Generalised fine and Wilf’s theorem for arbitrary number of periods

Sorin Constantinescu, Lucian Ilie∗,1

Department of Computer Science, University of Western Ontario, London, Ont., Canada N6A 5B7

Abstract

The well known Fine and Wilf’s theorem for words states that if a word has two periods and its length is at least as long as the sum of the two periods minus their greatest common divisor, then the word also has as period the greatest common divisor. We generalise this result for an arbitrary number of periods. Our bound is strictly better in some cases than previous generalisations. Moreover, we prove it optimal. We show also that any extremal word is unique up to letter renaming and give an algorithm to compute both the bound and a corresponding extremal word.

Keywords: Combinatorics on words; Periods; Fine and Wilf’s theorem

1. Introduction

Fine and Wilf’s theorem for words is one of the most widely used and known results on words. It was initially proved by Fine and Wilf [7] in connection with real functions but then adopted as a natural result for words, see [5,9,10]. We say that a word (string) has a certain integer as period if the word repeats itself after that period; e.g., the word \textit{abaabaaba} has periods 3, 6, and 8. It is not difficult to see that, given a set of periods, any long enough word which has those periods will have also their greatest common divisor as period. The essential question is how long the word should be. Fine and Wilf’s theorem states that length for two periods: it is the sum of the two periods minus their greatest common divisor. We

∗ Corresponding author.

E-mail addresses: sorinco@csd.uwo.ca (S. Constantinescu), ilie@csd.uwo.ca (L. Ilie).

1 Research supported in part by NSERC.

do:10.1016/j.tcs.2005.01.007
are interested in both upper and lower bounds for this length. While the bound stated by this theorem is a lower bound, it has been also proved to be an upper bound, that is, it has been proved to be optimal. The optimality has been rigorously proved by Choffrut and Karhumäki [5].

The problem of finding the (optimal) bound for arbitrary number of period has not been settled yet. The first generalisation was given by Castelli et al. [3] for three periods. Then, following the same ideas, a generalisation for an arbitrary number of periods was given by Justin [8]. Their bounds are proved to be lower bounds and were claimed to be optimal in some loose sense, see below. Further extensions and generalisations of Fine and Wilf’s theorem are given in [1,4,2,11].

First of all, we need to make it clear what we are looking for. Given a set of periods, we want the optimal bound (i.e., shortest length) which imposes the greatest common divisor as period. The above mentioned generalisations gave a lower bound which can be strictly improved in some cases. The loose optimality given by them essentially shows that for some, but not all, sets of periods, if 1 is subtracted from their bound, then it will no longer impose the greatest common divisor as period.

We shall give a new bound, in the general case of any arbitrarily fixed number of periods, and prove it optimal in the natural (strong) sense mentioned above. Our construction closely follows the previous generalisations and then modifies those by considering a case when their bound can be strictly improved. The modification proves to be essential as it brings the optimality. While the proof that the new bound is a good lower bound is not difficult, the optimality is a bit more involved. We give also an algorithm which computes simultaneously the bound and a word realising it.

The paper is organised as follows. Section 2 gives the basic definitions and the formal statement of Fine and Wilf’s theorem together with its counterpart for the optimality. Section 3 introduces the new bound which is proved to be good in Section 4. Section 5 introduces graphs associated with bounds and sets of periods and gives several results about those which are used in the optimality proof of Section 6. The results for the associated graphs are interesting by themselves and, in particular, after the optimality of the bound is proved the uniqueness of the extremal words follows immediately. The last section contains the algorithm which is a straightforward application of the results on graphs.

2. Fine and Wilf’s theorem

An alphabet is a finite non-empty set. For an alphabet A, the set of all finite words over A is denoted by $A^*$. For a word $w \in A^*$, the length of $w$, that is, the number of letters in $w$, is denoted by $|w|$. If $w = a_1 a_2 \ldots a_n$, where $a_i \in A$, for all $1 \leq i \leq n$, we say that $p \geq 1$ is a period of $w$ if $a_i = a_{i+p}$, for all $1 \leq i \leq n - p$. Notice that any $p \geq |w|$ is a period of $w$.

