Asymptotic behaviour of the nonoscillatory solutions of differential equations of second order with delay depending on the unknown function

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Received 16 June 1997; received in revised form 12 January 1998

Abstract

The asymptotic behaviour of the solutions of the differential equations

\[(r(t)x'(t))' + f(t,x(t),x(\Delta(t,x(t)))) = 0\]

in the case when \(\int_{-\infty}^{\infty} ds/r(s) = +\infty\) is considered. © 1998 Elsevier Science B.V. All rights reserved.

AMS classification: 34K15

Keywords: Asymptotic behaviour of the solutions; Differential equations of second order with delay

1. Introduction

The systematic investigation of the oscillatory properties and asymptotic behaviour of the solutions of functional-differential equations begins with the works [3, 5, 6].

For the last two decades the number of the papers dealing with these problems considerably increased. The monograph [4] published in 1987 is devoted to the systematic investigation of the oscillatory properties of the solutions of ordinary differential equations with deviating arguments.

The present paper deals with the asymptotic behaviour of the nonoscillatory solutions of a class of differential equations with delay depending on the unknown function of the form

\[(r(t)x'(t))' + f(t,x(t),x(\Delta(t,x(t)))) = 0\]

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under the condition that $\int_{0}^{\infty} \frac{ds}{r(s)} = +\infty$. In the case when $f$ is strongly sublinear (superlinear) function, conditions for oscillation of the solutions of Eq. (1) are found.

Let us note that the differential equations of form (1) with delay depending on the unknown function have been investigated only in the papers [1, 2].

2. Preliminary notes

Let $T \in \mathbb{R}_{+} = [0, +\infty)$. Define $T_{-1} = \inf\{\Delta(t,x): t \geq T, x \in \mathbb{R}\}$.

**Definition 1.** The function $x(t)$ is called a solution of the differential equation (1) in the interval $[T, +\infty)$, if $x(t)$ is defined for $t \geq T_{-1}$, it is twice differentiable and satisfies (1) for $t \geq T$.

**Definition 2.** The solution $x(t)$ of Eq. (1) is said to be regular, if it is defined on some interval $[T_{x}, +\infty)$ and $\sup\{|x(t)|: t \geq T\} > 0$ for each $T \geq T_{x}$.

**Definition 3.** The solution $x(t)$ of Eq. (1) is said to be:
(1) finally positive: if there exists $T \geq 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) > 0$ for $t \geq T$;
(2) finally negative: if there exists $T \geq 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) < 0$ for $t \geq T$;
(3) nonoscillatory: if it is either finally positive or finally negative;
(4) oscillatory: if it is neither finally positive nor finally negative.

Introduce the following conditions:

H1. $r \in C(\mathbb{R}_{+}, \mathbb{R}_{+})$ and $r(t) > 0$, $t \in \mathbb{R}_{+}$.
H2. $\int_{0}^{\infty} \frac{ds}{r(s)} = +\infty$.
H3. $f \in C(\mathbb{R}_{+} \times \mathbb{R}_{+}, \mathbb{R})$.
H4. There exists $T \in \mathbb{R}_{+}$ such that $u.f(t, u, v) > 0$ for $t \geq T$, $u, v > 0$ and $f(t, u, v)$ is nondecreasing function in $u$ and $v$ for each fixed $t \geq T$.
H5. $\Delta \in C(\mathbb{R}_{+} \times \mathbb{R}_{+}, \mathbb{R})$.
H6. There exist a function $\Delta_{*}(t) \in C(\mathbb{R}_{+}, \mathbb{R})$ and $T \in \mathbb{R}_{+}$ such that $\lim_{t \rightarrow +\infty} \Delta_{*}(t) = +\infty$ and $\Delta_{*}(t) \leq \Delta(t, x)$ for $t \geq T$, $x \in \mathbb{R}$.
H7. There exist a function $\Delta^{*}(t) \in C(\mathbb{R}_{+}, \mathbb{R})$ and $T \in \mathbb{R}_{+}$ such that $\Delta^{*}(t)$ is a nondecreasing function for $t \geq T$ and $\Delta(t, x) \leq \Delta^{*}(t) \leq t$ for $t \geq T$, $x \in \mathbb{R}$.

