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A New Computational Method for the Functional Inequality Constrained Minimax Optimization Problem

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Abstract—In this paper, we consider a general class of functional inequality constrained minimax optimization problems. This problem is first converted into a semi-infinite programming problem. Then, an auxiliary cost function is constructed based on a positive saturated function. The smallest zero of this auxiliary cost function is equal to the minimal cost of the semi-infinite programming problem. However, this auxiliary cost function is nonsmooth. Thus, a smoothing function is introduced. Then, an efficient computational procedure is developed to estimate the smallest zero of this auxiliary cost function. Furthermore, an error bound is obtained to validate the accuracy of the approximate solution. For illustration, two numerical examples are solved using the proposed approach.

Keywords—Functional inequality constraints, Optimization problems, Minimax, Estimation, Smoothing function, Computational methods.

1. INTRODUCTION

Consider the following functional inequality constrained minimax optimization problem:

$$\min_{x} \max_{y\in\Omega} f(x,y) \tag{1a}$$

$$\text{subject to:} \quad h_i(x,y) \le 0, \qquad \forall\, y \in \Omega, \quad i = 1,2,\dots,k, \tag{1b}$$

where Ω is a compact subset of R, $f(x,y)$ and $h_i(x,y)$, $i = 1,2,\dots,k$, are continuously differentiable functions.

For convenience, let $\Im$ be the feasible region of the problem (1) defined by

$$\Im = \{x \in R^n : h_i(x,y) \le 0,\ \forall\, y \in \Omega,\ i = 1,2,\dots,k\}.$$

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Throughout this paper, we assume the following condition is satisfied.

ASSUMPTION (A1). *The feasible set $\Im$ contains at least one interior point, i.e., there exists an x^0 such that $h_i(x^0, y) < 0$, $\forall\, y \in \Omega$, $i = 1, 2, \dots, k$.*

It is well known that the minimax problem (1) is equivalent to the following semi-infinite programming problem:

$$\min_{x \in \Im} z \tag{2a}$$

$$\text{subject to:} \quad f(x,y) \le z, \qquad \forall\, y \in \Omega. \tag{2b}$$

Suppose Ω is a discrete set. Let the number of elements in Ω be denoted by r. Then, the semi-infinite programming problem (2) reduces to a conventional inequality constrained minimization problem, and hence, can be solved by gradient-based algorithms such as those reported in [1] and the references therein. However, it has been found in [2] that the rate of convergence of these algorithms is, in general, not high. Furthermore, if r is large, or Ω is a continuous set, then these methods are most likely to perform poorly. This unfortunate outcome was also reported in [3], where Ω is a continuous set. The minimax problem considered in [3] arises from a constrained H_∞ optimal control problem, where the cost function is the maximum singular value of the transfer function matrix over a frequency range. This cost function is to be minimized over all stabilizing controllers. Consequently, a new computational procedure was proposed in [3] based on a well-known idea in functional analysis. More precisely, the cost function is approximated by the L_p-norm of the maximum eigenvalue of the transfer function matrix over the frequency range. For the functional inequality constraints, they are handled by the constraint transcription introduced in [4].

Thus, the functional inequality constrained minimax optimization problem is approximated by a sequence of conventional optimization problems, each of which is solvable by existing smooth optimization software packages. Further results and new applications were reported in [5,6]. The main disadvantage of this approach is that it is required to scale the integrand of the L_p-approximate cost function to avoid numerical overflow when the value of p is large. A scaling scheme will be different from one problem to another. Furthermore, there is no systematic way to choose these scaling schemes. They are basically chosen on the basis of trial-and-error. In addition, the value of the scaled approximate cost function is not allowed to vary too much during the optimization or we will encounter numerical overflow. Unfortunately, there is no way to know if the value of the scaling approximate cost function would vary slowly or not as a function of the decision variables.

