INJECTIVES IN VARIETIES OF COMPLETELY REGULAR SEMIGROUPS

P.G. TROTTER

Department of Mathematics, University of Tasmania, Hobart, Tasmania, Australia

Communicated by J. Rhodes
Received 25 June 1986
Revised 23 February 1988

Introduction

The injectives in some varieties of completely regular semigroups are known. In particular the injectives are known in the following varieties: any variety of bands (Gerhard [2]); any variety of groups (García-Larrión [1] and Kovacs-Newman [6]); any variety of completely simple semigroups (Trotter [12]); and the variety of all semilattices of abelian groups (Schein [11]). In this paper the semigroups of the title will be specified.

The paper begins with definitions and statements of the relevant results from [1, 2, 6, 12]. In the next section it is shown that any injective in a variety of completely regular semigroups is an orthodox normal band of groups with retractions for its structure maps.

In Section 3 the investigation is restricted to semilattices of groups that are injectives in a variety \( \mathcal{V} \) of completely regular semigroups. Necessary and sufficient conditions for a semilattice of groups to be a \( \mathcal{V} \)-injective are obtained. In Section 4 the semilattice of groups results are generalised so that a description of the remaining injectives in \( \mathcal{V} \) can be given.

1. Preliminary results and definitions

A semigroup is completely regular if and only if it is a union of its subgroups. The \( \mathcal{D} \) relation on a completely regular semigroup \( S \) is the least semilattice congruence on \( S \) and the \( \mathcal{D} \)-classes are completely simple subsemigroups of \( S \) (see [4]). We will write

\[
S = \bigcup \{ S_\alpha : \alpha \in \Gamma \} \quad \Gamma = S/\mathcal{D}, \quad S_\alpha \text{ is a } \mathcal{D} \text{-class of } S. \quad (1)
\]

Let \( E(S) \) denote the set of idempotents of \( S \). For \( x \in S \) let \( x^{-1} \in S \) be the inverse of \( x \) that is \( \mathcal{K} \)-related to \( x \) and \( x^0 = xx^{-1} \).

The completely regular semigroup \( S \) is a normal band of groups if and only if
there exist homomorphisms \( \psi_{\alpha, \beta} : S_\alpha \to S_\beta \) for each \( \alpha, \beta \in \Gamma \) where \( \alpha \geq \beta \), such that \( \psi_{\alpha, \alpha} \) is the identity map, \( \psi_{\alpha, \beta} \psi_{\beta, \gamma} = \psi_{\alpha, \gamma} \) if \( \beta \geq \gamma \), and

\[
xy = x\psi_{\alpha, \alpha \beta}(y\psi_{\beta, \alpha \beta}), \quad x \in S_\alpha, \ y \in S_\beta.
\]

Call \( \psi_{\alpha, \beta} \) a structure map of \( S \). If \( S \) is a normal band of groups write

\[
S = [\Gamma; S_\alpha, \psi_{\alpha, \beta}].
\]  

(2)

By [7] the class of all completely regular semigroups is the variety of universal algebras with an associative binary operation and a unary operation \( x \mapsto x^{-1} \) such that \( xx^{-1}x = x, x^{-1}xx^{-1} = x^{-1} \) and \( xx^{-1} = x^{-1}x \). The following notation will be used:

- \( \mathcal{CR} \) – the variety of all completely regular semigroups.
- \( \mathcal{CP} \) – the variety of all completely simple semigroups.
- \( \mathcal{B} \) – the variety of all bands.
- \( \mathcal{PL} \) – the variety of all semilattices.
- \( \mathcal{NB} \) – the variety of all normal bands.
- \( \mathcal{LNB}, \mathcal{RNB} \) – respectively the variety of all left normal bands and all right normal bands.
- \( \mathcal{BB} \) – the variety of all rectangular bands.
- \( \mathcal{O} \) – the variety of all orthodox completely regular semigroups.
- \( \mathcal{G} \) – the variety of all groups.
- \( \mathcal{L}(V) \) – the lattice of all subvarieties of a subvariety \( V \) of \( \mathcal{CR} \).
- \( \mathcal{WB} \) – the variety of all bands of groups \( S \) such that \( S/H \in \mathcal{U} \in \mathcal{L}(\mathcal{B}) \).
- \([\mathcal{U}, \mathcal{V}] \) – the interval \( \{ H \in \mathcal{L}(\mathcal{CR}) : \mathcal{U} \subseteq H \subseteq \mathcal{V} \} \), for \( \mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR}) \).

