time scales, respectively.

Analogous approximation formulae can be obtained directly for second-order equations in which the first derivative appears explicitly. The precise condition for the asymptotic validity of (3.2.20) and (3.2.21) is that

$$\int_a^b \omega^{-1/4}\left|\frac{d^2}{dt^2}\,\omega^{-1/4}\right| dt < M$$

where M is a constant and (a, b) is the interval considered. We can express this condition in terms of the characteristic roots x for the equations $(x^2 + \omega = 0)$ as

$$\int_a^b x^{+1/2}\left|\frac{d^2}{dt^2}\,x^{+1/2}\right| dt < M.$$

This latter form lends itself to analyzing higher-order equations. In particular, in Ref. 11 the corresponding criterion has been developed for third-order equations.

Third- and higher-order equations can be treated similarly. Therefore, we will now consider the nth-order case.

3.3. *Equations of* nth *Order*

We discuss some cases of equations of the nth order. Again, a small or a large parameter enters the coefficients in a simple way.

Case (1)—*Coefficients with Small Parameter:* $0 < |\epsilon| \ll 1$. We shall first study the parameterized canonical (i.e., $(n-1)$th derivative term is absent) equation,

$$y^{(n)} + \epsilon[\omega_{n-1}(t)\,y^{(n-2)} + \cdots + \omega_0(t)\,y] = 0 \tag{3.3.1}$$

in the limit as $|\epsilon| \to 0$. Direct perturbation can be shown to be nonuniform, depending on the nature of the coefficients. The extension $\tau_0 = t$; $\tau_1 = \epsilon k(t)$ leads to a set of $(n+1)$ partial differential equations. Using the results of Appendix A, the two leading equations are written as:

$$\frac{\partial^n \mathbf{y}}{\partial {\tau_0}^n} = 0 \tag{3.3.2}$$

$$k^{(n)} \frac{\partial \mathbf{y}}{\partial \tau_1} + \sum_{\gamma=1}^{n-1} \binom{n}{\gamma} k(n-\gamma) \frac{\partial^{\gamma+1}\mathbf{y}}{\partial {\tau_0}^2\, \partial \tau_1} = -\,\omega_{n-2} \frac{\partial^{n-2}\mathbf{y}}{\partial \tau_0^{n-2}} + \cdots + \omega_0 \mathbf{y}. \tag{3.3.3}$$

Integration gives:

$$\mathbf{y}(\tau_0\,, \tau_1) = A_{n-1}(\tau_1)\,\tau_0^{n-1} + A_{n-2}(\tau_1)\,\tau_0^{n-2} + \cdots + A_1(\tau_1)\,\tau_0 + A_0(\tau_1). \tag{3.3.4}$$

The terms on the right side are linearly independent with respect to τ_0 and thus generate corrections to the lowest-order result. For example, substituting $A_{n-1}(\tau_1)\,\tau_0^{n-1}$ into (3.3.3) and choosing an exponential τ_1 dependence of A_{n-1} results in the following equation for the clock:

$$\tau_0^{n-1} \frac{d^{n-1}(\dot{k}_{n-1})}{d\tau_0^{n-1}} + \binom{n}{1}(n-1)\,\tau_0^{n-2} \frac{d^{n-2}(\dot{k}_{n-1})}{d\tau_0^{n-2}} + \cdots (n!)\,(\dot{k}_{n-1})$$

$$= -\,(\omega_{n-2}\tau_0 + \omega_{n-3}{\tau_0}^2 + \cdots + \omega_0 \tau_0^{n-1}). \tag{3.3.5}$$

Even though this equation has variable coefficients, it can be recognized as the inhomogeneous Euler-Cauchy or equidimensional equation and can be solved exactly. The solution of the corresponding homogeneous equation can be expressed as $\dot{k}_{n-1} = {\tau_0}^m$, where m satisfies the algebraic equation:

$$m(m-1)(m-2)\cdots(m-n+2) + m(m-1)\cdots(m-n+3)\binom{n}{1}(n-1)$$

$$+ \cdots + n! = 0. \tag{3.3.6}$$

This can be written as

$$m^{n-1} + a_{n-2}m^{n-2} + \cdots + a_1 m + a_0 = 0 \tag{3.3.7}$$

having $(n-1)$ roots which are assumed to be distinct for this analysis. Let the homogeneous solution be given by:

$$(\phi_{n-1})_h = \sum_{i=1}^{n-1} c_i \tilde{\phi}_i; \qquad \tilde{\phi}_i = \tau_0{}^i. \tag{3.3.8}$$

