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Stability of fractional-order linear time-invariant systems with multiple noncommensurate orders☆

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ABSTRACT

Bounded-input bounded-output stability conditions for fractional-order linear time-invariant (LTI) systems with multiple noncommensurate orders have been established in this paper. The orders become noncommensurate orders when they do not have a common divisor. Sufficient and necessary conditions of stability for a fractional-order LTI system with multiple noncommensurate orders are presented in two cases. Based on the numerical inverse Laplace transform technique, time-domain responses for a fractional-order system with double noncommensurate orders are presented to illustrate the obtained stability results.

1. Introduction

Fractional calculus [1–3] generalizes the traditional (integer-order) calculus in which the order of derivatives and integrals can be any real or complex number. There are more and more research results announced verify the fact that the behavior of real systems can be described by fractional-order differential equations [4]. One resulting consequence is that fractional calculus is being extensively applied to control theory and applications, as documented the latest dedicated monograph [5]. On the one hand, using the notion of fractional-order, it may be a step closer to the real world life [6], because many real physical systems are well characterized by fractional-order differential equations. On the other hand, fractional calculus provides a powerful tool for the description of memory and hereditary effects in various substances, as well as for modeling dynamical processes in fractal media [7], this is why the fractional-order models are superior in comparison with integer-order models, in which, such effects or geometry are neglected.

Stability is a minimum requirement for control systems, certainly including fractional-order systems. In [8], the stability results on fractional-order linear time-invariant (FO-LTI) systems with commensurate orders were presented for the first time, it permits to check the asymptotic stability through the location of the system matrix eigenvalues of the pseudo state space representation of the fractional-order system in the complex plane. Henceforth, there were some systematic results on the robust stability of interval uncertain FO-LTI systems as presented in [9–13]. The BIBO-stability of fractional-order delay systems of retarded and neutral types has been studied in [14], in which necessary and sufficient conditions were presented for the retarded type, and only sufficient conditions were provided for the neutral type. In [15], necessary and sufficient conditions of stability were provided for an important special case fractional-order delay system of neutral type. However, such theorems obtained in [14,15] do not permit to conclude to system stability without computing the system's poles, which constitutes a tedious work, so based on Cauchy's integral theorem and by solving an initial-value

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problem, an effective numerical algorithm for testing the BIBO stability of fractional delay systems was presented in [16]. However, the fractional-order systems discussed in these papers are mostly commensurate orders, which means the orders can always converted to commensurate orders when they have a common divisor. To the best of our knowledge, there are few results concerning the stability analysis problems for fractional-order systems with noncommensurate orders. Based on Cauchy's theorem, a graphical test to evaluate fractional-order systems with noncommensurate orders are given in [17], however, this method is not very helpful because of the complicated procedures. Therefore motivated by the previous references, this paper addresses the bounded-input bounded-output stability for fractional-order systems with multiple discrete noncommensurate orders.

This paper is organized as follows. Some preliminaries for discussing fractional-order systems with multiple noncommensurate orders are given in Section 2, which includes definitions of fractional-order calculus operators, and Laplace transforms of fractional-order calculus operators are reviewed. In Section 3, main results on bounded-input and bounded-output stability for fractional-order systems with double noncommensurate orders and N-term noncommensurate orders are presented in detail. In Section 4, based on the numerical inverse Laplace transform technique, numerical examples for the double noncommensurate orders case are included to illustrate the main results established.

2. Preliminaries

In order to utilize fractional calculus for the discussion in this paper, the fundamental knowledge of fractional calculus is firstly recalled [3]. The unified formula of fractional-order integral is defined as follows:

$$ {}_0\mathrm{D}_t^{-\alpha} f(t) := \mathrm{D}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau $$

where $\alpha > 0, f(t)$ is an arbitrary integrable function, $\Gamma(\cdot)$ denotes the Gamma function.