Given an $n$-tuple of positive integers $\mathbf{p} = (p_1, \ldots, p_n)$ and a positive integer $k$, we say that $k$ is a good bound for $\mathbf{p}$ if any word of length $k$ which has periods $p_1, \ldots, p_n$ has also period $d = \gcd(p_1, \ldots, p_n)$; $k$ is the optimal bound for $\mathbf{p}$ if it is a good bound whereas $k - 1$ is not, that is, there exists a word $w$ of length $k - 1$ which has the periods in $\mathbf{p}$ but not $d$. Notice that the notion of optimal bound makes sense only if $d$ is not among the elements of $\mathbf{p}$. 
With the above notions, Fine and Wilf’s theorem for words \([5,9,10]\) is Theorem 1 below. The optimality \([5]\) is given in Theorem 2.

**Theorem 1.** For any positive \(p\) and \(q\), \(p + q - \gcd(p, q)\) is a good bound for \((p, q)\).

**Theorem 2.** For any positive \(p\) and \(q\) such that one is not a divisor of the other, \(p + q - \gcd(p, q)\) is the optimal bound for \((p, q)\).

3. The new bound

We give in this section the new generalisation of Fine and Wilf’s theorem. We closely follow the idea of Castelli et al. \([3]\) for arbitrary number of periods (they did it for three) and then add an improvement at the end. As already mentioned, Justin \([8]\) did a similar thing except that he did not do the improvement at the end. Also, his highly compact bound is harder to understand.

Let \(p = (p_1, \ldots, p_n)\) be an \(n\)-tuple of non-negative integers; \(p\) is called increasing if \(p_1 \leq p_2 \leq \cdots \leq p_n\) and strictly increasing if \(p_1 < p_2 < \cdots < p_n\). The sum of its elements is denoted by \(|p| = p_1 + p_2 + \cdots + p_n\). \(I(p)\) gives the increasing \(n\)-tuple with the same elements as \(p\). By convention, all tuples of integers we use have non-negative components and are increasing. For a tuple \(p = (p_1, \ldots, p_n)\) such that \(p_{n-1} > 0\), we define the operator \(R\) by

\[
R(p) = (0, 0, \ldots, 0, p_m, p_{m+1} - p_m, p_{m+2} - p_m, \ldots, p_n - p_m),
\]

where \(m = \min\{i \mid 1 \leq i \leq n - 1, p_m > 0\}\). We then define the sequence \((p^{(k)})_{k \geq 0}\) by \(p^{(0)} = p\), \(p^{(k+1)} = I(R(p^{(k)}))\), for any \(k \geq 0\) for which \(R(p^{(k)})\) is defined (that is, \(p_{n-1}^{(k)} > 0\)). Put \(m_0(p) = 0\) and, for any \(1 \leq i \leq n - 1\),

\[
m_i(p) = \min\{k \geq 0 \mid p_i^{(k)} = 0\}.
\]

When \(p\) is understood we simply write \(m_i\). With these notations, the last element of our sequence \((p^{(k)})_{k \geq 0}\) is \(p^{(mn-1)} = (0, 0, \ldots, 0, \gcd(p))\). We define then, for any \(n \geq 1\), the function \(f_n\) by

\[
f_n(p) = \sum_{i=1}^{n-1} \sum_{k=m_{i-1}}^{m_i-1} p_i^{(k)}.
\]

In words, \(f_n(p)\) is the sum of the first non-zero elements of the tuples \(p^{(i)}\), \(0 \leq i \leq m_{n-1} - 1\). Notice that, for \(n = 1\), we have \(f_1(p_1) = 0\) and, for \(n = 2\), we obtain the bound in Fine and Wilf’s theorem, i.e., \(f_2(p_1, p_2) = p_1 + p_2 - \gcd(p_1, p_2)\).

To simplify the notations, from now on, we shall write, for \(p = (p_1, \ldots, p_n)\) and for \(1 \leq m \leq n\), \(\gcd_m(p) = \gcd(p_1, \ldots, p_m)\) and \(f_m(p) = f_m(p_1, \ldots, p_m)\).

**Example 3.** For \(p = (17, 24, 26, 28, 32)\), the computation of \(f_5(p)\) is shown in Fig. 1; \(f_5(p) = 34\) is the sum of the numbers written in boldface.
The improvement we brought to $f_w$ than $f_n$ is the straightforward generalisation of what Castelli et al. [3] did. However, we can do better. In the case when $p_n$ leaves the greatest common divisor unchanged, that is, $d = \gcd_n(p) = \gcd_{n-1}(p)$, we have two bounds, $f_n(p)$ and $f_{n-1}(p)$ which impose the same period $d$. In the case when $f_n(p) > f_{n-1}(p)$, $f_n(p)$ is clearly not optimal. This happens for instance when $p_n > f_{n-1}(p)$. Here is an example.