The particular solution can then be written as:

$$-\phi_{n-1} \equiv -k_{n-1} = \phi_1 \int \frac{v_1 f}{W(\tau_0)}\, d\tau_0 - \phi_2 \int \frac{v_2 f}{W(\tau_0)}\, d\tau_0 \cdots + \cdots (-1)^n \phi_{n-1} \int \frac{v_{n-1} f}{W(\tau_0)}\, d\tau_0 . \tag{3.3.9}$$

We have used the notation $W \equiv W(\phi_1, \phi_2, ..., \phi_{n-1})$ for the Wronskian of the ϕ_i, and v_i for the determinant of the matrix formed by replacing the ith column of the $(n-1)$ square matrix

$$\begin{bmatrix} \tilde{\phi}_1 & \tilde{\phi}_2 & \cdot & \cdot & \tilde{\phi}_{n-1} \\ \tilde{\phi}_1' & \tilde{\phi}_2' & \cdot & \cdot & \tilde{\phi}_{n-1}' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \tilde{\phi}_1^{(n-2)} & \tilde{\phi}_2^{(n-2)} & \cdot & \cdot & \tilde{\phi}_{n-1}^{(n-2)} \end{bmatrix}$$

by the column

$$\begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ f \end{bmatrix}$$

where $f(\tau_0)$ denotes the right-hand side of (3.3.5). One more integration of ϕ_{n-1} gives the clock function k_{n-1}, thus obtaining the proper time scales. After restriction along $\tau_0 = t$, and $\tau_1 = \epsilon k(t)$, one approximate solution to (3.3.1) can be written as:

$$y_{n-1}(t) = c_{n-1} t^{n-1} \exp\left(\epsilon \int \phi_{n-1}\, dt\right).$$

The other independent approximations are similarly obtained by determining the clock functions $\phi_{n-2}, ..., \phi_1, \phi_0$. We note that $\dot{k}_{n-1}$ satisfies an $(n-1)$th order inhomogeneous linear equation; $\dot{k}_{n-2}$ satisfies an $(n-2)$th order equation, and so forth. Thus, $\dot{k}_2$ satisfies a second- and $\dot{k}_1$ a first-order equation. The case given above is therefore the most complicated and the analysis becomes simpler for the determination of successive clocks. The approximate general solution of (3.3.1) to order ϵ for small ϵ can be written as

$$\tilde{y}(t) = \sum_{0}^{n-1} c_i t^i \exp\left(\epsilon \int \phi_i \, dt\right) \tag{3.3.10}$$

(c_i being arbitrary constants).

For any given n the breakdown of this approximation can be determined by studying the equations that have been neglected. Higher-order corrections can be obtained by employing slower clocks.

Case (2)—*Coefficients with Large Parameter.* Consider the equation:

$$y^{(n)} + \lambda\omega_{n-1}(t)\, y^{(n-1)} + \lambda^2\omega_{n-2}(t)\, y^{(n-2)} + \cdots + \lambda^{n-1}\omega_1(t)\, y^{(1)} + \lambda^n\omega_0(t)\, y = 0 \tag{3.3.11}$$

where $|\lambda| \gg 1$. Equation (3.3.11) is equivalent to an equation of the form

$$y^{(n)} + \omega_{n-1}(\epsilon\tau)\, y^{(n-1)} + \omega_{n-2}(\epsilon\tau)\, y^{(n-2)} + \cdots + \omega_0(\epsilon\tau)\, y = 0 \tag{3.3.12}$$

with

$$0 < \epsilon \ll 1$$

for (3.3.12) can be written in the form (3.3.11) by letting $\epsilon\tau = t$ and $\lambda = 1/\epsilon$.