Suppose \( S = [\Gamma; S_\alpha, \psi_{\alpha, \beta}] \in \mathcal{NB} \mathcal{B} \). There is a natural partial ordering \( \geq \) on \( S \) defined as follows: for \( a \in S_\alpha \) and \( b \in S_\beta \), \( a \geq b \) if and only if \( \alpha \geq \beta \) and \( a\psi_{\alpha, \beta} = b \).

A subset \( H \) of \( S \) is called compatible if and only if for each \( a \in H \cap S_\alpha \) and \( b \in H \cap S_\beta \), \( a\psi_{\alpha, \beta} = b\psi_{\beta, \alpha \beta} \) for all \( \alpha, \beta \in \Gamma \).

A normal band of groups \( S \) is complete if and only if each compatible subset \( H \) of \( S \) has a least upper bound, denoted by \( \vee H \). A complete normal band of groups \( S \) is infinitely distributive if and only if for each pair of compatible subsets \( H \) and \( K \) of \( S \),

\[
(\vee H)(\vee K) = \vee (HK),
\]

where \( HK = \{ hk : h \in H, k \in K \} \). It should be noted that in the special cases where \( S \in \mathcal{NB} \) or \( S \in \mathcal{PL} \mathcal{B} \), these definitions coincide with those in [2] and [10] respectively.

Remark 1.1. If \( S \) is a complete and infinitely distributive normal band of groups then \( S = S^0 \). This is because the empty set \( \emptyset \) is a compatible subset of \( H \). Since for any \( s \in S \), \( \{ s \} \) is compatible and \( s = \vee \{ s \} \) then by infinite distributiivity \( s(\vee \emptyset) = (\vee \emptyset)s = \vee \emptyset \).
Suppose $\mathcal{V} \in \mathcal{L}(\mathcal{A})$ and $I \in \mathcal{V}$. Define $I$ to be a $\mathcal{V}$-injective if and only if for each pair $S, T \in \mathcal{V}$ such that $S$ is a subsemigroup of $T$, any homomorphism from $S$ into $I$ extends to a homomorphism from $T$ into $I$.

If $S$ and $T$ are semigroups such that $\phi: T \rightarrow S$ and $\theta: S \rightarrow T$ are homomorphisms and $\theta \phi$ is the identity map on $S$, then $S$ is a retract of $T$ and $\phi$ is a retraction.

It is easy to see that a direct product of $\mathcal{V}$-injectives is a $\mathcal{V}$-injective, and every retract of a $\mathcal{V}$-injective is a $\mathcal{V}$-injective. Furthermore, any $\mathcal{V}$-injective $I$ is an absolute retract; that is, $I$ is a retract of any $S \in \mathcal{V}$ in which it is embeddable.

**Theorem 1.2** [2]. Suppose $\mathcal{V} \in \mathcal{L}(\mathcal{B})$. Then $I$ is a $\mathcal{V}$-injective if and only if one of the following is satisfied:

(i) $\mathcal{V} \subseteq \mathcal{B}$ and $I \in \mathcal{V}$;

(ii) $\mathcal{A} \subseteq \mathcal{V} \subseteq \mathcal{B}$ and $I$ is a complete infinitely distributive semilattice;

(iii) $\mathcal{V} \subseteq \mathcal{A}$, $I \in \mathcal{V}$ and $I$ is complete and infinitely distributive with retractions for its structure maps.

This theorem summarises the results of [2]. Condition (iii) is by [2, Theorem 4.4] with the observation that a homomorphism of rectangular bands is surjective if and only if it is a retraction.

Suppose $\mathcal{V} \in \mathcal{L}(\mathcal{A})$. Define the index of $\mathcal{V}$ to be the least positive integer $n$ such that $(x^{-1}yx^{-1})^n = xy(x y)^{-1}$ is a law in $\mathcal{V} \cap \mathcal{B}$; if there is no such integer, we say the index is infinite. Clearly $\mathcal{V} \cap \mathcal{B} \subseteq \mathcal{B}$ (or equivalently by [8, IV.3.2], $\mathcal{V} \subseteq \mathcal{B}$) if and only if the index of $\mathcal{V}$ is 1.