Based on the fractional-order integral definition, two well-known fractional-order derivative operators definitions with $n-1 < \alpha < n,\ n \in N$ are presented as follows:

Riemann–Liouville fractional-order derivative:

$$ {}_0^R\mathrm{D}_t^{\alpha} f(t) := \mathrm{D}_R^{\alpha} f(t) = \mathrm{D}^n \mathrm{D}^{\alpha-n} f(t) = \frac{d^n}{dt^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \right]. $$

Caputo fractional-order derivative:

$$ {}_0^C\mathrm{D}_t^{\alpha} f(t) := \mathrm{D}_C^{\alpha} f(t) = \mathrm{D}^{\alpha-n} \mathrm{D}^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. $$

Note that Laplace transform [18] is a fundamental tool in systems and control engineering, in the following Laplace transforms for the fractional-order calculus operators defined above are given.

Laplace transform for fractional-order integral:

$$ L\left\{\mathrm{D}^{-\alpha} f(t)\right\} = s^{-\alpha} F(s). $$

Laplace transform for Riemann–Liouville fractional-order derivative:

$$ L\left\{\mathrm{D}_R^{\alpha} f(t)\right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k \left[\mathrm{D}_R^{\alpha-k-1} f(t)\right]_{t=0}. $$

Laplace transform for Caputo fractional-order derivative:

$$ L\left\{\mathrm{D}_C^{\alpha} f(t)\right\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). $$

3. Main results

In this section, bounded-input bounded-output (BIBO) stability in the context of fractional-order systems with multiple noncommensurate orders are established for two cases.

3.1. Case 1: double noncommensurate orders

For the fractional-order system with double noncommensurate orders described by

$$ \begin{aligned} &\mathrm{D}^{\beta_1} x(t) + \mathrm{D}^{\beta_2} x(t) = Ax(t) + Bu(t) \\ &y(t) = Cx(t) + Du(t) \end{aligned} \tag{1} $$

where $0 < \beta_1, \beta_2 \leq 1$ are noncommensurate orders, which means that they do not have a common divisor. D^{α} denotes the Caputo fractional-order derivative operator.

Remark 1. The fractional-order system (1) can always be converted to a fractional-order system with commensurate orders if both β_1 and β_2 are rational numbers [5]. Two fractional orders can be noncommensurate orders when at least one of β_1 and β_2 is not a rational number, they do not have common divisors. Based on Cauchy's theorem, a graphical test to evaluate fractional-order systems with noncommensurate orders are given in [17], and the fractional-order system is considered in frequency domain, however, this method is not very helpful because of the complicated procedures. In the following, sufficient and necessary conditions for the fractional-order system with double noncommensurate orders is proposed first.

Under the assumption of zero initial conditions, taking the Laplace transform of (1), and let $u(t) = \delta(t)$, we have

$$s^{\beta_1}X(s) + s^{\beta_2}X(s) = AX(s) + B.$$

Assume that $D = 0$, the transfer function of (1) is

$$H(s) = C\left((s^{\beta_1} + s^{\beta_2})I - A\right)^{-1}B.$$

Paralleled with the BIBO stability for traditional control systems, we have the following definition.

Definition 1. Fractional-order system (1) defined by its impulse response $h(t) = L^{-1}\{H(s)\}$ is BIBO stable if and only if $\forall u \in L^{\infty}(R^+)$, $h * u \in L^{\infty}(R^+)$, where $*$ stands for the convolution product and $L^{\infty}(R^+)$ stands for the Lebesgue space of measurable function h such that $\text{ess}\sup_{t\in R^+} |h(t)| < \infty$.

Theorem 1. *Fractional-order system (1) is BIBO stable if and only if $H(s)$ has no poles in the closed right half plane of the complex plane, i.e., the transcendental characteristic equation $\left|(s^{\beta_1} + s^{\beta_2})I - A\right| = 0$ has no zeros in the closed right half plane of the complex plane.*

Proof. (if part) Note that the final value theorem implies that $\lim_{t\to\infty} h(t) = sH(s) \to 0$, if all poles of $sH(s)$ lie in the left half-plane of complex plane. Note that s^{β} defines a multi-valued function of the complex variable s whose domain can be seen as a Riemann surface of a number of sheets which is infinite when β is an irrational number, or is finite when β is a rational number. In multiple-valued functions only the principal sheet defined by $-\pi < \arg s < \pi$ has its physical significance [19], so fractional-order system (1) is BIBO stable if $H(s)$ has no poles lie in the closed right half plane of the principal Riemann sheet, i.e., the transcendental characteristic equation $\left|(s^{\beta_1} + s^{\beta_2})I - A\right| = 0$ has no zeros in the closed right half plane of the complex plane.