**Example 4.** Consider $p = (4, 6, 10)$. The bound given by $f_5(17, 24, 26, 28, 32)$ is not optimal since there is no word of length 9 which has the periods 4, 6, 10 but not 2. The optimal bound is 8.

However, $p_n$ might leave the greatest common divisor unchanged but strictly improve the bound, as in the following example.

**Example 5.** Consider $p = (10, 14, 16)$. Here $f_3(10, 14) = 22$ is the optimal bound for $(10, 14)$ and $f_3(10, 14, 16) = 20$ is the optimal bound for $p$. Both impose the period $2 = \gcd(10, 14) = \gcd(10, 14, 16)$.

Notice further that, whenever $p_n$ changes the greatest common divisor, the relation between $p_n$ and $f_{n-1}(p)$ is irrelevant since the two bounds, $f_{n-1}(p)$ and $f_n(p)$ have different goals.

Next is the bound we give and which we shall prove optimal. We define, for any $p = (p_1, \ldots, p_n)$, the function $f_w(p)$ by $f_{w_2}(p_1, p_2) = f_2(p_1, p_2)$ and, for $n \geq 3$, by (we use the same notation: $f_{w_n}(p) = f_{w_m}(p_1, \ldots, p_m)$):

$$ f_{w_n}(p) = \begin{cases} f_n(p) & \text{if } \gcd_n(p) \neq \gcd_{n-1}(p), \\ f_{n-1}(p) & \text{if } \gcd_n(p) = \gcd_{n-1}(p) \text{ and } p_n < f_{w_{n-1}}(p), \\ f_{w_{n-1}}(p) & \text{if } \gcd_n(p) = \gcd_{n-1}(p) \text{ and } p_n \geq f_{w_{n-1}}(p). \end{cases} $$

The improvement we brought to $f_w$ over $f$ is in the third line of the definition. It looks simple but what is hidden there is the following assertion which seems to be difficult to prove directly: In the case $p_n$ does not change the greatest common divisor but is smaller than $f_{n-1}(p)$, we should have, intuitively, $f_n(p) \leq f_{n-1}(p)$.

As seen from the above definition, each $f_{w_n}(p)$ equals some $f_r(p)$, for some $r \leq n$. Let us closely investigate this correspondence between $f_{w_n}$ and $f_s$. Assume that, for some $r$, $p_r$
Lemma 7. \[ \gcd_i(p) \neq \gcd_{i-1}(p) \] changes the greatest common divisor, that is, \( \gcd_{r-1}(p) \neq \gcd_r(p) \). (Actually, \( \gcd_{r-1}(p) > \gcd_r(p) \), but this is irrelevant for our considerations.) Then, according to the definition of \( f_w \), we have \( f_w(p) = f_r(p) \). Now, we may have, for a number of indices \( i = r + 1, r + 2, \ldots \), that \( p_i < f_w_i(p) \) and \( f_w_i(p) = f_i(p) \) until some \( s \) is found with \( p_{s+1} \geq f_w_s(p) \). As \( p \) is increasing, we get that, for all indices \( i = s + 1, s + 2, \ldots \) such that \( p_i \) does not change the greatest common divisor, \( f_w_i(p) = f_s(p) \) and also \( f_i(p) = p_i \). We summarise these observations in Lemma 6. Consider \( m \geq 2 \) such that \( \gcd_m(p) \neq p_1 \) and denote

\[
\alpha_{p,m} = \max \{ i \mid 2 \leq i \leq m, \ \gcd_i(p) \neq \gcd_{i-1}(p) \},
\]

\[
\beta_{p,m} = \begin{cases} 
\min \{ i \mid \alpha_{p,m} + 1 \leq i \leq m, \ p_i \geq f_{w_{i-1}}(p) \} & \text{if } \alpha_{p,m} < m, \\
\alpha_{p,m} & \text{if } \alpha_{p,m} = m.
\end{cases}
\]

**Lemma 6.** Given a tuple \( p = (p_1, \ldots, p_n) \), for any \( m \) with \( 2 \leq m \leq n \) such that \( \gcd_m(p) \neq p_1 \), the assertions (i) and (ii) below are true:

(i) for any \( i \) with \( \alpha_{p,m} \leq i \leq \beta_{p,m} - 1 \), we have \( p_{i+1} < f_{w_i}(p) = f_i(p) \), and \( f_{w_{\beta_{p,m} - 1}}(p) = f_{p_{\beta_{p,m}}}(p) \),

(ii) for any \( i \) with \( \beta_{p,m} + 1 \leq i \leq m \), we have \( f_i(p) = p_i \geq f_{w_{i-1}}(p) \) and \( f_{w_i}(p) = f_{p_{\beta_{p,m}}}(p) \).