Introduce the functions

$$R(t) = \int_{0}^{t} \frac{ds}{r(s)}, \quad R(t, T) = \int_{T}^{t} \frac{ds}{r(s)}.$$

3. Main results

**Lemma 4.** Let the conditions H1–H7 hold and $x(t)$ be a nonoscillatory solution of Eq. (1).

Then:
1. If \( x(t) \) is a finally positive function then finally \( x(t) \) is an increasing function and \( r(t)x'(t) \) is a positive and decreasing function.

2. If \( x(t) \) is a finally negative function then finally \( x(t) \) is a decreasing function and \( r(t)x'(t) \) is a negative and increasing function.

3. \( x(t) \) possesses one of the following properties:

   \[
   \begin{align*}
   (P1) & \quad \lim_{t \to +\infty} \frac{x(t)}{R(t)} = \text{const} \neq 0. \\
   (P2) & \quad \lim_{t \to +\infty} \frac{x(t)}{R(t)} = 0, \quad \lim_{t \to +\infty} |x(t)| = +\infty. \\
   (P3) & \quad \lim_{t \to +\infty} \frac{x(t)}{R(t)} = 0, \quad \lim_{t \to +\infty} |x(t)| = \text{const} \neq 0.
   \end{align*}
   \]

**Proof.** Let \( x(t) > 0 \) for \( t \geq T_0 \geq 0 \). It follows from condition H6 that there exists \( T_1 \geq T_0 \) such that \( x(A(t,x(t))) > 0 \) for \( t \geq T_1 \) and from H4 and (1) we conclude that \( (r(t)x'(t))' < 0 \) for \( t \geq T_2 \geq T_1 \). Therefore, \( r(t)x'(t) \) is a decreasing function for \( t \geq T_2 \).

The case \( r(t)x'(t) \leq 0 \) is not possible. If we suppose that there exist \( k > 0 \) and \( T_3 \geq T_2 \) such that \( r(t)x'(t) \leq -k, \ t \geq T_3 \), then we obtain that \( x(t) \to -\infty \) as \( t \to +\infty \), which is a contradiction.

Thus, \( r(t)x'(t) > 0 \) for \( t \geq T_2 \). Then there exists the limit

\[
\lim_{t \to +\infty} r(t)x'(t) = L \in [0, \infty). \tag{2}
\]

It is not difficult to prove that

\[
\lim_{t \to +\infty} \frac{x(t)}{R(t)} = L. \tag{3}
\]

If \( L > 0 \), then \( x(t) \) possesses property (P1). Let \( L = 0 \). Since \( x'(t) > 0 \) for \( t \geq T_2 \) and \( x(t) \) is an increasing function then either \( \lim_{t \to +\infty} x(t) = \text{const} \neq 0 \) (and \( x(t) \) possesses property (P3)), or \( \lim_{t \to +\infty} x(t) = +\infty \) (and \( x(t) \) possesses property (P2)). \( \square \)

**Theorem 5.** Let the conditions H1–H7 hold.

Then:

1. If Eq. (1) has a nonoscillatory solution with the property (P1), then there exists a constant \( c \neq 0 \) such that
   \[
   \int_{-\infty}^{\infty} |f(s,cR(s),cR(A_+(s)))| \, ds < +\infty. \tag{4}
   \]

2. If for some \( c \neq 0 \) we have
   \[
   \int_{-\infty}^{\infty} |f(s,cR(s),cR(A_+(s)))| \, ds < +\infty, \tag{5}
   \]

then Eq. (1) has a solution with the property (P1).
Proof. (1) Let Eq. (1) have a solution $x(t)$ for which

$$\lim_{t \to +\infty} \frac{x(t)}{R(t)} = L \neq 0.$$ 

Without loss of generality, we suppose that $L > 0$. Then there exist $c > 0$ and $T_0 \geq 0$ such that

$$x(t) \geq cR(t),$$

$$x(A(t,x(t))) \geq cR(A_*(t)), \quad t \geq T_0. \quad (6)$$

The integration of Eq. (1) shows that

$$\int_{T_0}^{\infty} f(s,x(s),x(A(s,x(s)))) \, ds \leq r(T_0)x'(T_0). \quad (7)$$