(only if part) Since terms s^{β_1} and s^{β_2} are multi-valued functions whose domain is a Riemann surface of a number of sheets which is infinite in the case of $\beta_1, \beta_2 \in \mathrm{R}^+ \setminus \mathrm{Q}^+$. Let $\{s_k\}_{k=1,2,\ldots,n,\ldots}$ denotes the set of all poles of $H(s)$ such that $\left|(s^{\beta_1} + s^{\beta_2})I - A\right| = 0$. The multi-valued function $s \mapsto s^{\beta}$ becomes analytical in the complement of its branch cut line of complex plane when the branch cut line is chosen to be along the negative real axis including the branching points 0 and ∞. Hence, all arguments of s are restricted to the principal Riemann sheet $-\pi < \arg s < \pi$. It is obviously known from the Definition 1 that $H(s)$ lies in H_{∞}, which is the space of bounded analytic functions on the closed right half plane of complex plane, which means that there is not any pole of $H(s)$ lies in the closed right half plane of the principal Riemann sheet, i.e., there is not any zero of the transcendental characteristic equation $\left|(s^{\beta_1} + s^{\beta_2})I - A\right| = 0$ lies in the closed right half plane of the complex plane. □

Remark 2. It can be known from [3] that the analytical solution of $h(t)$ can be obtained based on the three-term fractional Green's function:

$$h(t) = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^{\beta_2(k+1)-1} E^{(k)}_{\beta_2-\beta_1,\beta_2+k\beta_1}\left(-t^{\beta_2-\beta_1}\right)$$

where $E^{(k)}_{\alpha,\beta}$ denotes the Mittag-Leffler function with three parameters [20].

Remark 3. The stability of n-dimensional linear fractional differential equation with time delays is studied in [21], when the delays are set to be zeros, the n dimensional linear fractional differential system with multiple time delays will become the normal sequential dimensional linear fractional differential system. While in this paper, the fractional-order systems with multiple noncommensurate orders cannot be transformed into the normal dimensional linear fractional differential systems.

Remark 4. It is not easy to calculate the zeros of $\left|(s^{\beta_1} + s^{\beta_2})I - A\right| = 0$ when both β_1 and β_2 are irrational numbers, so we have the following theorem.

Theorem 2. *The fractional-order system (1) is BIBO stable if and only if all the eigenvalues of A lie on the left of curve $l := l_a \bigcup l_b$, where l_a and l_b are symmetrical with respect to the real axis, and l_a is defined as $l_a := \{x + iy | x = x_{\omega}, y = y_{\omega}, \omega \in [0, \infty)\}$, with x_{ω} and y_{ω} are defined as $x_{\omega} := \omega^{\beta_1} \cos \frac{\beta_1}{2}\pi + \omega^{\beta_2} \cos \frac{\beta_2}{2}\pi$, $y_{\omega} := \omega^{\beta_1} \sin \frac{\beta_1}{2}\pi + \omega^{\beta_2} \sin \frac{\beta_2}{2}\pi$.*

Proof. From Theorem 1, we know that all the zeros of the transcendental characteristic equation $\left|(s^{\beta_1} + s^{\beta_2})I - A\right| = 0$ should lie in the open left half plane of the complex plane to ensure the BIBO stability of $H(s) = C\left((s^{\beta_1} + s^{\beta_2})I - A\right)^{-1}B$.