Let us now return to consider the problem (1) with Ω being a discrete set. Many papers are devoted to the development of methods for this constrained minimax problem. For example, see [1,2,5,7]. In [7], the minimax problem (1) is first converted into an equivalent problem similar to (2). Then, an exact penalty function is introduced as follows:

$$P(x,y,z) = z + \mu p(f(x,y) - z) + \sum_{i=1}^{k} \mu_i p\left(h_i(x,y)\right), \tag{3}$$

where μ and μ_i, $i = 1, 2, \dots, k$ are appropriate positive parameters, and $p(t)$ is a positive saturated function such that

$$p(t) = \begin{cases} 0, & \text{for } t \le 0, \\ t, & \text{for } t > 0. \end{cases}$$

Now, the minimax optimization problem is converted into the unconstrained nonsmooth minimization problem of the function $P(x,y,z)$ with respect to x, z for all y. In order to solve this nonsmooth minimization problem, many smoothing functions (see [7–9]) can be introduced to approximate $p(t)$. The one used in [7] is, in fact, similar to that reported earlier in [8,9]. Let this approximate function be denoted by $g_\varepsilon(t)$. It will also be used in this paper, and hence, will be defined later. The main advantage of using the approximate function $g_\varepsilon(t)$ is that the suboptimal solution obtained by solving the corresponding approximate cost function $P_\varepsilon(x,z)$ will satisfy the constraints if the parameters μ and μ_i, $i = 1, 2, \ldots, k$, are chosen to be sufficiently large.

In this paper, the functional inequality constrained minimax optimization problem will also be converted into a semi-infinite programming problem. Then, based on the constraint transcription introduced in [8], a smoothing auxiliary function is constructed for approximating this semi-infinite programming problem. A computational procedure is developed as the result of this approximate method. Two numerical examples are also given to illustrate our method.

2. EQUIVALENT AND APPROXIMATE PROBLEMS

Consider the problem (1) in Section 1. Let $J_0(\alpha)$ be an auxiliary function defined as

$$J_0(\alpha) = \min_{x\in R^n} \int_\Omega \left[p(f(x,y)-\alpha) + \sum_{i=1}^{k} p\left(h_i(x,y)\right)\right] dy. \tag{4}$$

Then, it is clear that the smallest zero of the auxiliary function is equal to the optimal solution of the original functional constrained minimax optimization problem (1). Let us adopt the smoothing function $g_\varepsilon(t)$ introduced in [8,9] to construct a new smooth approximate auxiliary cost function to estimate the smallest zero of the nonsmooth function J_0. For any $\varepsilon > 0$,

$$g_\varepsilon(t) = \begin{cases} 0, & \text{if } t \le -\varepsilon, \\ \dfrac{(t+\varepsilon)^2}{4\varepsilon}, & \text{if } -\varepsilon < t \le \varepsilon, \\ t, & \text{if } t > \varepsilon. \end{cases}$$

From [8,9], we know that this function has several desirable properties. These properties, which can be easily obtained from its definition, are listed in the following remark.

REMARK 1. The function $g_\varepsilon(t)$ has the following properties [8,9].

(a) It is monotonically increasing, though not strictly.
(b) $0 \le g_\varepsilon(t) - t \le \varepsilon/4$, $\forall\, t \in R$.
(c) For each $t \in R$, $g_\varepsilon(t)$ considered as a function of ε is nonnegative and continuous. Furthermore, if $0 < \varepsilon_1 < \varepsilon_2$, then $g_{\varepsilon_1}(t) \le g_{\varepsilon_2}(t)$, for any $t \in R$.
(d) For each $\varepsilon > 0$, $g_\varepsilon(t)$, as a function of t, is continuously differentiable and has a quadratic zero at $t = -\varepsilon$.
(e) $g_\varepsilon(t) \le \varepsilon/4$ if and only if $t < 0$.

Instead of searching for the smallest zero of the nonsmooth function $J_0(\alpha)$ directly, we consider the following smooth auxiliary cost function defined by

$$J(\varepsilon,\alpha) = \min_{x\in R^n} \Phi(x,\varepsilon,\alpha), \tag{5a}$$

where

$$\Phi(x,\varepsilon,\alpha) = \int_\Omega \left[g_\varepsilon(f(x,y)-\alpha) + \sum_{i=1}^{k} g_\varepsilon\left(h_i(x,y)\right)\right] dy. \tag{5b}$$

Obviously, the parameter ε in the $(f-\alpha)$-term and h_i-term can be different. For the theoretical analysis in this paper, we will only discuss the case that all of them are the same. However, all results can be extended to general cases.