Inequalities (6) and (7) imply that

$$\int_{T_0}^{\infty} f(s,cR(s),cR(A_*(s))) \, ds < +\infty.$$

(2) Let inequality (5) be fulfilled with a constant $c \neq 0$. Without loss of generality, we suppose that $c > 0$. We choose $m$ such that $0 < m < c/2$ and $T > 0$ such that

$$\int_T^\infty f(s,2mR(s),2mR(A_*(s))) \, ds \leq m. \quad (8)$$

Define the set

$$D = \left\{ x \in C([T_{-1}, +\infty), \mathbb{R}) \mid x(t) = 0, \quad T_{-1} \leq t < T, \right. $$

$$\left. mR(t,T) \leq x(t) \leq 2mR(t,T), \quad t \geq T \right\}$$

and the operator $S : D \to C([T_{-1}, +\infty), \mathbb{R})$ by the formula

$$Sx(t) = \begin{cases} 
0, & T_{-1} \leq t < T, \\
\int_T^t \frac{1}{r(s)} \left[m + \int_s^\infty f(u,x(u),x(A(u,x(u)))) \, du\right] \, ds, & t \geq T. 
\end{cases} \quad (9)$$

Let $x \in D$. It follows from (9) and conditions H1, H4 that $Sx(t) \geq mR(t,T), \quad t \geq T$. Since $x(t) \leq 2mR(t,T)$ implies $x(t) \leq 2mR(t)$, then using conditions H4 and H7 we get

$$\int_T^\infty f(s,x(s),x(A(s,x(s)))) \, ds \leq \int_T^\infty f(s,2mR(s),2mR(A_*(s))) \, ds. \quad (10)$$

It follows from (8)–(10) that $Sx(t) \leq 2mR(t,T), \quad t \geq T$. Therefore, $SD \subseteq D$.

It is standard to verify that the other conditions of the Schauder–Tychonov fixed point theorem are fulfilled and therefore there exists $z \in D$ such that $z(t) = Sz(t)$. It is easy to see that $z(t)$ is a solution of Eq. (1).
The equality
\[ r(t)z'(t) = m + \int_{t}^{\infty} f(u, z(u), z(A(u, z(u)))) \, du \]
implies that \( \lim_{t \to +\infty} r(t)z'(t) = m \). Therefore, \( \lim_{t \to +\infty} z(t)/R(t) = m \), i.e., \( z(t) \) has the property (P1).

**Theorem 6.** Let the conditions H1–H7 hold. Then Eq. (1) has a nonoscillatory solution having the property (P3) if and only if there exists a constant \( c \neq 0 \) such that
\[ \int_{0}^{\infty} R(s)|f(s, c, c)| \, ds < +\infty. \]  

**Proof.** (1) Let Eq. (1) have a solution \( x(t) \) with the property (P3): \( \lim_{t \to +\infty} x(t) = c \neq 0 \). Without loss of generality, we suppose that \( c > 0 \). Then \( x(t) \) is a finally positive solution, \( x'(t) > 0 \) finally, \( \lim_{t \to +\infty} r(t)x'(t) = 0 \) and there exist positive constants \( c_1, c_2 \) and \( T_1 \) such that \( c_1 \leq x(t) \leq c_2, \ c_1 \leq x(A(t, x(t))) \leq c_2, \ t \geq T_1. \)

Integrating (1) from \( t \) to \( +\infty \), we get
\[ r(t)x'(t) = \int_{t}^{\infty} f(s, x(s), x(A(s, x(s)))) \, ds \]
\[ \geq \int_{t}^{\infty} f(s, c_1, c_1) \, ds, \]
i.e.,
\[ \frac{1}{r(t)} \int_{t}^{\infty} f(s, c_1, c_1) \, ds \leq x'(t). \]  

Integrating (12) from \( T_1 \) to \( +\infty \), we obtain
\[ \int_{T_1}^{\infty} \frac{1}{r(t)} \left[ \int_{t}^{\infty} f(s, c_1, c_1) \, ds \right] \, dt \leq x(+\infty) - x(T_1), \]
and therefore \( \int_{T_1}^{\infty} R(s, T_1)f(s, c_1, c_1) \, ds < +\infty. \) The last inequality implies (11).