A concise description of the above result is shown in Fig. 2.

### 4. Generalised fine and Wilf’s theorem

We give in this section our generalisation of Fine and Wilf’s theorem. That means we prove that the bound we introduced in the previous section, \( f_w_n(p) \), imposes the period \( d = \gcd_n(p) \) given that \( p_1, \ldots, p_n \) are periods already (that is, \( f_w_n(p) \) is a good bound for \( p \)).

We start with two lemmata which are useful. The first contains some properties of the functions \( (f_n)_n \).

**Lemma 7.** If \( p = (p_1, \ldots, p_n) \), \( n \geq 2 \), and \( p' = I(R(p)) = (p'_1, \ldots, p'_n) \), then

(i) if \( p_1 > 0 \), then \( f_n(p) = f_n(p') + p_1 \),

(ii) if \( p_1 = 0 \), then \( f_n(p) = f_{n-1}(p_2, \ldots, p_n) \),

(iii) \( f_n(p) \geq p_n \),

(iv) if \( p_1 < p_n \), then \( f_n(p) \geq 2p_1 \).
Proof. (i) and (ii) are clear from the definition of $f_n$.

We prove (iii) by double induction; first on $n \geq 2$ and second on $|p|$. For $n = 2$, the statement follows from the definition of $f_2$. Assume it is true for all values smaller than $n \geq 3$ and let us prove it for $n$. If $p_1 = 0$, then, using (ii) and the inductive hypothesis, we have $f_n(p) = f_{n-1}(p_2, \ldots, p_n) \geq p_n$. If $p_1 > 0$, then, using (i) and the inductive hypothesis, $f_n(p) = f_n(p') + p_1 \geq p_n' + p_1 \geq p_n - p_1 + p_1 = p_n$.

(iv) If $p_1 = 0$ there is nothing to prove. Assume $p_1 > 0$. Then, by (i) and (iii), $f_n(p) = f_n(p') + p_1 \geq p_n' + p_1 \geq 2p_1$. □

The next lemma is a well known property. We prove it for completeness sake.

Lemma 8. If $w$ is a word having two periods $p$ and $q$, then
(i) suff$_{|w| - p}(w)$ has period $q - p$ (assuming $p < q$) and
(ii) pref$_p(w)w$ has period $p + q$ (assuming $|w| \geq p$).

Proof. Assume $|w| = n$ and put $w = a_1a_2 \ldots a_n$.
(i) We have suff$_{|w| - p}(w) = a_{p+1} \ldots a_n$. For any $i$ with $p + 1 \leq i \leq n - q + p$, $a_i = a_{i-p} = a_{i+q-p}$.
(ii) We have pref$_p(w)w = a_1 \ldots a_p a_1 \ldots a_n = b_1 \ldots b_{n+p}$. For any $i$ with $1 \leq i \leq n - q$, $b_i = b_{i+p} = b_{i+p+q}$. □

We prove next that our bound is good.

Theorem 9. For any $p = (p_1, \ldots, p_n)$, $f_{w_n}(p)$ is a good bound for $p$.

Proof. We notice first that it is enough to show that $f_n(p)$ is a good bound. Indeed, this is very clear when we look at the definition of $f_{w_n}$. In the first two cases, it is equal to $f_n$, so it is good if $f_n$ is. In the third, it is equal to some $f_m(p)$, for some $m < n$, for which $\gcd_n(p) = d$. But if $f_n(p)$ imposes the period $d$, then so does $f_{w_n}(p)$.

Thus, let us show $f_n(p)$ is a good bound. As with the construction, our proof closely follows the ideas in [3]. The proof is by double induction; first on $n \geq 2$ and second on the integer $p_1|p| \geq 0$.