For the function $\Phi(x,\varepsilon,\alpha)$ and $J(\varepsilon,\alpha)$ defined in (6), several desirable properties can be easily obtained from the properties of $g_\varepsilon(t)$ and the definition of $J(\varepsilon,\alpha)$. These properties are listed in the following lemma.

LEMMA 1.

(a) *For any* $\varepsilon > 0$, $\Phi(x,\varepsilon,\alpha)$ *is continuously differentiable in* x *and* α.
(b) *For* $\alpha \in R$, *if* ε_1 *and* ε_2 *are such that* $0 \le \varepsilon_1 \le \varepsilon_2$, *then* $J(\varepsilon_1,\alpha) \le J(\varepsilon_2,\alpha)$.
(c) *For any* $\varepsilon > 0$, *if* α_1 *and* α_2 *are such that* $\alpha_1 \le \alpha_2$, *then* $J(\varepsilon,\alpha_1) \ge J(\varepsilon,\alpha_2)$.
(d) $J(\varepsilon,\alpha)$ *is nonnegative and continuous.*
(e) *For any* $\varepsilon > 0$, *let* $\alpha_\varepsilon^\star := \min\{\alpha \in R : J(\varepsilon,\alpha) = 0\}$. *Then,* $\alpha_\varepsilon^\star$ *is a quadratic zero of* $J(\varepsilon,\alpha)$ *as a function of* α.

To continue, we assume that there exist constants M_1, M_2 such that the following conditions are satisfied.

ASSUMPTION (A2). $\left|\frac{\partial f(x,y)}{\partial y}\right| \le M_1$, $\left|\frac{\partial h_i(x,y)}{\partial y}\right| \le M_1$ *for any* $(x,y) \in \Im \times \Omega$, $i = 1,2,\dots,k$.

ASSUMPTION (A3). $\left|\frac{\partial f(x,y)}{\partial x_j}\right| \le M_2$ *for any* $(x,y) \in \Im \times \Omega$ *and* $j = 1,2,\dots,n$.

LEMMA 2. *For any* $\varepsilon > 0$ *and* $\alpha \in R$, *if* x_ε *satisfies* $\Phi(x_\varepsilon,\varepsilon,\alpha) = J(\varepsilon,\alpha) < \varepsilon^2/(32M_1)$, *then*

(a) $h_i(x_\varepsilon,y) < 0$, $\forall\, y \in \Omega$, $i = 1,2,\dots,k$, *and*
(b) $f(x_\varepsilon,y) < \alpha$, $\forall\, y \in \Omega$.

PROOF. Assume the contrary. Without loss of generality, we assume that (b) is not satisfied. Then, there exists at a point $y^0 \in \Omega$ such that $f(x_\varepsilon,y^0) \ge \alpha$.

Let $Y_\varepsilon(y^0)$ be a neighbourhood of y^0 defined by

$$Y_\varepsilon\left(y^0\right) = \left\{y \in \Omega : \left|y - y^0\right| \le \frac{\varepsilon}{2M_1}\right\}.$$

Then, by Assumption (A2) and the Taylor Theorem, we have

$$f\left(x_\varepsilon,y\right) \ge f\left(x_\varepsilon,y^0\right) - \sup_{y\in Y_\varepsilon(y^0)}\left|\frac{\partial f\left(x_\varepsilon,y\right)}{\partial y}\right|\left|y-y^0\right| \ge \alpha - M_1\left|y-y^0\right|,$$

and hence,

$$f\left(x_\varepsilon,y\right) - \alpha \ge -\frac{\varepsilon}{2}, \qquad \forall\, y \in Y_\varepsilon\left(y^0\right).$$