(2) Let (11) hold with a constant \( c \neq 0 \). Without loss of generality, we suppose that \( c > 0 \). Choose \( T \geq 0 \) such that
\[ \int_{T}^{\infty} R(s)f(s, c, c) \, ds \leq \frac{c}{2}. \]  

Define the set
\[ D = \left\{ x \in C([T_-, +\infty), \mathbb{R}) \mid \begin{array}{l} x(t) = c/2, \ T_- \leq t < T, \\ c/2 \leq x(t) \leq c, \ t \geq T \end{array} \right\} \]
and the operator $S: D \to C([T_-, +\infty), \mathbb{R})$ by the formula

$$Sx(t) = \begin{cases} \frac{c}{2} & T_- \leq t < T, \\
\frac{c}{2} + \int_T^t \frac{1}{r(s)} \left[ \int_s^\infty f(u, x(u), x(A(u, x(u)))) \, du \right] \, ds, & t \geq T.\end{cases}$$

Let $x \in D$. Then $Sx(t) \geq c/2$ for $t \geq T$. Moreover, it follows from (13) and from the definition of the operator $S$ that $Sx(t) \leq c$.

Therefore, $SD \subseteq D$. Analogously to the proof of Theorem 5, we conclude that the operator $S$ has a fixed point $z \in D$, i.e., $z(t) = Sz(t)$, $t \geq T_-$. It can be verified immediately that $z(t)$ is a solution of Eq. (1).

The equality

$$r(t)z'(t) = \int_t^\infty f(u, z(u), z(A(u, z(u)))) \, du$$

implies that

$$\lim_{t \to +\infty} \frac{z(t)}{r(t)} = \lim_{t \to +\infty} r(t)z'(t) = 0$$

and

$$\lim_{t \to +\infty} z(t) = \text{const} \in \left[ \frac{c}{2}, c \right].$$

**Theorem 7.** Let the conditions $H_1$–$H_7$ hold.

Then Eq. (1) has a nonoscillatory solution having the property (P2) if

$$\int_0^\infty |f(s, cR(s), cR(A^*(s)))| \, ds < +\infty$$

for some nonzero constant $c$ and

$$\int_0^\infty R(s) |f(s, d, d)| \, ds = +\infty$$

for each nonzero constant $d$ for which $cd > 0$.

**Proof.** Without loss of generality, we suppose that $c > 0$ in (14). Let $m \in (0, c]$ and $T \geq 0$ be such that

$$\int_T^\infty f(s, cR(s), cR(A^*(s))) \, ds \leq m$$

and $R(T) \geq 1$. Then

$$m + mR(t, T) \leq cR(t), \quad t \geq T.$$ (17)

Define the set

$$D = \left\{ x \in C([T_-, +\infty), \mathbb{R}) \mid \begin{array}{ll}
x(t) = m, & T_- \leq t < T, \\
m \leq x(t) \leq m + mR(t, T), & t \geq T\end{array} \right\}$$
and the operator $S : D \to C([T_{-1}, +\infty), \mathbb{R})$ by the formula

$$Sx(t) = \begin{cases} m, & T_{-1} \leq t < T, \\
 m + \int_{T}^{t} \frac{1}{r(s)} \left[ \int_{s}^{\infty} f(u, x(u), x(A(u, x(u)))) \, du \right] \, ds, & t \geq T. 
\end{cases}$$

Let $x \in D$. Then $Sx(t) \geq m$. It follows from (16) and (17) that

$$Sx(t) \leq m + \int_{T}^{t} \frac{1}{r(s)} \left[ \int_{s}^{\infty} f(u, cR(u), cR(A(u))) \, du \right] \, ds$$

$$\leq m + m \int_{T}^{t} \frac{1}{r(s)} \, ds = m + mR(t, T).$$

Therefore, $SD \subseteq D$. Analogous to the proof of Theorem 5, we conclude that the operator $S$ has a fixed point $z \in D$, $z(t) = Sz(t)$, $t \geq T_{-1}$. It can be verified immediately that $z(t)$ is a solution of Eq. (1).