From transcendental characteristic equation $\left|(s^{\beta_1}+s^{\beta_2})I-A\right|=0$, we know that $s^{\beta_1}+s^{\beta_2}=\sigma_i(A)$ $(i=1,\dots,n)$, where $\sigma(A)$ denotes the set of eigenvalues of A. It is necessary to derive the range of $\lambda=s^{\beta_1}+s^{\beta_2}$ when s belongs to the open left half-plane of the complex plane, so it is naturally to derive the range of $\lambda=s^{\beta_1}+s^{\beta_2}$ when s lies on the imaginary axis, then for $s=j\omega$ $(0\le\omega<\infty)$, we have

$$\lambda=\left(\omega^{\beta_1}\cos\frac{\beta_1}{2}\pi+\omega^{\beta_2}\cos\frac{\beta_2}{2}\pi\right)+j\left(\omega^{\beta_1}\sin\frac{\beta_1}{2}\pi+\omega^{\beta_2}\sin\frac{\beta_2}{2}\pi\right)$$

for $s=j\omega(-\infty<\omega<0)$, we have

$$\lambda=\left((-\omega)^{\beta_1}\cos\frac{\beta_1}{2}\pi+(-\omega)^{\beta_2}\cos\frac{\beta_2}{2}\pi\right)-j\left((-\omega)^{\beta_1}\sin\frac{\beta_1}{2}\pi+(-\omega)^{\beta_2}\sin\frac{\beta_2}{2}\pi\right)$$

which means that the imaginary axis is mapped to a curve denoted by l, which is symmetrical with respect to the real axis. By choosing a s randomly which lies on the left of the imaginary axis, the range of $\lambda=s^{\beta_1}+s^{\beta_2}$ lies on the left of curve l, which means that the range of λ is the open left half plane of l when s belongs to the open left half complex plane. For simple illustration, the stable boundary of fractional-order system (1) with $\beta_1=\sqrt{2}-1$ and $\beta_2=\sqrt{3}-1$ is plotted in Fig. 1. □

Remark 5. The asymptotic stability of the zero solution of the linear homogeneous differential system with Riemann–Liouville fractional derivative is studied in [22], while the fractional-order systems with multiple noncommensurate orders discussed in current paper are with Caputo fractional derivative.

Remark 6. For fractional-order systems described by $D^{\beta_i}x(t)=Ax(t)+Bu(t)$ $(i=1,2)$, the stable boundary is defined as $l_{\beta_i}:=l_{a_i}\bigcup l_{b_i}$, where l_{a_i} and l_{b_i} are symmetrical with respect to the real axis, and l_{a_i} is defined as $l_{a_i}:=\left\{x+iy\,\middle|\,x=\omega^{\beta_i}\cos\frac{\beta_i}{2}\pi,\ y=\omega^{\beta_i}\sin\frac{\beta_i}{2}\pi,\ \omega\in[0,\infty)\right\}$. l_{β_1} and l_{β_2} are also plotted in Fig. 1, in which note that curve l lies between l_{β_1} and l_{β_2}.

Remark 7. We assume that $0<\beta_1,\beta_2\le 1$ in (1), which means that both β_1 and β_2 cannot be equal to 0. Without loss of generality, if $\beta_2=0$, β_1 is a irrational number, the fractional-order system (1) will become $D^{\beta_1}x(t)=A_0x(t)+Bu(t)$, with $A_0=A-I$. In this case the stable boundary is l_{β_1} which is defined in Remark 6 with respect to A_0.

3.2. Case 2: N-term noncommensurate orders

For the fractional-order system with multiple noncommensurate orders described by

$$\begin{aligned}&\sum_{i=1}^{n}D^{\beta_i}x(t)=Ax(t)+Bu(t)\\&y(t)=Cx(t)+Du(t)\end{aligned}\tag{2}$$

where β_i $(i=1,2,\dots,n)$ are noncommensurate orders, which means they do not have a common divisor.

Similar to the analysis in Case 1, under the assumption of zero initial conditions, and let $u(t)=\delta(t)$, then the transfer function of (2) is

$$H_1(s)=C\left(\sum_{i=1}^{n}s^{\beta_i}I-A\right)^{-1}B.$$

And we have the following parallel results.