Thus, by the definition of the function $g_\varepsilon(t)$ and Remark 1(a), it follows that

$$\begin{aligned} J(\varepsilon,\alpha) &> \int_{Y_\varepsilon(y^0)} g_\varepsilon(f(\varepsilon,y)-\alpha)\,dy \\ &> \frac{\varepsilon}{16}\int_{Y_\varepsilon(y^0)} dy = \frac{\varepsilon^2}{32M_1}. \end{aligned}$$

This is clearly a contradiction. Therefore, the proof is complete. ∎

Define

$$\Im_\varepsilon := \{x \in R^n : h_i(x,y) < -\varepsilon, \text{ for all } y \in \Omega\}.$$

Since Ω is a compact set, and $f(x,y)$ is continuous on $\Im \times \Omega$, the function

$$\max_{y\in\Omega} f(x,y)$$

is a continuous function on $\Im$. Similarly, the function

$$\bar{h}(x) := \max_{i=1,2,\dots,k.y\in\Omega} h_i(x,y)$$

is a continuous function of $x \in \Im$. Therefore, $h(x,\bar{h}(x))$ is a continuous function on $\Im$. Thus, the set $\Im_\varepsilon$ can be expressed as

$$\Im_\varepsilon = \left\{x \in \Im : h\left(x,\bar{h}(x)\right) < -\varepsilon\right\}.$$

We need the following assumption.

ASSUMPTION (A4). *The boundary* $\partial\Im := \{x \in \Im : \bar{h}(x) = 0\}$ *of the set* $\Im$ *has no interior point nor isolated point.*

In view of (A4), it follows that for any $x' \in \partial\Im$, i.e., $\bar{h}(x') = 0$, there is a sequence $\{x_i\}_{i=1}^{\infty} \subset \Im$, such that $x_i \to x'$ and $\bar{h}(x_i)$ is strictly monotonically increasing as $i \to \infty$. Then, we have the following result.

THEOREM 1. *Let* α_ε *and* $\alpha^\star$ *be defined, respectively, by*

$$\alpha_\varepsilon = \min_{x\in\Im_\varepsilon} \max_{y\in\Omega} f(x,y)$$

and

$$\alpha^\star = \min_{x\in\Im} \max_{y\in\Omega} f(x,y).$$

Then,

$$\alpha^\star \le \alpha_\varepsilon$$

and

$$\alpha_\varepsilon \to \alpha^\star \qquad \text{as } \varepsilon \to 0^+.$$

PROOF. Let $(x^0, y^0) \in \Im \times \Omega$ be such that

$$f\left(x^0, y^0\right) = \min_{x\in\Im} \max_{y\in\Omega} f(x,y).$$

There are two cases to be considered.

(1) If (x^0, y^0) is an interior point of $\Im$, let

$$\delta = \min_{i=1,2,\dots,k} \left|h_i\left(x^0, y^0\right)\right| > 0.$$

Then, for any ε, $0 < \varepsilon < \delta$,

$$\min_{x\in\Im} \max_{y\in\Omega} f(x,y) = \min_{x\in\Im_\varepsilon} \max_{y\in\Omega} f(x,y).$$

(2) (x^0, y^0) is a boundary point of $\Im$. For any given $\varepsilon > 0$, since

$$\max_{y\in\Omega} f(x,y)$$

is a continuous function of $x \in \Im$, there exists a $\delta > 0$, such that

$$\max_{y\in\Omega} f(x,y) < \max_{y\in\Omega} f\left(x^0, y\right) + \varepsilon = f\left(x^0, y^0\right) + \varepsilon$$

for all $x \in \Theta_\delta(x^0)$, where

$$\Theta_\delta\left(x^0\right) = \left\{x \in \Im : \left|x - x^0\right| < \delta\right\}.$$

On the other hand, from Assumption (A4), there exists a point $\bar{x} \in \Im$ such that

$$\left|\bar{x} - x^0\right| < \delta, \qquad \bar{h}\left(\bar{x}\right) < 0.$$

Let $\bar{\varepsilon} = -(\bar{h}(\bar{x}))/2$. The following inequalities hold:

$$\min_{x\in\Im} \max_{y\in\Omega} f(x,y) \le \min_{x\in\Im_{\bar{\varepsilon}}} \max_{y\in\Omega} f(x,y) \le \max_{y\in\Omega} f\left(\bar{x}, y\right) < \min_{x\in\Im} \max_{y\in\Omega} f(x,y) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. ∎