The equality

$$r(t)z'(t) = \int_{T}^{t} f(s, z(s), z(A(s, z(s)))) \, ds$$

implies that

$$\lim_{t \to +\infty} z(t) = \lim_{t \to +\infty} r(t)z'(t) = 0$$

and the inequality

$$z(t) \geq m + \int_{T}^{t} \frac{1}{r(s)} \left[ \int_{s}^{\infty} f(u, m, m) \, du \right] \, ds$$

implies in view of condition (15) that $\lim_{t \to +\infty} z(t) = +\infty$. Therefore, $z(t)$ has the property (P2).

**Definition 8.** The function $f(t, u, v)$ is said to be strongly sublinear if there exists a constant $\beta \in (0, 1)$ such that $|f(t, u, v)|/|u|^\beta$ is a nonincreasing function in $|u|$, $|v|$ for each fixed $t \geq 0$.

**Definition 9.** The function $f(t, u, v)$ is said to be strongly superlinear if there exists a constant $\gamma > 1$ such that $|f(t, u, v)|/|u|^\gamma$ is a nondecreasing function in $|u|$, $|v|$ for each fixed $t \geq 0$.

**Theorem 10.** Let the conditions H1–H7 hold and the function $f(t, u, v)$ be strongly sublinear with a constant $\beta \in (0, 1)$.

Then the following assertions are equivalent:

(i) For each nonzero constant $c$

$$\int_{0}^{\infty} |f(s, cR(s), cR(A^*(s)))| \, ds = +\infty.$$

(ii) Each regular solution of Eq. (1) is oscillatory.
Proof. If we suppose that
\[
\int_0^\infty |f(s, cR(s), cR(\Delta^*(s)))| \, ds < +\infty
\]
for some \( c \neq 0 \), then by Theorem 5, Eq. (1) has a nonoscillatory solution with the property (P1).

Let \( x(t) \) be a nonoscillatory solution of Eq. (1). Without loss of generality, we suppose that \( x(t) \) is a finally positive solution: \( x(t) > 0, t \geq T_1 \geq 0 \). Then, as in Lemma 4 we obtain (2) and integrating from \( t \) to \( +\infty \) we get
\[
x'(t) \geq \frac{1}{r(t)} \int_t^\infty f(s, x(s), x(\Delta(s, x(s)))) \, ds, \quad t \geq T_1.
\]
(18)

Let \( T_2 \geq T_1 \) be such that \( R(t, T_1) \geq \frac{1}{2} R(t) \) for \( t \gg T_2 \). Then, integrating (18) from \( T_1 \) to \( t \geq T_2 \) we conclude that
\[
x(t) \geq x(t) - x(T_1)
\]
\[
\geq \int_{T_1}^t \frac{1}{r(s)} \left[ \int_s^\infty f(u, x(u), x(\Delta(u, x(u)))) \, du \right] \, ds
\]
\[
\geq \int_{T_1}^t \frac{1}{r(s)} \left[ \int_s^\infty f(u, x(u), x(\Delta(u, x(u)))) \, du \right] \, ds
\]
\[
\geq \frac{1}{2} R(t) \int_t^\infty f(u, x(u), x(\Delta(u, x(u)))) \, du.
\]
(19)

Since there exists a constant \( c > 0 \) such that \( x(t) \leq cR(t), x(\Delta(t, x(t))) \leq cR(\Delta^*(t)) \) for \( t \geq T_2 \), the strong sublinearity of \( f \) implies
\[
f(t, x(t), x(\Delta(t, x(t)))) = \frac{f(t, x(t), x(\Delta(t, x(t))))}{x^\beta(t)} x^\beta(t)
\]
\[
\geq \frac{f(t, cR(t), cR(\Delta^*(t)))}{c^\beta} \left[ \frac{x(t)}{R(t)} \right]^\beta.
\]
(20)