For $n = 2$ we have the ordinary Fine and Wilf theorem. Consider $n \geq 3$ and assume the property true up to $n - 1$. Then consider an $n$-tuple $p = (p_1, \ldots, p_n)$ of non-negative integers and a word $w$ with periods $p_i$, $1 \leq i \leq n$, such that $|w| \geq f_n(p)$.

If $p_1|p| = 0$, then $p_1 = 0$. Thus $w$ has periods $p_2, p_3, \ldots, p_n$ and $|w| \geq f_n(0, p_2, p_3, \ldots, p_n)$. By Lemma 7(ii), $|w| \geq f_{n-1}(p_2, \ldots, p_n)$, hence, by the inductive hypothesis, $w$ has period $\gcd_{n-1}(p_2, \ldots, p_n) = \gcd_{n-1}(0, p_2, p_3, \ldots, p_n)$.

Assume $p_1|p| > 0$ and assume the property true for all $n$-tuples $q = (q_1, \ldots, q_n)$ with $q_1|q| < p_1|p|$. If $p_1 = p_n$, there is nothing to prove, so assume $p_1 < p_n$.

Lemma 7(iii) gives $|w| \geq p_n$ and put $w = uv$ with $|u| = p_1$. By Lemma 8(i), $v$ has periods $p_1, p_2 - p_1, \ldots, p_n - p_1$. Moreover, Lemma 7(i) implies $|v| = |w| - p_1 \geq f_n(p) - p_1 = f_n(I(R(p)))$. By the inductive hypothesis, $v$ has also period $\gcd_n(R(p)) = \gcd_n(p)$. Since, by Lemma 7(iv), $|v| = |w| - p_1 \geq p_1$, it follows that $w$ has also period $\gcd_n(p)$, as claimed. □
5. The associated graph

We introduce in this section graphs associated with words and periods. We prove some results about those which are going to be useful for the optimality proof in the next section.

Given a word $w$ with $|w| = k$ and the periods $p = (p_1, \ldots, p_n)$, we construct the undirected graph

$$G(k, p) = ([1, \ldots, k], \{(i, j) | 1 \leq i \neq j \leq k, |i - j| = p_r, \text{ for some } 1 \leq r \leq n\}).$$

Notice that the graph depends only on the length of the word and the periods. We shall assume $k \geq 2, p_1$. Let us see next how the graph changes from $G = G(k, p)$ to $G' = G(k - p_1, R(p))$. We assume $k \geq 2p_1 + 1$. We claim first that two vertices $x, y$ in $G$ such that $x > p_1, y > p_1$, are connected in $G$, if and only if $x - p_1$ and $y - p_1$ are connected in $G'$.

Consider an arbitrary edge on a path in $G$ between $x$ and $y$. If this edge is $(i, i + p_1)$ with $i > p_1$, then we have the edge $(i - p_1, i)$ in $G'$; if $i \leq p_1$, then there must be a path $(i + p_1, i, i + p_j), j \geq 2$, in $G$ and we have the edge $(i, i + p_j - p_1)$ in $G'$. If the edge is $(i, i + p_j)$, $j \geq 2$, with $i > p_1$, then we have in $G'$ the path $(i - p_1, i, i + p_j - p_1)$; in the case $i \leq p_1$, we must have a path $(i + p_j, i, i + p_k)$, and we may assume $k \geq 2, k \neq j$; the corresponding path in $G'$ is $(i + p_j - p_1, i, i + p_k - p_1)$. The above transformations from $G$ to $G'$ are shown in Fig. 4(i)–(iv).

Conversely, for the edge $(i - p_1, i)$ in $G'$, we have $(i, i + p_1)$ in $G$. For $(i, i + p_j - p_1)$, $j \geq 2$, in $G'$, we have in $G$ the path $(i + p_1, i, i + p_j)$; see Fig. 4(i)–(ii).

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1 This type of graph was first introduced by Castelli et al. [3].
Assuming that \( k - p_1 \) disconnects \( G' \), this means there are two vertices \( x \) and \( y \) in \( G' \) connected by a path in \( G' \) but any such path contains \( k - p_1 \). By the above, there is a path in \( G \) between the vertices \( x + p_1 \) and \( y + p_1 \). We claim that any path between \( x + p_1 \) and \( y + p_1 \) in \( G \) must contain \( k \). Indeed, if there is one such path which does not contain \( k \), then, using the transformations in Fig. 4(i)–(iv), we can construct a path in \( G' \) connecting \( x \) and \( y \) that does not contain \( k - p_1 \), a contradiction. Consequently, \( k \) disconnects \( G \). We have proved the following lemma; Corollary 13 shows how Lemma 12 can be used in the optimality proof.