Now, define

$$\alpha^\star = \min_{x \in \Im \subset R^n} \max_{y \in \Omega \subset R} f(x,y). \tag{6}$$

THEOREM 2. *For any $\varepsilon > 0$, let*

$$0 < J(\varepsilon,\alpha) < \frac{\varepsilon^2}{32M_1}.$$

Then,

$$\min_{x \in \Im_\varepsilon} \max_{y \in \Omega} f(x,y) \le \alpha \le \min_{x \in \Im_\varepsilon} \max_{y \in \Omega} f(x,y) + \varepsilon.$$

PROOF. Since $J(\varepsilon,\alpha) < \varepsilon^2/(32M_1)$, it follows from Lemma 2 that

$$\alpha \ge \min_{x \in \Im_\varepsilon} \max_{y \in \Omega} f(x,y).$$

On the other hand, let $x_\varepsilon \in \Im_\varepsilon$ such that

$$\max_{y \in \Omega} f(x_\varepsilon, y) = \min_{x \in \Im_\varepsilon} \max_{y \in \Omega} f(x,y).$$

Then,

$$J\left(\varepsilon, \bar{\alpha}_\varepsilon\right) \le \Phi\left(x_\varepsilon, \varepsilon, \bar{\alpha}_\varepsilon\right) = 0,$$

where

$$\bar{\alpha}_\varepsilon = \max_{y \in \Omega} f(x_\varepsilon, y) + \varepsilon.$$

Therefore,

$$\alpha \le \bar{\alpha}_\varepsilon.$$

The proof is complete. ∎

THEOREM 3. *For any $\varepsilon > 0$, let $x_\varepsilon \in R^n$ and α a real number such that $\Phi(x_\varepsilon, \varepsilon, \alpha) = J(\varepsilon, \alpha)$, where $|\Omega|$ denotes the Lebesgue measure of the set Ω. If the following condition is satisfied:*

$$J(\varepsilon,\alpha) \ge \frac{(k+1)\varepsilon}{4}|\Omega|, \tag{7}$$

then

$$\alpha \le \alpha^\star.$$

PROOF. Since

$$\alpha^\star = \min_{x \in \Im} \max_{y \in \Omega} f(x,y)$$

and $\Im, \Omega$ are compact sets by assumption, there exists a point $(x^\star, y^\star)$ such that

$$f\left(x^\star, y^\star\right) = \min_{x \in \Im} \max_{y \in \Omega} f(x,y).$$

Assuming the contrary, suppose $\alpha > \alpha^\star$. It follows from Remark 1(a),(e) that

$$\int_\Omega g_\varepsilon\left(f(x^\star, y) - \alpha\right) dy < \frac{\varepsilon}{4}|\Omega|$$

and

$$\int_\Omega g_\varepsilon\left(h_i\left(x^\star, y\right)\right) dy \le \frac{\varepsilon}{4}|\Omega|.$$

Therefore,

$$J(\varepsilon,\alpha) \le \Phi\left(x^\star, \varepsilon, \alpha\right) < \frac{(k+1)\varepsilon}{4}|\Omega|.$$

This is a contradiction to the condition (7). Thus the proof is complete. ∎

3. A COMPUTATIONAL PROCEDURE

On the basis of the results established in Section 2, we are in the position to present a procedure for solving the functional inequality constrained minimax optimization problem.

Computational Procedure

STEP 0. Select parameters ε and δ.

STEP 1. Choose $y_1, y_2 \in \Omega$ and calculate the values of

$$\min_{x\in\Im} f(x, y_1) \text{ and } \min_{x\in\Im} f(x, y_2).$$

Denote the values as α_1, α_2, respectively. Without loss of generality, we assume that $\alpha_1 < \alpha_2$.