We now choose the extension

$$\tau_0 = t, \qquad \tau_1 = \lambda k(t) \tag{3.3.12}$$

and obtain (Appendix A) the two leading equations:

$$\dot{k}^n \frac{\partial^n \mathbf{y}}{\partial \tau_1{}^n} + \omega_{n-1}\dot{k}^{n-1} \frac{\partial^{n-1}\mathbf{y}}{\partial \tau_1^{n-1}} + \cdots + \omega_1 \dot{k} \frac{\partial \mathbf{y}}{\partial \tau_1} + \omega_0 \mathbf{y} = 0 \tag{3.3.13}$$

$$n(\dot{k})^{n-1} \frac{\partial^n \mathbf{y}}{\partial \tau_0\, \partial \tau_1^{n-1}} + (n-1)\, \omega_{n-1}(\dot{k})^{n-2} \frac{\partial^{n-1}\mathbf{y}}{\partial \tau_0\, \partial \tau_1^{n-2}} + \cdots$$

$$+ 2\omega_2 \dot{k} \frac{\partial^2 \mathbf{y}}{\partial \tau_0\, \partial \tau_1} + \omega_1 \frac{\partial \mathbf{y}}{\partial \tau_0} + \frac{n(n-1)}{2} (\dot{k})^{n-2}\, \ddot{k} \frac{\partial^{n-1}\mathbf{y}}{\partial \tau_1^{n-1}}$$

$$+ \frac{(n-1)(n-2)}{2}\, \omega_{n-1}(\dot{k})^{n-3}\, \ddot{k} \frac{\partial^{n-2}\mathbf{y}}{\partial \tau_1^{n-2}} + \cdots$$

$$+ \omega_2 \ddot{k} \frac{\partial \mathbf{y}}{\partial \tau_1} = 0. \tag{3.3.14}$$

As before, we seek solutions of the form:

$$\mathbf{y}(\tau_0, \tau_1) = \alpha(\tau_0)\,\beta(\tau_1) = \alpha(\tau_0)\exp(\tau_1).$$

The derivative of the clock function satisfies the algebraic equation:

$$(\dot{k})^n + \omega_{n-1}(\dot{k})^{n-1} + \cdots + \omega_1\dot{k} + \omega_0 = 0. \tag{3.3.15}$$

On substitution into (3.3.14) and simplifying, the explicit amplitude variation is given by

$$\frac{d}{d\tau_0}(\ln\alpha) = -\frac{1}{2}\frac{\partial}{\partial(\dot{k})}\left\{\ln\left(\frac{\partial F}{\partial\dot{k}}\right)\right\}\ddot{k}$$

where

$$F(\dot{k}, \tau_0) \equiv \dot{k}^n + \omega_{n-1}\dot{k}^{n-1} + \cdots + \omega_1\dot{k} + \omega_0 \tag{3.3.16}$$

i.e.,

$$\alpha(\tau_0) = \left(\frac{\partial F}{\partial(\dot{k})}\right)^{-1/2}\exp\left(\int\frac{\partial}{\partial\tau_0}\ln\left(\frac{\partial F}{\partial\dot{k}}\right)^{1/2}d\tau_0\right). \tag{3.3.17}$$

Thus, the approximate general solution to (3.3.11) is obtained after restriction and is given by

$$\hat{y}(t) = \sum_{i=1}^{n} c_i\left(\frac{\partial F}{\partial\dot{k}_i}\right)^{-1/2}\exp\left(\int\frac{\partial}{\partial\tau_0}\ln\left(\frac{\partial F}{\partial\dot{k}_i}\right)^{1/2}d\tau_0\right)\exp(\lambda k_i) \tag{3.3.18}$$

where F is given by (3.3.16) and k_i by (3.3.15). The c_i are arbitrary constants.

The roots of the equation (3.3.15) are called "characteristic" roots and are assumed to be distinct. When ω_1, ω_2,... ω_{n-1} vary more slowly than ω_0, a simpler expression is obtained for the approximation, viz.,

$$\hat{y}(t) = \sum_{i=1}^{n} c_i\left(\frac{\partial F}{\partial\dot{k}_i}\right)^{-1/2}\exp(\lambda k_i). \tag{3.3.19}$$

We see that the results of the standard Liouville-Green theory can be recovered for second-order equations.

4. Examples

We shall now consider some examples of the present approach. It will be seen that our results hold well even in some cases for which the formal conditions of validity of the approximation are not strictly met.