A group $G \in \mathcal{V} \cap \mathcal{G}$, $\mathcal{V} \in \mathcal{L}(\mathcal{A})$, is a $\mathcal{V}$-injective group if and only if $G$ is a $\mathcal{V} \cap \mathcal{B}$-injective such that

(i) if $\mathcal{V}$ has infinite index, then $|G| = 1$, and

(ii) if $\mathcal{V}$ has finite index $n$, then $G$ has no non-identity element of order dividing $n$.

**Theorem 1.3** (Trotter, [12, Theorem 3]). Suppose $\mathcal{V} \in \mathcal{L}(\mathcal{A})$. Then $I$ is a $\mathcal{V}$-injective if and only if $I$ is a direct product of a rectangular band in $\mathcal{V}$ by a $\mathcal{V}$-injective group.

In [1, 6] the injectives in a variety of groups are described; they are all abelian groups. Use will be made of the following corollary of the results of [1, 6]. Define the exponent of a variety of groups $\mathcal{V}$ to be the least positive integer $n$ such that $x^n = 1$ is a law in $\mathcal{V}$; if there is no such $n$ we say the exponent is infinite.

**Theorem 1.4.** Suppose $\mathcal{V} \in \mathcal{L}(\mathcal{B})$ and $G$ is a non-trivial $\mathcal{V}$-injective.

(i) If $\mathcal{V}$ has infinite exponent, then $G$ is a divisible group.

(ii) If $\mathcal{V}$ has exponent $p_1^{e_1} \cdots p_n^{e_n}$ where $p_1, \ldots, p_n$ are distinct prime numbers, then $G$ is a direct product of cyclic groups whose orders are from $\{p_1^{e_1}, \ldots, p_n^{e_n}\}$.

The next result follows easily from the definitions.
Proposition 1.5. Suppose $\mathcal{V} \in \mathcal{L}(GR)$ and $I$ is a $\mathcal{V}$-injective. Then the $\mathcal{H}$-classes and the $\mathcal{B}$-classes of $I$ are respectively $(\mathcal{V} \cap \mathcal{B})$-injectives and $(\mathcal{V} \cap \mathcal{E})$-injectives. If $I \in \mathcal{O}$, then $E(I)$ is a $(\mathcal{V} \cap \mathcal{B})$-injective.

\[ \square \]

2. Orthodox normal bands of groups

In this section it is shown that for $\mathcal{V} \in \mathcal{L}(GR)$, any $\mathcal{V}$-injective is an orthodox normal band of groups with retractions for structure maps.

Proposition 2.1. Let $\mathcal{V} \in \mathcal{L}(GR)$ and $I$ be a $\mathcal{V}$-injective. Then $I \in \mathcal{NBG} \cap \mathcal{O}$. If $\mathcal{V} \cap \mathcal{B} \subseteq \mathcal{NB}$, then $I \in \mathcal{NBG}$.

Proof. By [12, Lemma 1], a $(\mathcal{V} \cap \mathcal{E})$-injective is orthodox. So by Proposition 1.5, the $\mathcal{D}$-classes of $I$ are orthodox semigroups; therefore $I$ is orthodox by [8, IV.3.2].

Since $I \in \mathcal{O}$, it follows from Proposition 1.5 that $E(I)$ is a $(\mathcal{V} \cap \mathcal{B})$-injective. Hence by Theorem 1.2, $E(I) \in \mathcal{NB}$. It can be readily shown that any completely regular semigroup whose set of idempotents is a normal band, is a normal band of groups (in particular, this is an immediate corollary of the description of a normal band of groups in terms of idempotent properties given in [8, Exercise IV.4.9.1]). It follows that $I \in \mathcal{NBG}$.

The last statement of the proposition follows from Theorem 1.2. \[ \square \]

We may now assume for any $\mathcal{V}$-injective, $\mathcal{V} \in \mathcal{L}(GR)$, that

\[ I = [\Gamma; I_0, \psi_{\alpha, \beta}] \in \mathcal{NBG} \cap \mathcal{O}. \]  \hspace{1cm} \text{(3)}

The following is a generalisation of the well-known homomorphism theorem for semilattices of groups.