Theorem 3. *The fractional-order system (2) is BIBO stable if and only if one of the following is satisfied:*

- *$H_1(s)$ has no poles in the closed right half plane of the complex plane, i.e., the transcendental characteristic equation $\left|\sum_{i=1}^{n}s^{\beta_i}I-A\right|=0$ has no zeros in the closed right half plane of the complex plane.*
- *all the eigenvalues of A lie on the left of curve $l_1:=l_{1a}\bigcup l_{1b}$, where l_{1a} and l_{1b} are symmetrical with respect to the real axis, and $l_{1a}:=\{x+iy|x=x_\omega, y=y_\omega, \omega\in(0,\infty)\}$, with x_ω and y_ω defined as $x_\omega:=\sum_{i=1}^{n}\omega^{\beta_i}\cos\frac{\beta_i}{2}\pi$, $y_\omega:=\sum_{i=1}^{n}\omega^{\beta_i}\sin\frac{\beta_i}{2}\pi$.*

Proof. The proof of this theorem is similar to the proof of Theorems 1 and 2. □

Remark 8. It can be also known from [3] that the analytical solution of $h(t)$ can also be obtained on the base of the n-term fractional Green's function.

Remark 9. From calculation and plotting through MATLAB, we know that curve l_1 must lies between $l_{\min}$ and $l_{\max}$ which are defined as

$$l_{\min}:=\left\{re^{i\theta}\,\middle|\,\theta=\pm\frac{\min\{\beta_i\}}{2}\pi,\ r\ge 0\right\},\qquad l_{\max}:=\left\{re^{i\theta}\,\middle|\,\theta=\pm\frac{\max\{\beta_i\}}{2}\pi,\ r\ge 0\right\}.$$

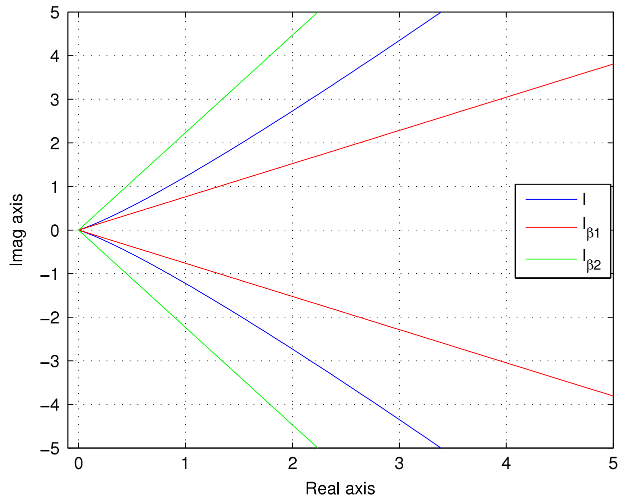
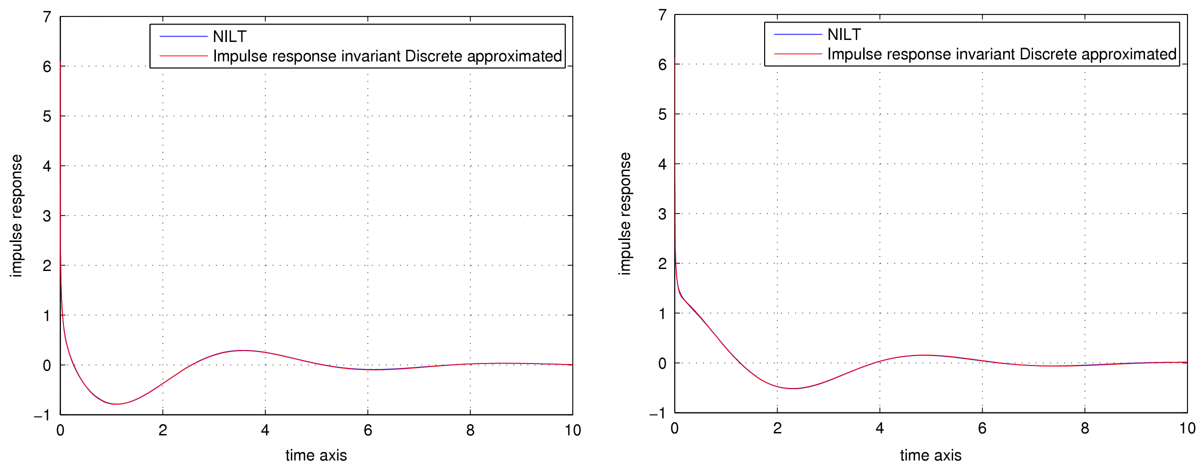


Fig. 1. Stable boundary of fractional-order system (1) with noncommensurate orders $\beta_1 = \sqrt{2} - 1$ and $\beta_2 = \sqrt{3} - 1$.