4.1. *Kummer's Equation*

Consider Kummer's equation

$$ty'' + (b - t)y' - ay = 0 \tag{4.1.1}$$

where a and b are fixed constants. As $t \to \infty$, the solutions asymptotically behave as $e^t t^{a-b}$ and t^{-a}, and so have only a finite number of zeros. This information is obtained from the preceding theory as follows. The asymptotic behavior of the characteristic roots is obtained from the equation

$$\dot{k}^2 + \left(\frac{b}{t} - 1\right)\dot{k} - \frac{a}{t} = 0 \tag{4.1.2}$$

as $\dot{k}_1 \sim 1 + (a - b)/t$ and $\dot{k}_2 \sim - a/t$. From (3.3.18) the asymptotic solutions are obtained as

$$\hat{y}_1 = \alpha_1(\tau_0) \exp(\tau_1) \mid_t = c_1 e^t t^{a-b} \tag{4.1.3}$$

$$\hat{y}_2 = c_2 t^{-a}. \tag{4.1.4}$$

4.2. *A Third-order Equation*

Consider

$$ty''' + 3y'' + ty = 0. \tag{4.2.1}$$

The characteristic equation can be written as:

$$\dot{k}^3 + 3\dot{k}^2 + 1 = 0. \tag{4.2.2}$$

For large t the characteristic roots t can be determined by a perturbation expansion, yielding

$$\dot{k} = \dot{k}_0 + \delta\dot{k}_0; \qquad \dot{k}_{0i} = -1, \frac{+1 \pm i\sqrt{3}}{2}; \qquad \delta k_{0i} = -t. \tag{4.2.3}$$

Thus

$$\dot{k}_i = \left[-1 - \frac{1}{t}\right], \left[\frac{+1 + i\sqrt{3}}{2} - \frac{1}{t}\right], \left[\left(\frac{+1 - i\sqrt{3}}{2}\right) - \frac{1}{t}\right]. \tag{4.2.4}$$

The approximation is given by

$$\begin{aligned}\hat{y}(t) &= \sum_{i=1}^{3} c_i \exp(ik_i) = \sum_{i=1}^{3} c_i \exp[k_{0i} + \delta k_{0i}] \\ &= \frac{c_1}{t}\exp(-t) + \frac{c_2}{t}\exp\left[\left(\frac{+1 + i\sqrt{3}}{2}\right)t\right] + \frac{c_3}{t}\exp\left[\left(\frac{1 - i\sqrt{3}}{2}\right)t\right]\end{aligned} \tag{4.2.5}$$

which can be recognized as the exact solution of (4.2.1).

4.3. "*Double Airy*" *Equation*

Consider the equation

$$y''' - 4ty' - 2y = 0 \tag{4.3.1}$$

which we call the "Double Airy" equation. The characteristic roots for large t are obtained from the characteristic equation

$$\dot{k}^3 - 4t\dot{k} - 2 = 0 \tag{4.3.2}$$

as $\dot{k}_1 = 2t^{1/2}$; $\dot{k}_2 = -2t^{1/2}$; $\dot{k}_3 = 0$. The approximate solutions to (4.2.1) are given by

$$\hat{y}_i = (3k_i^2 - 4t)^{-1/2} \exp(k_i)$$

i.e.,

$$\hat{y}_{1,2} = t^{-1/2} \exp(\pm \tfrac{4}{3} t^{3/2}), \qquad \tilde{y}_3 = t^{-1/2}. \tag{4.3.3}$$

Now, (4.3.1) has the products of Airy functions as exact solutions, $Ai^2(t)$, $Bi^2(t)$, and $Ai(t)\,Bi(t)$. Thus, the solutions (4.3.3) agree with the asymptotic behavior of the exact solutions [13].

4.4. *An Equation of the* nth *Order*

We shall now consider a special equation of the nth order, viz., the Euler-Cauchy or equidimensional equation:

$$y^{(n)} + \frac{\lambda^n}{t^n} y = 0 \tag{4.4.1}$$

in the limit $|\lambda| \to \infty$. The equation conforms to the conditions of the approximation. The characteristic equation is

$$\dot{k}^n + \frac{1}{t^n} = 0 \tag{4.4.2}$$

with the characteristic roots:

$$\dot{k}_j = \left\{\exp \pi \left(\frac{i + 2j}{n}\right)\right\} \left(\frac{1}{t}\right), \qquad j = 1, 2, \ldots n,\ i = \sqrt{-1}. \tag{4.4.3}$$

The asymptotic solution of (4.4.1) as $|\lambda| \to \infty$ is given by our theory as:

$$\begin{aligned} y_j(t, \lambda) &= c_j(\dot{k}_j^{n-1})^{-1/2} \exp(\lambda k_j(t)) \\ &= c_j \exp\left\{-\left(\frac{(n-1)}{2n}\right) \pi(i + 2j)\right\} \left(\frac{1}{t}\right)^{-(n-1)/2} \exp(\lambda k_j(t)). \end{aligned} \tag{4.4.4}$$