Proposition 2.2. Suppose $S, T \in \mathcal{NBG} \cap \mathcal{O}$, $S = [\Lambda; S_\alpha, \phi_{\alpha, \beta}]$ and $T = [\Omega; T_\alpha, \psi_{\alpha, \beta}]$. Let $\eta : \Lambda \rightarrow \Omega$ be a homomorphism and for each $\alpha \in \Lambda$ let $\chi_\alpha : S_\alpha \rightarrow T_{\eta \alpha}$ be a homomorphism. Assume $\chi_\alpha \psi_{\alpha, \eta \alpha} = \phi_{\alpha, \beta} \chi_\beta$ for all $\alpha, \beta \in \Lambda$ such that $\alpha \succeq \beta$. Then the map $\chi : S \rightarrow T$ given by $s \chi = s \chi_\alpha$ for $s \in S_\alpha$ is a homomorphism. Conversely, every homomorphism of $S$ into $T$ can be so constructed.

Proof. It is easy to see that $\chi$ is a homomorphism (for a detailed proof when $S, T \in \mathcal{NBG}$, see [9, Proposition II.2.8]).

Conversely, assume $\chi : S \rightarrow T$ is a homomorphism. Since $\chi$ maps $\mathcal{D}$-classes into $\mathcal{D}$-classes, it induces homomorphisms $\eta : \Lambda \rightarrow \Omega$ and $\chi_\alpha : S_\alpha \rightarrow T_{\eta \alpha}$ such that $s \chi_\alpha = s \chi$ for each $s \in S_\alpha$, $\alpha \in \Lambda$. Let $s \in S_\alpha$, $e \in E(S_\beta)$ and $\alpha \succeq \beta$. Since $S \in \mathcal{NBG} \cap \mathcal{O}$, $s \phi_{\alpha, \beta} = ses^{-1}s$. So

\[ s \chi_\alpha \psi_{\alpha, \eta \alpha} = s \chi_\alpha (\phi_{\alpha, \beta} (ses^{-1}s) \chi_\alpha) = (ses^{-1}s) \chi = s \phi_{\alpha, \beta} \chi = s \phi_{\alpha, \beta} \chi_\beta. \]  \[ \square \]
Proposition 2.3. Let $V \in \mathcal{L}(\mathcal{C})$ and $I$ be a $V$-injective. Then the structure maps of $I$ are retractions.

Proof. Let $I$ be as in (3) with $|\Gamma| > 1$; otherwise the result is trivial. Select $e \in E(I)$ and let $S = \{e\} \cup I_\beta$ be a subsemigroup of $I$ for some $\alpha \geq \beta$ in $\Gamma$. We may select a completely simple semigroup $C$ with an idempotent $e$ such that there is an isomorphism $\theta : C \to I_\beta$ with $e\theta = ew_{\alpha, \beta}$. Let $T = C \cup I_\beta$ be a normal band of groups and $\theta$ be its non-identity structure map. Clearly $S \subseteq T$ and $S$ is a subsemigroup of $T$. Since $I$ is a $V$-injective, the natural embedding $\phi : S \to I$ extends to a homomorphism $\psi : I \to I$. Since $\psi$ maps $D$-classes into $D$-classes, it maps $C$ into $I_\beta$ and maps $I_\beta$ identically onto itself. By Proposition 2.2, for any $c \in C$, $c\psi_w \alpha, \beta = c\psi_w$. But $\theta : C \to I_\beta$ is an isomorphism, so $(\theta \psi)^{-1}\psi_w \alpha, \beta$ is the identity on $I_\beta$. Thus $\psi_w \alpha, \beta$ is a retraction. □

Remark 2.4. All of Green’s relations on a normal band of groups $I$ are congruences. If $I$ is also orthodox, then the natural homomorphism $\mathcal{H}^*$ induced by $\mathcal{H}$ is a retraction onto $E(I)$. Hence by Proposition 2.1, if $I$ is a $V$-injective for $V \in \mathcal{L}(\mathcal{C})$ then, so is $E(I)$.

Proposition 2.5. Suppose there exists $V \in \mathcal{L}(\mathcal{C})$ such that $R$ is not a congruence on $V$. If $I$ is a $V$-injective, then $I \in \mathcal{R} \mathcal{A} \mathcal{R} \mathcal{G}$.