**Lemma 12.** If \( k \geq 2p_1 + 1 \) and \( k - p_1 \) disconnects \( G' = G(k - p_1, R(p)) \), then \( k \) disconnects \( G = G(k, p) \).

**Corollary 13.** If \( f_w(p) = p_1 + f_w(R(p)) \) and \( \gcd_n(p) = \gcd_m(p) \), then the optimality of \( f_w(R(p)) \) implies the optimality of \( f_w(p) \).

Notice also that, in the case \( k = 2p_1 = f_w(p) \), the vertex \( p_1 \) is adjacent in the graph \( G(k, p) \) only with \( 2p_1 \). But \( 2p_1 \) is connected also to other vertices (here we assume \( p \) strictly increasing such that \( \gcd_n(p) < p_1 \) and use Lemma 7(iii); \( 2p_1 \) is adjacent to \( 2p_1 - p_2 \)). Therefore \( 2p_1 \) disconnects \( G(2p_1, p) \). We proved therefore the following result:

**Lemma 14.** If \( f_w(p) = 2p_1 \), then it is optimal.

**Remark 15.** Another observation which will be useful is that whenever two consecutive arguments of \( f_w \) or \( f_n \) are the same, one can be omitted. That is, if \( p_i = p_{i+1} \), then \( f_w(p) = f_w(p_1, \ldots, p_i, p_{i+2}, \ldots, p_n) \) and similarly for \( f_n \).

### 6. Optimality

Besides the results from the previous section, we shall need several lemmata for the optimality proof. Essentially, we are interested to see how the situation in Lemma 6 changes from \( p \) to \( I(R(p)) \). For all lemmata below, we assume \( p = (p_1, \ldots, p_n) \) is an increasing tuple and \( p' = I(R(p)) \). We shall distinguish and investigate two cases, depending on whether \( r \), with \( p'_r = p_1 \), is smaller than \( z_{p,n} \) or in between \( z_{p,n} \) and \( b'_{p,n} \). The first case is simpler and dealt with in Lemma 16. The second is somewhat more complicated and discussed in Lemma 19; Lemmata 17 and 18 are used for its proof.
Lemma 16. If $p'_r = p_1$, for some $1 \leq r < \alpha_{p,n}$, then
(i) $\alpha_{p',n} = \alpha_{p,n}$ and
(ii) for any $i$ with $\alpha_{p,n} \leq i \leq n$, we have $\beta_{p',i} = \beta_{p,i}$.

Proof. Notice first that, for any $i$ with $r + 1 \leq i \leq n$, $\gcd_i(p) = \gcd_i(p')$. Since $r < \alpha_{p,n}$, this implies (i). We have also $fw_{2p,n}(p') = f_{2p,n}(p') = f_{2p,n}(p) - p_1 = fw_{2p,n}(p) - p_1$.

Inductively, we get, for any $i$ with $\alpha_{p,n} \leq i \leq n$, that $fw_i(p') = fw_i(p) - p_1$; since also $p'_i = p_i - p_1$, we get $p'_i < fw_i(p')$ iff $p_i < fw_i(p)$. This implies (ii). □

Lemma 17. If $fw_n(p) \geq 2p_1 + 1$ and both $fw_n(p)$ and $fw_n(p')$ are optimal bounds, then $fw_n(p') = fw_n(p) - p_1$.

Proof. Denote $d = \gcd_n(p) = \gcd_n(p')$. We prove first that $fw_n(p) - p_1$ is a good bound for $p'$. Consider an arbitrary word $w$ with $|w| = fw_n(p) - p_1$ which has periods $p_1, p_2 - p_1, \ldots, p_n - p_1$. Take $x = \text{pref}_{p_1}(w)$ (notice that $|w| > p_1$). We have $|x| = fw_n(p)$ and, by Lemma 8, $x$ has periods $p_1, \ldots, p_n$. Since $fw_n(p)$ is a good bound for $p$, it follows that $x$ has also period $d$ and so does $w$.