Simplifying:

$$\hat{y}(t\lambda) = \sum_{j=1}^{n} c_j t^{i\, n-1/2}\, t^{\lambda \exp[(i+2j)/n]\pi}. \tag{4.4.5}$$

The exact solution can be determined as follows. Since (4.3.1) is a homogeneous equation, we look for a solution in the form $y = t^m$. On substitution, m is found to satisfy the equation:

$$m(m-1)(m-2)\cdots(m-n+1) + \lambda^n = 0. \tag{4.4.6}$$

The asymptotic behavior of the roots of this equation can be seen rather easily in the graphical root locus form [14]:

$$1 + \lambda^n G(m) \tag{4.4.7}$$

where

$$G(m) = \frac{1}{m(m-1)\cdots(m-n+1)}.$$

The locus of the roots of (4.4.7) as λ increases from 0 to ∞ is governed by the sign of λ and whether n is odd or even. For example, when $\lambda \to \infty$, the locus of the roots of (4.4.7) on the complex m plane as $|\lambda|$ increases from 0 to ∞ is shown in Fig. 4. When $\lambda \to \pm\infty$, the locus lies on the part of the real

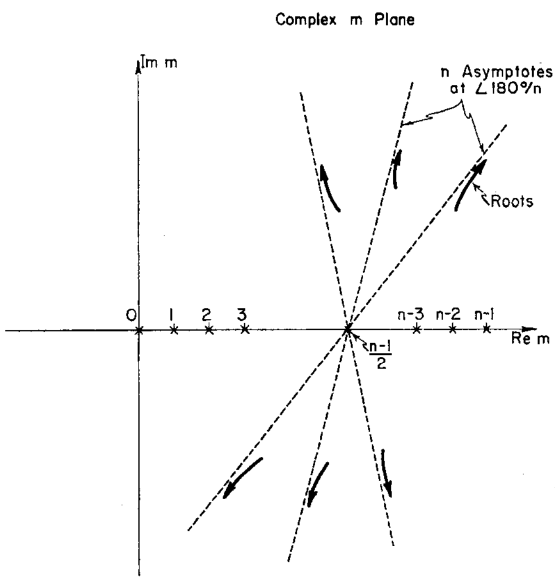

FIG. 4. Root locus for $y^{(n)} + (\lambda/t)^n y = 0$; with $y = t^n$: $m(m-1)\cdots(m-n+1) + \lambda^n = 0$.

axis which has an odd/even number of poles of $G(m)$ to the right of the point considered. For the present, however, the behavior of the roots as $|\lambda| \to \infty$ is of interest. As $|\lambda| \to \infty$, the root loci become asymptotic to the rays emanating from the point $m = (n-1)/2$ on the real axis, at angles $180°/n$ or $360°/n$ (according as $\lambda \to +\infty$ or $-\infty$, respectively); i.e.,

$$m \sim \left(\frac{n-1}{2}\right) + \lambda \exp\left(\frac{\pi(i+2j)}{n}\right); \qquad j = 1, 2, \ldots, n; \; i = \sqrt{-1}.$$

Thus, the exact solution has the same asymptotic behavior for $|\lambda| \to \infty$ as predicted by our approximation.

The application to some other equations is given in Table 1. An application of the small ϵ result, viz., (3.2.8) and (3.2.9), to the problem of wave propagation in a turbulent plasma is discussed in Ref. 12. The motion of a vertical take-off and landing aircraft during the hover-forward flight transition has been analyzed by means of the multiple time scales technique and uniformly valid asymptotic solutions have been obtained [11, 13].