Proof. There exist $\mathcal{R}$-classes $R_1$ and $R_2$ of $V$ such that $R_1R_2$ is not contained in an $\mathcal{R}$-class. Let $S$ be the $\mathcal{D}$-class of $V$ that contains $R_1R_2$, and let $T = R_1 \cup R_2 \cup S$ be a subsemigroup of $V$. Clearly $S \subseteq T \subseteq V$. Select $r \in R_1$ and let $R$ be the $\mathcal{R}$-class in $S$ that contains $rR_2$. $R$ exists since $\mathcal{R}$ is a left congruence on $V$. Since $R_1R_2 \notin R$, there exists $s \in R_1$ such that $sR \cap R$ is empty.

Suppose $I \not\subseteq \mathcal{R} \mathcal{A} \mathcal{R} \mathcal{G}$; then $I$ contains a two-element left-zero subsemigroup $\{a, b\}$. Since $S \subseteq \mathcal{D}$, there is a homomorphism $\phi : S \to I$ given by

$$x\phi = \begin{cases} 
    a & \text{if } x \in R, \\
    b & \text{if } x \notin R.
\end{cases}$$

This map extends to a homomorphism $\psi : T \to I$ by injectivity. Because $\mathcal{R}$ is a congruence on $I$ and $r, s \in R_1$, we have that $(ry)\phi R(sy)\phi$ for any $y \in R_2$. But then $a\mathcal{R}b$ in $I$, which is a contradiction. □

3. Semilattices of groups

The aim initially is to find necessary conditions for a semilattice of groups $I \in \mathcal{V} \subseteq \mathcal{L}(\mathcal{C})$ to be a $V$-injective. It will be shown that a direct product of $V$-injective groups with adjoined zeros is a $V$-injective and that a semigroup satisfying the necessary conditions is a retract of such a direct product. So the necessary conditions are also sufficient.
For $I = [I; I_a, \psi_{a, \beta}] \in \mathcal{PDG}$ and $\alpha \geq \beta$ in $\Gamma$ let $\ker \psi_{a, \beta}$ denote the kernel of the group homomorphism $\psi_{a, \beta}$.

**Proposition 3.1.** Let $I = [I; I_a, \psi_{a, \beta}] \in \mathcal{PDG}$ be a $\mathcal{V}$-injective for $\mathcal{V} \in \{\mathcal{PD}, \mathcal{CR}\}$. Then $I$ is complete and infinitely distributive, $\psi_{a, \beta}$ is a retraction and $\ker \psi_{a, \beta}$ is a $\mathcal{V}$-injective group for each $\alpha, \beta \in \Gamma$, $\alpha \geq \beta$.

**Proof.** By [10, 1.10, 1.15 and 1.33], $I$ is embedded in a complete infinitely distributive semigroup $\mathcal{H}(I) \in \mathcal{PDG}$ and the $\mathcal{H}$-classes of $\mathcal{H}(I)$ are subgroups of direct products of $\mathcal{H}$-classes of $I$ and hence lie in $\mathcal{V} \cap \mathcal{G}$. So $\mathcal{H}(I) \in \mathcal{V}$. As a $\mathcal{V}$-injective embedded in $\mathcal{H}(I)$, $I$ is a retract of $\mathcal{H}(I)$. Hence there are homomorphisms $\tau: I \to \mathcal{H}(I)$ and $\theta: \mathcal{H}(I) \to I$ such that $\tau \theta$ is the identity map on $I$. Let $H$ be a compatible subset of $I$; so $H \tau$ is compatible in $\mathcal{H}(I)$. Since homomorphisms preserve the natural partial order on inverse semigroups, $(\vee (H \tau)) \theta$ is an upper bound of $H$. If $h \in H$, $i \in I$ and $i \geq h$, then $i \tau \geq h \tau$, so $i \tau \geq (\vee (H \tau)) \theta$, whence $i \geq (\vee (H \tau)) \theta$. Therefore, $(\vee (H \tau)) \theta = \vee H$.

We have now shown that $I$ is complete. By Proposition 1.5 and Theorem 1.2, $E(I)$ is infinitely distributive, so by [10, 1.13] $I$ is infinitely distributive.

The structure maps are retractions by Proposition 2.3.