STEP 2. Calculate $J(\varepsilon, \alpha_1)$. Suppose $J(\varepsilon, \alpha_1) \neq 0$; otherwise, decrease ε and recalculate $J(\varepsilon, \alpha_1)$ again.

STEP 3. Calculate $J(\varepsilon, \alpha_2)$.

STEP 4. Calculate α_3 in the following two cases:

(a) if $J(\varepsilon, \alpha_2) = 0$, find α_3 according to Golden section search formula:

$$\alpha_3 = \alpha_1 + 0.618\,(\alpha_2 - \alpha_1)\,;$$

(b) if $J(\varepsilon, \alpha_2) \geq \delta$, then, use the method reported in [10] to calculate α_3:

$$\alpha_3 = \alpha_2 + \frac{\alpha_2 - \alpha_1}{\left(J(\varepsilon, \alpha_1)/J(\varepsilon, \alpha_2)\right)^{1/2} - 1}.$$

Then, replace α_1 and α_2 by α_2 and α_3, respectively. For the sake of simplicity, we still denote them as α_1 and α_2, respectively. Then, return to Step 3.

STEP 5. If $J(\varepsilon, \alpha_2) < \delta$, check whether the solution satisfies the constraints and $f(x, y) \leq \alpha$ or not. If not, decrease δ and return to Step 3.

END of Procedure.

The following theorem can be easily proved on the basis of the results obtained in Section 2, and hence, the proof is omitted.

THEOREM 4.

(a) *The Computational Procedure terminates in a finite number of iterations.*

(b) *Let all the α_1's obtained in the successive order as $\bar{\alpha}_i$, $i = 1, 2, \ldots, L-1$. Let the last α_2 be denoted by $\bar{\alpha}_L$. Then, $\bar{\alpha}_i$ is increasing and $J(\bar{\alpha}_i)$ is decreasing.*

(c) *If there is an integer i_0 such that $J(\bar{\alpha}_{i_0}) \geq ((k+1)\varepsilon/4)|\Omega|$ and*

$$J\left(\bar{\alpha}_{i_0+1}\right) < \frac{(k+1)\varepsilon}{4}|\Omega|,$$

then, $\alpha^* \in (\alpha_{i_0}, \alpha_L]$.

By Theorem 4, it is clear that the precision of solution is $\alpha_L - \alpha_{i_0}$. To increase the accuracy, we can reduce the value of ε and then repeat the procedure. If a better lower bound α_{i_0} is required, one can modify Step 4(b) in the Computational Procedure to

$$\alpha_3 = \alpha_2 + \frac{\alpha_2 - \alpha_1}{\left(J(\varepsilon, \alpha_1)/J(\varepsilon, \alpha_2)\right) - 1}$$

or develop another procedure according to Theorem 3.

Let the vector x_ε^L be the decision vector of $J(\varepsilon, \alpha_L)$, i.e.,

$$\Phi\left(x_\varepsilon^L, \varepsilon, \alpha_L\right) = J\left(\varepsilon, \alpha_L\right).$$

We have the following result.

THEOREM 5. *If the decision vector $x^\star$ of the minimax optimization problem, i.e.,*

$$\max_{y\in\Omega} f\left(x^\star, y\right) = \min_{x\in\Im}\max_{y\in\Omega} f(x,y),$$

is unique, then,

$$x_\varepsilon^L \to x^\star \qquad \text{as } \varepsilon \to 0.$$

PROOF. Assume the contrary. Then there exists a positive real number ν such that, for any positive real number δ, there is a x_ε^L such that

$$\varepsilon < \delta, \qquad \left\|x^\star - x_\varepsilon^L\right\| > \nu.$$

Since $\Im$ is compact, there is a convergent subsequence of these x_ε^L. Let $x_{\varepsilon_i}^L$, $i = 1, 2, \dots$ be this subsequence and let x° be its limit vector. Then,

$$\Phi\left(x^\circ, 0, \alpha^\star\right) = \lim_{i\to\infty} \Phi\left(x_{\varepsilon_i}^L, \varepsilon_i, \alpha_{\varepsilon_i}^L\right) = \lim_{\varepsilon\to 0} \Phi\left(x_\varepsilon^L, \varepsilon, \alpha_\varepsilon^L\right) = 0.$$

Therefore, x° is another decision vector of the minimax optimization problem. This is a contradiction, and hence, the proof is complete. ∎

4. NUMERICAL EXAMPLES

For illustration of the proposed computational procedure, we consider two minimax optimization problems.