TABLE 1

Examples of Some Classical Equations

Name	Equation	Asymptotic Behavior	
1. Bessel's eq. of zeroth order	$y'' + \frac{1}{t} y' + y = 0$	$t^{-1/2} \exp(\pm it)$;	$t \to \infty$
2. Kummer's confluent hypergeometric	$ty'' + (b-t)y' - ay = 0$	$\left.\begin{matrix} t^{a-b} e^t \\ t^{-a} \end{matrix}\right\}$;	$t \to \infty$
3. A third-order equation	$ty''' + 3y'' + ty = 0$	$t^{-1} \exp(-t)$ $t^{-1} \exp\left(\frac{1 \pm i\sqrt{3}}{2} t\right)$	$t \to \infty$
4. "Double Airy"	$y''' - 4ty' - 2y = 0$	$\left.\begin{matrix} t^{-1/2} \\ t^{-1/2} \exp[\pm 4/3\, t^{3/2}] \end{matrix}\right\}$	$t \to \infty$
5. Euler's equation	$y^{(n)} + \left(\frac{\lambda}{t}\right)^n y = 0$	$t^{\frac{n-1}{2}} t^{\lambda \exp[(\frac{i+2p}{n})\pi]}$; $1 \leqslant p \leqslant n$	$\lvert\lambda\rvert \to \infty$ $i = \sqrt{-1}$

5. Concluding Remarks

The main theme of this paper has been to demonstrate with a class of linear differential equations that when direct perturbation theory fails, natural (nonlinear) scales can be found on which the solution can be described more uniformly. The approach is mathematically straightforward and allows for the injection of physical insight.

We have developed here first-order theory. Higher approximations can be obtained in two ways [11]: (i) by employing more scales, and (ii) by expanding the dependent variable. Multiplicative and additive corrections to the approximate solutions are obtained, respectively. Both are of interest, as well as combinations.

Error bounds can be found for our approximation formulae. Several results are discussed in Ref. 11.

For the $|\lambda| \gg 1$ cases, the compact form of our formula (3.3.18) enables one to write the approximation by inspection and ties in neatly with the concept of variable characteristic roots. Of course, special precautions are needed when characteristic roots coalesce and turning point problems arise [5].

It may not be amiss to remark that even though the coefficients have been taken to be either small and rapidly varying or large and slowly varying, special cases of a different nature are included in our treatment. Thus, for example, $y'' + \lambda\omega(\lambda t)y = 0$ with $\lambda \gg 1$ is readily transformed to (3.2.1) with $\epsilon = 1/\lambda$.

Appendix A. Extension of the nth-Order Derivative

With the two time scale extension

$$t \to \{\tau_0, \tau_1\}, \qquad \tau_0 = t, \qquad \tau_1 = \epsilon k(t). \tag{A.1}$$

An extension of the derivative operator is

$$\frac{d}{dt} \to \frac{\partial}{\partial\tau_0} + \frac{d\tau_1}{dt}\frac{\partial}{\partial\tau_1} = \frac{\partial}{\partial\tau_0} + \epsilon\dot{k}\frac{\partial}{\partial\tau_1}. \tag{A.2}$$

Similarly, we have, for the second derivative

$$\frac{d^2}{dt^2} \to \left(\frac{\partial}{\partial\tau_0} + \epsilon\dot{k}\frac{\partial}{\partial\tau_1}\right)\left(\frac{\partial}{\partial\tau_0} + \epsilon\dot{k}\frac{\partial}{\partial\tau_1}\right)$$

$$= \frac{\partial^2}{\partial\tau_0{}^2} + \epsilon\left(\ddot{k}\frac{\partial}{\partial\tau_1} + 2\dot{k}\frac{\partial^2}{\partial\tau_0\,\partial\tau_1}\right) + \epsilon^2\left(\dot{k}^2\frac{\partial^2}{\partial\tau_1{}^2}\right). \tag{A.3}$$

We now consider the extension of the nth derivative:

$$\frac{d^n}{dt^n} \to \left(\frac{\partial}{\partial \tau_0} + \epsilon \dot{k} \frac{\partial}{\partial \tau_1}\right)^n. \tag{A.4}$$

We order the terms in (A.4) with powers of ϵ

$$\left(\frac{\partial}{\partial \tau_0} + \epsilon \dot{k} \frac{\partial}{\partial \tau_1}\right)^n = \sum_{p=0}^{n} \epsilon^p A_p . \tag{A.5}$$