Since $\psi_{a, a}$ is the identity map, $\ker \psi_{a, a}$ is the trivial group (a $\mathcal{V}$-injective group). Suppose $\alpha \geq \beta$ in $\Gamma$. Choose $S, T \in \mathcal{V} \cap \mathcal{G}$ such that $S$ is a subgroup of $T$ and suppose $\phi: S \to \ker \psi_{a, \beta}$ is a homomorphism. Then $S^0, T^0 \in \mathcal{V} \cap \mathcal{PDG}$ and there is a homomorphism $\phi_1: S^0 \to I$ given by

$$s \phi_1 = \begin{cases} s \phi & \text{if } s \in S, \\ e = \text{identity of } I \beta & \text{if } s = 0. \end{cases}$$

By the injectivity of $I$ there is a homomorphism $\psi_1: T^0 \to I$ that extends $\phi_1$. Then $0\psi_1 = e$ and since $T$ is an $\mathcal{H}$-class of $T^0$, $T\psi_1 \subseteq I_a$. With $\psi_{T^0, I} : T \to \{0\}$ being a structure map of $T^0$, by Proposition 2.2

$$t \psi_{T^0, I} \psi_1 = t \psi_1, \psi_{a, \beta}, \quad t \in T.$$ 

But $t \psi_{T^0, I} = 0$ so $t \psi_1 \in \ker \psi_{a, \beta}$ for all $t \in T$. Thus there is a homomorphism $\psi : T \to \ker \psi_{a, \beta}$ such that $t \psi = t \psi_1$ for all $t \in T$ and $\psi$ extends $\phi$. Therefore $\ker \psi_{a, \beta}$ is a $\mathcal{V} \cap \mathcal{G}$-injective. But by Proposition 1.5 and Theorem 1.3, $I_a$ is a $\mathcal{V}$-injective group; since the $\mathcal{V} \cap \mathcal{G}$-injective $\ker \psi_{a, \beta}$ is a subgroup of $I_a$, it is also a $\mathcal{V}$-injective group. \qed

It should be noticed that the conditions of Proposition 3.1 are necessary and sufficient if $\mathcal{V} \subseteq \mathcal{PD}$ (by Theorem 1.2) or if $\mathcal{V}$ is the variety of semilattices of abelian groups (by [11, Theorem 2]).

A partially ordered set $\Gamma$ is **directed** if and only if each pair of elements of $\Gamma$ has an upper bound in $\Gamma$. Suppose $I = [I; I_a, \psi_{a, \beta}] \in \mathcal{PDG}$ where $\Gamma$ is a directed semilattice. The family of groups $I_a$, indexed by elements of $\Gamma$, along with the structure
Injectives in varieties 295

maps form an inverse family of groups (see [3, §21]). The inverse limit of the family is the subgroup of the direct product $\prod \{I_\alpha : \alpha \in \Gamma \}$ given by

$$\text{inv lim} \left[ \Gamma; I_\alpha, \psi_{\alpha, \beta} \right] = \{ p \in \prod \{ I_\alpha : \alpha \in \Gamma \} : p(\alpha)\psi_{\alpha, \beta} = p(\beta) \text{ if } \alpha \geq \beta \}. \quad (4)$$

Of course, $p(\alpha)$ denotes the component of $p$ in $I_\alpha$.

Let $\alpha$ denote the principal ideal of $\Gamma$ generated by $\alpha$. Define

$$\bar{\Gamma} = \{ \alpha : \alpha \in \Gamma \}. \quad (5)$$

Under set intersection, $\bar{\Gamma} \in \mathcal{P}\mathcal{D}$ and $\bar{\Gamma} \cong \Gamma$. Define

$$I_\alpha = \text{inv lim} \left[ \alpha : I_\beta, \psi_{\beta, \gamma} \right], \quad \alpha \in \Gamma. \quad (6)$$

For $\alpha \geq \beta$, define $\psi_{\alpha, \beta}: I_\alpha \rightarrow I_\beta$ to be the restriction to $I_\alpha$ of the projection from $\prod \{ I_\gamma : \gamma \in \alpha \}$ onto $\prod \{ I_\gamma : \gamma \in \beta \}$. Then $\psi_{\alpha, \beta}$ is a homomorphism, $\psi_{\alpha, \alpha}$ is the identity map on $I_\alpha$, and $\psi_{\alpha, \beta} \psi_{\beta, \gamma} = \psi_{\alpha, \gamma}$ for $\alpha \geq \beta \geq \gamma$. By [8, III.7.7] we may define

$$\bar{I} = [\bar{\Gamma} : I_\alpha, \psi_{\alpha, \beta}] \in \mathcal{P}\mathcal{P}\mathcal{G}. \quad (7)$$

**Proposition 3.2.** Let $I = [I^I : I_\alpha, \psi_{\alpha, \beta}] \in \mathcal{P}\mathcal{P}\mathcal{G}$ and $I$ be a directed set. Then $I \cong \bar{I}$.