(a) State x_1.

(b) State x_2.

Fig. 2. The states of impulse response with null initiations.

4. Numerical examples

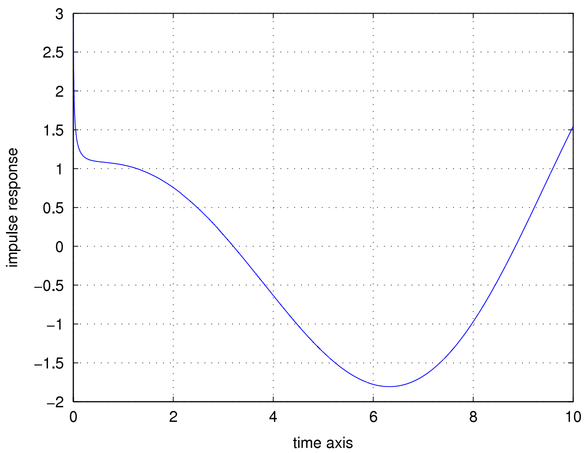
In this section, numerical examples are shown to demonstrate the effectiveness of the proposed concepts for the first case in Section 3.

Example 1. Consider a fractional-order system (1) with double noncommensurate orders described with parameters as $\beta_1 = \sqrt{2} - 1$, $\beta_2 = \sqrt{3} - 1$, $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $D = 0$.

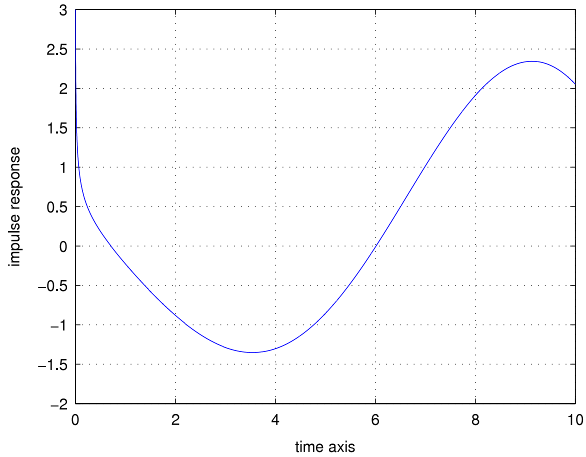
The eigenvalues of A are $\lambda_1 = 1 + 2j$ and $\lambda_2 = 1 - 2j$, it can be known from Fig. 1 in Theorem 2 that this fractional-order system is bounded-input bounded-output stable. It is not easy to plot the analytical solution, so based on the numerical inverse Laplace transform (NILT) technique [23], impulse response for $H(s) = C\left((s^{\beta_1} + s^{\beta_2})I - A\right)^{-1} B$ with null initiations are shown in Fig. 2.

Example 2. Consider a fractional-order system (1) with double noncommensurate orders described with parameters as $\beta_1 = \sqrt{2} - 1$, $\beta_2 = \sqrt{3} - 1$, $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 \end{bmatrix}$ and $D = 0$.

The eigenvalues of A are $\lambda_1 = 1 + j$ and $\lambda_2 = 1 - j$, it can be known from Fig. 1 in Theorem 2 that this fractional-order system is not bounded-input bounded-output stable. Based on the NILT technique, the states of impulse response for $H(s) = C\left((s^{\beta_1} + s^{\beta_2})I - A\right)^{-1} B$ with null initiations are shown in Fig. 3.



(a) State x_1.



(b) State x_2.

Fig. 3. The states of impulse response with null initiations.

5. Conclusions

The stability issues for fractional-order linear time-invariant systems with multiple noncommensurate orders are solved in this paper. The double noncommensurate orders and N-term noncommensurate orders cases are analyzed respectively, sufficient and necessary conditions of stability are obtained. In addition, based on the numerical inverse Laplace transform technique, time-domain responses for the double noncommensurate order case are presented to verify the main results as illustrative numerical examples.

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