Clearly, the right-hand side of (A.5) contains terms due to the binomial expansion of the operator as well as terms with successive derivatives of the clock function. For purposes of the present approximation scheme, only terms of order ϵ^n and ϵ^{n-1} are needed, in addition to the lowest-order terms. The corresponding operator coefficients are:

$$A_0 = \frac{\partial^n}{\partial {\tau_0}^n} \tag{A.6}$$

$$A_1 = k^{(n)} \frac{\partial}{\partial \tau_1} + \sum_{r=1}^{n-1} \binom{n}{r} k^{(n-r)} \frac{\partial^{r+1}}{\partial {\tau_0}^r \, \partial \tau_1} \tag{A.7}$$

$$A_{n-1} = n(\dot{k})^{n-1} \frac{\partial^n}{\partial \tau_0 \, \partial \tau_1^{n-1}} + \frac{n(n-1)}{2} (\dot{k})^{n-2} \ddot{k} \frac{\partial^{n-1}}{\partial \tau_1^{n-1}} \tag{A.8}$$

$$A_n = (\dot{k})^n \frac{\partial^n}{\partial {\tau_1}^n} . \tag{A.9}$$

That these are, indeed, the coefficients is proven by mathematical induction as follows. We shall prove that if (A.6)–(A.9) is true for n, then it is true for $n + 1$. Letting the derivative operator (A.2) act again on (A.5), we find:

$$\begin{aligned}
&\left(\frac{\partial}{\partial \tau_0} + \epsilon \dot{k} \frac{\partial}{\partial \tau_1}\right)^{n+1} \\
&= \left(\frac{\partial}{\partial \tau_0} + \epsilon \dot{k} \frac{\partial}{\partial \tau_1}\right) \left\{ \frac{\partial^n}{\partial {\tau_0}^n} + \epsilon k^{(n)} \frac{\partial}{\partial \tau_1} + \sum_{r=1}^{n-1} \binom{n}{r} k^{(n-r)} \frac{\partial^{r+1}}{\partial {\tau_0}^r \, \partial \tau_1} \right. \\
&\qquad + \cdots + \epsilon^{n-1} \left(n(\dot{k})^{n-1} \frac{\partial^n}{\partial \tau_0 \, \partial \tau_1^{n-1}} \right. \\
&\qquad\qquad \left. + \frac{n(n-1)}{2} (\dot{k})^{n-2} \ddot{k} \frac{\partial^{n-1}}{\partial \tau_1^{n-1}} \right) \\
&\qquad \left. + \epsilon^n \left((\dot{k})^n \frac{\partial^n}{\partial {\tau_1}^n} \right) \right\} = \sum_{p=0}^{n+1} \epsilon^p B_p .
\end{aligned} \tag{A.10}$$

The relevant coefficients can be written as:

$$B_0 = \frac{\partial^{n+1}}{\partial\tau_0^{n-1}} \tag{A.11}$$

$$B_1 = k^{(n+1)} \frac{\partial}{\partial\tau_1} + k^{(n)} \frac{\partial^2}{\partial\tau_0\,\partial\tau_1} + \dot{k} \frac{\partial^{n+1}}{\partial\tau_0{}^n\,\partial\tau_1} + \sum_{r=1}^{n-1} \binom{n}{r} \left[k^{(n-r+1)} \frac{\partial^{r+1}}{\partial\tau_0{}^r\,\partial\tau_1} + k^{(n-r)} \frac{\partial^{r+2}}{\partial\tau_0^{r+1}\,\partial\tau_1}\right] \tag{A.12}$$

$$= k^{(n+1)} \frac{\partial}{\partial\tau_1} + \sum_{r=1}^{n} \binom{n+1}{r} k^{(n-r+1)} \frac{\partial^{r+1}}{\partial\tau_0{}^r\,\partial\tau_1}$$

$$B_n = n(\dot{k})^{n-1} k'' \frac{\partial^n}{\partial\tau_1{}^n} + \dot{k}^n \frac{\partial^{n+1}}{\partial\tau_1\,\partial\tau_1{}^n} + n(\dot{k})^n \frac{\partial^{n+1}}{\partial\tau_0\,\partial\tau_1{}^n} + \frac{n(n-1)}{2} (\dot{k})^{n-1} k'' \frac{\partial^n}{\partial\tau_1{}^n}$$

$$= (n+1)(\dot{k})^n \frac{\partial^{n+1}}{\partial\tau_0\,\partial\tau_1{}^n} + \frac{(n+1)\,n}{2} (\dot{k})^{n-1} k'' \frac{\partial^n}{\partial\tau_1{}^n} \tag{A.13}$$

$$B_{n+1} = (\dot{k})^{n+1} \frac{\partial^{n+1}}{\partial\tau_1^{n+1}}. \tag{A.14}$$

On examining (A.6)–(A.9) and (A.11)–(A.14), it is seen that the latter are obtained from the former by replacing n by $(n+1)$. Hence, if (A.6)–(A.9) are true for n, they are true for $(n+1)$. It is easily verified from (A.5)–(A.9) and (A.3) that the validity for $n = 2$; thus, the induction is complete.