EXAMPLE 1. Find a 4$^{\text{th}}$ order polynomial to approximate the function s^6 on the closed interval $[0,1]$ such that the maximal difference is minimized. More precisely,

$$\min_{x\in R^5}\max_{s\in[0,1]} \left|x_1 + x_2 s + x_3 s^2 + x_4 s^3 + x_5 s^4 - s^6\right|.$$

Let

$$P(x,s) := x_1 + x_2 s + x_3 s^2 + x_4 s^3 + x_5 s^4.$$

To solve this problem, we construct an auxiliary cost function as

$$J(\varepsilon,\alpha) = \int_0^1 \left[g_\varepsilon\left(P(x,s) - s^6 - \alpha\right) + g_\varepsilon\left(-\alpha - P(x,s) - s^6\right)\right] ds.$$

Let α^l (respectively, α^u) be the largest α (respectively, the first value of α) obtained by the procedure such that $J(\varepsilon,\alpha) > (k+1)\varepsilon/4$ (respectively, $J(\varepsilon,\alpha) < \delta$). The results listed in Table 1 are obtained in 4 iterations, where x_1, x_2, x_3, x_4, x_5 are corresponding to α^u.

EXAMPLE 2. The harmonic filter design problem can be posed as a minimax optimization problem. In [5], the C-type filter is considered, where the the following weighted cost function J is selected to reflect the requirements of the filter

$$J = \lambda_1 G_S(s) + \frac{\lambda_2}{G_F(s)} + \lambda_3 G_V(s), \tag{8}$$

Table 1.

ε	10^{-4}	10^{-6}
δ	10^{-6}	10^{-8}
α^l	5.87×10^{-3}	5.87×10^{-3}
α^u	6.00×10^{-3}	5.95×10^{-3}
x_1	5.95×10^{-3}	5.81×10^{-3}
x_2	-0.27	-2.73
x_3	2.01	2.01
x_4	-4.92	-4.93
x_5	4.18	4.18
$\max(f(x,s) - \alpha^u)$	-5.68×10^{-5}	-2.73×10^{-6}
$f(x,s) \leq \alpha^u$ violated at $s =$	—	—

where $s = \omega j$ is the frequency variable, $\lambda_1, \lambda_2, \lambda_3$ are positive weights, and $G_S(s)$, $G_F(s)$, $G_V(s)$ are defined, respectively, as

$$\begin{aligned} G_S(s) &= \left\| \frac{\Omega_S(s)}{\Omega_F(s) + \Omega_S(s)} \right\|, \\ G_F(s) &= \left\| \frac{\Omega_F(s)}{\Omega_F(s) + \Omega_S(s)} \right\|, \\ G_V(s) &= \left\| \frac{1}{\Omega_F(s) + \Omega_S(s)} \right\|. \end{aligned}$$

These functions depend on some parameters R, L, c_1, c_2. The objective is to minimize the H_∞ norm of J, i.e.,

$$\|J\|_\infty = \max_{\omega \in [\alpha,\beta]} J(\omega j) \tag{9}$$

with respect to these parameters.

In [5], the H_∞ norm of J is estimated by the minimization of L_p-norm as $p \to +\infty$. The decision vector x obtained is that of the minimization problem of L_p-norm. Only a subsequence of these vectors tends to the decision vector of the original minimax optimization problem. Thus, the decision vector or its estimate is not provided by the method reported in [5].