Next let us prove that $fw_n(p) - p_1$ is the optimal bound for $p'$. As $fw_n(p)$ is optimal, there is $x$ with $|x| = fw_n(p) - 1$ such that $x$ has periods $p_1, \ldots, p_n$ but not $d$. Put $w = \text{suff}_{|x|-p_1}(x)$. We have $|w| = fw_n(p) - p_1 - 1$ and, by Lemma 8(i), $w$ has periods $p_1, p_2 - p_1, \ldots, p_n - p_1$. Also, $w$ does not have period $d$ since this would imply, by the fact that $|w| \geq p_1$, that $x$ has period $d$, a contradiction.

Therefore, $fw_n(p) - p_1$ is the optimal bound for $p'$. Since $fw_n(p')$ is also the optimal bound for $p'$, they must coincide and the result is proved. □

Lemma 18. If $fw_n(p) = f_n(p)$, then either
(i) $\beta_{p,n} = n$ or
(ii) for any $i$ with $\beta_{p,n} + 1 \leq i \leq n$, $p_i = p_n$ and, for any $i$ with $\beta_{p,n} \leq i \leq n$, $fw_i(p) = f_i(p) = p_n$.

Proof. Assume $\beta_{p,n} < n$. Then, using Lemmata 6 and 7(iii), we have

$$p_n \geq p_{n-1} \geq \cdots \geq p_{\beta_{p,n} - n} = fw_{p_{\beta_{p,n}}}(p) = f_{p_{\beta_{p,n}}}(p) = fw_{p_n}(p) = f_n(p) \geq p_n.$$ 

Thus, all the above values are the same. By Lemma 6, we have, for any $i$ with $\beta_{p,n} \leq i \leq n$, that $fw_i(p) = f_i(p) = f_{p_{\beta_{p,n}}}(p) = p_n$. □

Lemma 19. Assume that $p'_r = p_1$, for some $\alpha_{p,n} \leq r < \beta_{p,n}$, $p_1 < p_n$, $p_{r+1}' > p_r'$ and both $fw_r(p)$ and $fw_r(p')$ are optimal bounds. Then, for any $i$ with $r \leq i \leq n$, $\beta_{p',i} = \beta_{p,i}$.

Proof. We show first that $fw_r(p) \geq 2p_1 + 1$. Using Lemmata 6 and 7(iv), we have $fw_r(p) = f_r(p) \geq 2p_1$. If $fw_r(p) = 2p_1$, then $p_{r+1} = p_{r+1}' + p_1 > p_r' + p_1 = 2p_1 = fw_r(p)$ implies, by Lemma 6, that $r + 1 \geq \beta_{p_{r+1},n} + 1$, a contradiction.

Now, using Lemma 17, we have, $fw_r(p') = fw_r(p) - p_1 = f_r(p) - p_1 = f_r(p')$. Therefore, by Lemma 18, we have that either $\beta_{p',r} = r$ or $fw_r(p') = p_r'$. In the latter case,
\( f_w(p) = f_w(p') + p_1 = p'_r + p_1 \leq p_1 \), a contradiction. Therefore, \( \beta_{p', r} = \beta_{p, r} = r \) and the claim follows by a reasoning similar to the one in the proof of Lemma 16(ii). \( \square \)

**Theorem 20.** For any \( p = (p_1, \ldots, p_n) \) with \( d = \gcd_n(p) < p_1 \), \( f_w(n) \) is the optimal bound for \( p \).

**Proof.** The proof is by induction on both \( n \) and \( |p| \). For \( n = 2 \) the optimality has been proved in Theorem 2 but follows also from the proof below by noticing that \( n = 2 \) implies \( p \) is strictly increasing and \( \alpha_{p, 2} = \beta_{p, 2} = 2 \).

Consider \( p \) with \( d < p_1 \). If \( p \) is not strictly increasing, then the optimality follows from the inductive hypothesis and Remark 15. So, we may assume \( p \) is strictly increasing. Also, if \( \beta_{p, n} < n \), then \( f_w_n(p) = f_w_n(p) \), for some \( m < n \) and the optimality follows again by the inductive hypothesis. Therefore, we assume \( \beta_{p, n} = n \) which implies \( f_w_n(p) = f_n(p) \).

Also, let \( r \) be such that \( p'_r = p_1 \) and \( p'_{r+1} > p'_r \) (for the case when \( r < n \)).