Appendix B. A Criterion of Validity

We derive here the criterion for validity of our approximation when applied to the second-order equation with small parameter. Failure of the approximation occurs when the term neglected in the analysis ceases to be small:

$$\left(\dot{k}^2 \frac{\partial^2 y}{\partial\tau_1{}^2}\right)\left(\frac{1}{\omega y}\right) \sim \frac{1}{\epsilon}. \tag{B.1}$$

The condition for failure is then given by

$$(\dot{k}^2) = \frac{\omega}{\epsilon}. \tag{B.2}$$

Substituting from (3.2.8) and simplifying, we obtain:

$$\sqrt{\epsilon}\left(\int t^2\omega\, dt\right) = \omega^{\frac{1}{2}} t^2. \tag{B.3}$$

Differentiation of (B.3) yields

$$\sqrt{\epsilon}\, t^2\omega = \tfrac{1}{2}\,\omega^{-\frac{1}{2}}\dot{\omega} t^2 + 2t\omega^{\frac{1}{2}} \tag{B.4}$$

which can be rearranged to read

$$\frac{1}{2}\,\omega^{-\frac{3}{2}}\dot{\omega} + \frac{2}{t}\,\omega^{-\frac{1}{2}} = \sqrt{\epsilon}. \tag{B.5}$$

We recognize the first term to be a total derivative; therefore

$$\frac{d}{dt}\,(\omega^{-\frac{1}{2}}) - \frac{2}{t}\,(\omega^{-\frac{1}{2}}) = -\sqrt{\epsilon}. \tag{B.6}$$

Using t^{-2} as an integrating factor, the above condition becomes:

$$\frac{d}{dt}\left(\frac{\omega^{-\frac{1}{2}}}{t^2}\right) = -\frac{\sqrt{\epsilon}}{t^2}. \tag{B.7}$$

Integration of (B.7) now gives

$$\frac{1}{\sqrt{\omega\, t^2}} = \frac{\sqrt{\epsilon}}{t} \tag{B.8}$$

i.e., $\omega t^2 \simeq 1/\epsilon$. The approximation will therefore fail when

$$\omega t^2 \sim \frac{1}{\epsilon}. \tag{B.9}$$

The criterion can be derived in a different way, also. The approximating functions $A(\tau_1)\,\tau_0$ and $B(\tau_1)$ are linearly independent with respect to τ_0. Upon restriction along $\tau_0 = t$, $\tau_1 = \epsilon k(t)$, this property may not be satisfied throughout the domain. Our approximations can therefore be expected to fail in a region where the constancy of the Wronskian is destroyed. From (3.2.8) and (3.2.9), the Wronskian can be written as

$$W(\tilde{y}_1, \tilde{y}_2) = \left[-1 + \epsilon\left(\frac{1}{t}\int t^2\omega\, dt - t\int \omega\, dt\right)\right]$$
$$\times \exp\left\{\epsilon\left(\frac{1}{t}\int t^2\omega\, dt - \int t\omega\, dt - \iint \omega\, dt^2\right)\right\} \tag{B.10}$$

i.e., to lowest order in ϵ, $W(\tilde{y}_1, \tilde{y}_2)$ is a constant. Hence, failure is indicated when either the exponent is of order unity or

$$\frac{1}{t}\int t^2\omega\, dt - t\int \omega\, dt \sim \frac{1}{\epsilon}. \tag{B.11}$$

Therefore, we have

$$-2\epsilon \int t \int \omega\, dt\, dt \sim t. \tag{B.12}$$

Differentiating Eq. (B.12), we obtain

$$-2\epsilon t \int \omega\, dt \sim 1. \tag{B.13}$$

Differentiating again,

$$2\epsilon\omega \sim \frac{1}{t^2}. \tag{B.14}$$

Thus, the approximations (3.2.8) and (3.2.9) fail near a value of t for which $\omega t^2 \sim 1/\epsilon$ as obtained earlier. Substituting this shows that the exponent in the exponential function of $W(\tilde{y}_1, \tilde{y}_2)$ is of order unity.

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