**Proof.** Define a homomorphism $\psi_\alpha : I_\alpha \rightarrow I_\bar{\alpha}$ for each $\alpha \in \Gamma$ by

$$a\psi_\alpha(\beta) = a\psi_{\alpha, \beta}, \quad \beta \leq \alpha, \quad a \in I_\alpha.$$ 

If $f \in I_\alpha$, then $f(\alpha) \in I_\alpha$ and $f(\alpha)\psi_{\alpha, \beta} = f(\beta)$ so $f(\alpha)\psi_\alpha = f$; thus $\psi_\alpha$ is surjective. If $a, b \in I_\alpha$ and $a\psi_\alpha = b\psi_\alpha$, then $a = a\psi_\alpha(a) = b\psi_\alpha(a) = b$; so $\psi_\alpha$ is injective. Define a map $\psi : I \rightarrow \bar{I}$ by $a\psi = a\psi_\alpha$ for each $\alpha \in \Gamma$ and $a \in I_\alpha$. So $\psi$ is bijective. Since

$$a\psi_\alpha \psi_{\alpha, \beta}(\gamma) = a\psi_\alpha(\gamma) = a\psi_{\alpha, \gamma} = a\psi_{\alpha, \beta} \psi_{\beta, \gamma} = a\psi_{\alpha, \beta} \psi_\beta(\gamma),$$

$$\alpha \geq \beta \geq \gamma, \quad a \in I_\alpha,$$

then by Proposition 2.2, $\psi$ is a homomorphism. Thus $\psi$ is an isomorphism. \[\square\]

**Corollary 3.3.** If $S \subseteq \mathcal{Y} \subseteq \mathcal{P}\mathcal{P}\mathcal{G}$, then $S$ is embeddable in a complete infinitely distributive semigroup $T \subseteq \mathcal{Y}$ that has retractions for its structure maps.

**Proof.** By [10; 1.10, 1.15 and 1.33], $S$ is embeddable in a complete infinitely distributive semigroup $C \subseteq \mathcal{Y}$. Let $C = [\Delta : C_\alpha, \phi_{\alpha, \beta}]$. Since $\Delta$ is complete, it is directed, so by Proposition 3.2, $C \cong \bar{C}$. By [10, 1.18], the set $\mathcal{A}$ of all ideals of $\Delta$ is a complete infinitely distributive semilattice under intersection. Let $D_\Omega = [\{ C_\alpha : \alpha \in \Omega \}$ for $\Omega \in \mathcal{A}$ and $\phi_{\Omega, \alpha} : D_\Omega \rightarrow D_\alpha$ be a projection for $\Omega \supseteq \alpha$. Then $D = [\mathcal{A} : D_\Omega, \phi_{\Omega, \alpha}] \in \mathcal{P}\mathcal{P}\mathcal{G} \cap \mathcal{Y}$ and by (4)-(7), $D$ embeds $\bar{C}$. It is easy to see that $D$ is complete, $D$ is infinitely distributive by [10, 1.13] and the structure maps of $D$ are retractions. \[\square\]
The following is the converse of Proposition 3.1 for semilattices of groups with just two \( \mathcal{Q} \)-classes:

**Proposition 3.4.** Let \( \mathcal{V} \in \mathcal{FL, CR} \) and \( G \) be a \( \mathcal{V} \)-injective group. Then \( G^0 \) is a \( \mathcal{V} \)-injective.

**Proof.** Suppose \( S, T \in \mathcal{V} \), \( S \) is a subsemigroup of \( T \) and \( \phi : S \rightarrow G^0 \) is a homomorphism. It must be shown that there is a homomorphism \( \psi : T \rightarrow G^0 \) that extends \( \phi \). By (1),

\[
S = \bigcup \{ S_\alpha \in \mathcal{V} \cap \mathcal{EP}: \alpha \in \Lambda \}, \quad T = \bigcup \{ T_\alpha \in \mathcal{V} \cap \mathcal{EP}: \alpha \in \Omega \}
\]

where \( \Lambda \) is a subsemilattice of \( \Omega \) and \( S_\alpha \) is a subsemigroup of \( T_\alpha \) for each \( \alpha \in \Lambda \). Let \( \phi_\alpha \) denote the restriction of \( \phi \) to \( S_\alpha \). Let

\[
\Delta = \{ \alpha \in \Lambda: S_\alpha \phi = \{ 0 \} \}, \quad S_\Delta = \bigcup \{ S_\alpha : \alpha \in \Delta \}.
\]

Then \( S_\Delta \) is an ideal of \( S \), and \( S \setminus S_\Delta \) is a completely regular subsemigroup of \( S \).