It follows from (19) and (20) that
\[
\frac{x(t)}{R(t)} \geq \frac{1}{2c^\beta} \int_t^\infty f(s, cR(s), cR(\Delta^*(s))) \left[ \frac{x(s)}{R(s)} \right]^\beta \, ds \equiv z(t)
\]
and therefore
\[
z'(t) \leq -\frac{1}{2c^\beta} f(t, cR(t), cR(\Delta^*(t))) z^\beta(t).
\]
Thus, we obtain
\[
[z^{1-\beta}(t)]' = (1 - \beta)z^{-\beta}(t)z'(t)
\]
\[
\leq -\frac{1}{2} (1 - \beta)c^{-\beta} f(t, cR(t), cR(\Delta^*(t))).
\]
(21)
Integrating (21), we obtain
\[ z^{1-\beta}(t) - z^{1-\beta}(T_2) \leq -\frac{1}{2}(1 - \beta)c^{-\beta} \int_{T_2}^{t} f(s, cR(s), cR(A^*(s))) \, ds. \]

Having in mind that \( \lim_{t \to +\infty} z(t) = 0 \), we conclude that
\[ \int_{T_2}^{\infty} f(s, cR(s), cR(A^*(s))) \, ds < +\infty. \]

**Theorem 11.** Let the conditions H1–H7 hold and the function \( f \) be strongly superlinear with a constant \( \gamma > 1 \). Then the following assertions are equivalent:

(i) For each nonzero constant \( c \)
\[ \int_{0}^{\infty} R(s) |f(s, c, c)| \, ds = +\infty. \]

(ii) Each regular solution of Eq. (1) is oscillatory.

**Proof.** If we suppose that
\[ \int_{0}^{\infty} R(s) |f(s, c, c)| \, ds < +\infty \]
for some \( c \neq 0 \), then by Theorem 6, Eq. (1) has a nonoscillatory solution having the property (P3).

Let Eq. (1) have a nonoscillatory solution \( x(t) \). Without loss of generality, we suppose that \( x(t) \) is a finally positive solution: \( x(t) > 0, t \geq T_1 \gg 0 \). Then by Lemma 4 it follows that \( x(t) \) is an increasing function for \( t \geq T_1 \) and for some \( c > 0 \) the inequalities
\[ x(t) \geq c > 0, \quad x(A(t, x(t))) \geq c \]  
are valid for \( t \geq T_2 \geq T_1 \).

As in Theorem 10 we obtain the inequality
\[ x'(t) \geq \frac{1}{r(t)} \int_{t}^{\infty} f(s, x(s), x(A(s, x(s)))) \, ds \quad \text{for } t \geq T_2. \]  
(23)

Inequalities (22) and the strong superlinearity of \( f \) imply that
\[ f(t, x(t), x(A(t, x(t)))) \geq \frac{f(t, c, c)}{c^\gamma} x'(t). \]  
(24)

Since \( x(t) \) is an increasing function, it follows from (23) and (24) that
\[ x'(t) \geq \frac{1}{r(t)} \int_{t}^{\infty} f(s, c, c) \, ds \ \frac{x'(t)}{c^\gamma}. \]  
(25)

Dividing (25) by \( x'(t) \) and integrating from \( T_2 \) to \( +\infty \), we obtain the inequality
\[ c^{-\gamma} \int_{T_2}^{\infty} \frac{1}{r(t)} \left[ \int_{t}^{\infty} f(s, c, c) \, ds \right] \, dt \leq \frac{x^{1-\gamma}(T_2)}{\gamma - 1} < +\infty, \]
which implies
\[ \int_0^\infty R(s) f(s,c,c) \, ds < +\infty. \]

Acknowledgement

The present investigation was partially supported by the Bulgarian Ministry of Education and Science under Grant MM-702.

References