First assume \( r < n \). Then, by the above assumption, \( r < \beta_{p, n} \). Let us see that \( d < p'_1 \). Indeed, \( d = p_2 - p_1 \) implies, on one hand, that \( p'_n \geq f_{n-1}(p') \) and so \( p_n = p'_n + p_1 \geq f_{n-1}(p') + p_1 = f_{n-1}(p) \). On the other hand, \( d = p_2 - p_1 \) implies that \( \gcd_n(p) = \gcd_{n-1}(p) \). Together, these give \( \beta_{p, n} \leq n - 1 \), a contradiction.

Now, if \( r < \alpha_{p, n} \), then Lemma 16 gives \( \beta_{p', n} = \beta_{p, n} = n \). If \( r \geq \alpha_{p, n} \), then Lemma 19 implies \( \beta_{p', n} = \beta_{p, n} = n \). In either case, we have \( f_w_n(p') = f_n(p') \). Therefore \( f_w_n(p) = f_w_n(p') + p_1 \) and, since \( f_w_n(p') \) is optimal by the inductive hypothesis, the optimality of \( f_w_n(p) \) follows from Corollary 13.

Assume next \( r = n \). If \( d = p_2 - p_1 \), then \( f_w_n(p) = 2p_1 \) which is optimal by Lemma 14 so assume \( d < p_2 - p_1 \). If \( \beta_{p', n} = n \), then we use the inductive hypothesis and Corollary 13 as above to derive the optimality of \( f_w_n(p) \). If \( \beta_{p', n} < n \), then \( f_w_n(p) = f_n(p') + p_1 = 2p_1 \) and is optimal by Lemma 14. \( \square \)

As an immediate consequence of Theorem 20 and Lemma 10 we obtain the following uniqueness result concerning extremal words for the optimal bound.

**Corollary 21.** The word \( w \) of length \( f_w_n(p) - 1 \) which has all periods in \( p \) but not \( d = \gcd_n(p) \) and which has the highest number of letters is unique up to a renaming of letters.

7. **Computing the bound and extremal words**

We give in this section an algorithm which, given a tuple \( p = (p_1, \ldots, p_n) \), computes simultaneously \( f_w_n(p) \) and an extremal word as in Corollary 21. The idea is to construct the graph \( G(k, p) \) for \( k = p_1, p_1 + 1, p_1 + 2, \ldots \) until the number of connected components decreases to \( \gcd_n(p) \).

**Algorithm 22.**
- **input:** a tuple \( p = (p_1, \ldots, p_n) \)
- **output:** \( f_w_n(p) \) and a word \( w \) of length \( f_w_n(p) - 1 \) which has all periods in \( p \) but not \( d = \gcd_n(p) \)
1. compute \( d = \gcd_n(p) \)
2. if \( d = p_1 \) then
3. output ‘error: trivial tuple’; return
4. \( G \leftarrow ([1, 2, \ldots, p_1], \emptyset) \)
5. \( \text{node} \leftarrow p_1 \)
6. while the number of connected components of \( G \) is more than \( d \) do
7. \( \text{node} \leftarrow \text{node} + 1 \)
8. add \( \text{node} \) to \( G \)
9. \( \text{node} \) is a new vertex
10. \((\text{node}, \text{node} − p_i)\) is a new edge whenever \( \text{node} − p_i \geq 1 \)
11. update the number of connected components of \( G \)
12. output \( f_{\text{fw}}(p) = \text{node} \)
13. remove \( \text{node} \) from \( G \)
14. construct \( w = a_1 a_2 \ldots a_{\text{node}−1} \) such that
15. \( a_i = a_j \) iff \( i \) and \( j \) are in the same connected component of \( G \)
16. output extremal word = \( w \); return

The correctness of Algorithm 22 follows directly from our discussion on associated graphs. The complexity of the algorithm is essentially proportional with \( n f_{\text{fw}}(p) \) assuming good data structures are used for the connected components of \( G \); see [6].

7.1. Note added in proofs

Tijdeman and Zamboni [12] investigated, independently and simultaneously, the same problems we studied in this paper. They proved, using different methods, essentially the same results as our Theorems 9 and 20. They gave also an algorithm corresponding to our Algorithm 22.

Their work, even though based on the method introduced in [3], is substantially different from ours: a different stopping condition in the computation of the value of the \( f_n \) functions directly gives the value of the \( f_{\text{fw}} \) function. Their algorithm closely follows this line of reasoning. The other results, concerning uniqueness and optimality, are proved using specific (and different) methods.

Acknowledgements

We would like to thank the anonymous referee for pointing out Tijdeman and Zamboni’s paper.

References