For illustration, let us consider the following model:

$$\begin{aligned} \Omega_S(s) &= \frac{N(s)}{D(s)}, \\ N(s) &= 4.605 + 5208.3 \times 10^{-9} s + 166.9 s^2 + 0.7 \times 10^{-3} s^3 + 0.49 \times 10^{-6} s^4, \\ D(s) &= 1.071 \times 10^{11} + 4.467 \times 10^9 s + 2267 \times 10^{-4} s^2 + 2833 \times 10^{-7} s^3 \\ &\quad + 2.736 \times 10^{-6} s^4 + 1.835 \times 10^{-7} s^5, \\ \Omega_F(s) &= \frac{s c_1 \left(s^2 L c_2 + R c_2 s + 1\right)}{s^2 L c_2 (1 + R) 10^{-2} + R c_2 s + R + 1}, \end{aligned}$$

where

$$\begin{gathered}300\Omega \leq R \leq 700\Omega,\\ 0.15H \leq L \leq 0.3H,\\ 3\times 10^{-4}F \leq c_1 \leq 15\times 10^{-4}F,\\ 15\times 10^{-6}F \leq c_2 \leq 50\times 10^{-6}F,\\ \lambda_1 = 1, \quad [\alpha,\beta] = [250, 800]Hz,\\ \lambda_2 = 0.1, \quad \lambda_3 = 0.0005.\end{gathered}$$

In addition, there is a functional constraint on the filter

$$\|\Omega_F(s) + \Omega_S(s)\|_\infty \geq \delta'.$$

For this specific problem, we choose $\varepsilon = 10^{-6}$, $\delta = 10^{-8}$, and $\delta' = 2.5\times 10^{-4}$. We define an auxiliary cost function as

$$\tilde{J}(\varepsilon,\alpha) = \int_{0.25}^{0.8} k_1 g_{\varepsilon_1}(f(x,s)-\alpha) + k_2 g_{\varepsilon_2}\left(\delta' - \|\Omega_S(x,s)+\Omega_F(x,s)\|\right) ds, \tag{10}$$

where k_1 and k_2 are the weighting factors which are free to be chosen. Choose $\alpha_1 = 0$, $\alpha_2 = 10$, $k_1 = 10$, $k_2 = 1$, $\varepsilon = 10^{-3}\varepsilon_1$. By the Computational Procedure presented in Section 3, we use the minimization subroutine E04VDF of Nag library to find the minimum of $\tilde{J}$. Now the criteria for determining the lower bound α^l becomes

$$J(\varepsilon,\alpha) \geq \frac{k_1\varepsilon_1 + k_2\varepsilon_2}{4}|\Omega|.$$

The results are listed in the Table 2.

Table 2.

ε	$[10^{-4}, 10^{-7}]$	$[10^{-6}, 10^{-9}]$
δ	10^{-6}	10^{-8}
α^l	0.19610	0.20623
α^u, iterations	0.20768, 14	0.20795, 19
x_1	0.3	0.3
x_2	300	300
x_3	0.15×10^{-4}	0.15×10^{-4}
x_4	0.5×10^{-4}	0.5×10^{-4}
$\max(f(x,s)-\alpha^u)$	3.045×10^{-4}	2.7527×10^{-5}
constraint violated at:	—	—

In this table, one can see that $0.206 \leq \alpha^* \leq 0.208$, where α^* is defined by (6). Because of the different tolerances and search steps, the estimation of upper bound α^u for α^* does not decrease as the true argument of $\min_x \Phi(x,\varepsilon,\alpha)$ does. However, they are very close. If one can obtain the bound M_1 as that in Assumption (A2), an upper bound of α^* can be determined.

5. CONCLUSION

In this paper, we have considered a functional inequality constrained minimax optimization problem. Using a constraint transcription, we constructed an auxiliary cost function. The smallest zero of this auxiliary cost function is the solution to the original constrained minimax optimization problem. However, the auxiliary cost function is nonsmooth. Thus, a smoothing technique was used to approximate the smallest zero of that auxiliary function. We have also developed a computational procedure to estimate the smallest zero via solving several minimization problems. These minimization problems can be solved easily and efficiently by any optimization routine in the NAG library. Numerical studies demonstrate that the proposed computational procedure works well.

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