The following hierarchy of cases will be considered:

(i) \( S \) and \( T \) are as above;
(ii) \( S = \{ \alpha \}; S_\alpha, \phi_{\alpha, \beta} \subseteq T = \{ \alpha \}; T_\alpha, \psi_{\alpha, \beta} \in \mathcal{FLG} \);
(iii) \( S \) and \( T \) satisfy (ii), \( S \) and \( T \) are complete and infinitely distributive and \( T \) has retractions for structure maps;
(iv) \( S \) satisfies (ii), \( T \) satisfies (iii) and \( \Lambda = \Omega \);
(v) \( S \) and \( T \) satisfy (i) and \( \psi_\alpha = T_\alpha \) for all \( \alpha \in \Delta \);
(vi) \( S \) and \( T \) satisfy (v) and

\[
\{ t \in T: t \equiv a \text{ for some } a \in S \setminus S_\Delta \} \subseteq S.
\]

We proceed by showing that if \( \psi \) can be chosen whenever \( S \) and \( T \) satisfy (j), then \( \psi \) can be chosen when \( S \) and \( T \) satisfy (j - 1), \( j \leq j \leq vi \). The proof will be completed by showing that \( \psi \) can be chosen in case (vi).

**Case (i).** Let \( \tau \) and \( \sigma_\alpha \) denote respectively the least inverse congruence on \( T \) and the left group congruence on \( T_\alpha \). Define

\[
\tau' = \tau \cap (S \times S), \quad \sigma'_\alpha = \sigma_\alpha \cap (S \times S), \quad \alpha \in \Lambda.
\]

By [5], \( \tau = \bigcup \{ \sigma_\alpha: \alpha \in \Omega \} \) so \( \tau' = \bigcup \{ \sigma'_\alpha: \alpha \in \Lambda \} \). Since \( G \) and \( \{ 0 \} \) are \( \mathcal{V} \cap \mathcal{EP} \)-injectives by Theorem 1.3, there is a homomorphism \( \psi_\alpha : T_\alpha \rightarrow G^0 \) such that \( T_\alpha \psi_\alpha \subseteq G \) if \( \alpha \in \Lambda \setminus \Delta \) and \( T_\alpha \psi_\alpha = \{ 0 \} \) if \( \alpha \in \Delta \), and such that \( \psi_\alpha \) extends \( \phi_\alpha \). It follows that \( \psi_\alpha \circ \psi_\alpha^{-1} \geq \sigma_\alpha \), so \( \phi_\alpha \circ \phi_\alpha^{-1} \geq \sigma'_\alpha \), whence \( \psi \circ \phi^{-1} \geq \tau' \). Thus \( \psi = (\tau')^\# \phi_1 \) for some homomorphism \( \phi_1 : S/\tau' \rightarrow G^0 \). Since we may identify \( S/\tau' \) with a subsemigroup of \( T/\tau \in \mathcal{FLG} \), if we can extend \( \phi_1 \) to a homomorphism \( \psi_1 : T/\tau \rightarrow G^0 \), then \( \psi = \tau \# \psi_1 : T \rightarrow G^0 \) extends \( \phi \).

**Case (ii).** If \( H \) is a compatible subset of \( S \setminus S_\Delta \) and \( a \in H \setminus S_\beta \), \( b \in H \setminus S_\beta \), then \( a \phi, a \beta = b \phi, a \beta \in S \setminus S_\Delta \). Since the structure maps for \( G^0 \) are either trivial or identity maps, by Proposition 2.2 \( a \phi = a \phi, a \beta \phi = b \phi, a \beta \phi = b \phi \). Thus

\[
a \phi = b \phi, \quad a, b \in H, \quad H \subseteq S \setminus S_\Delta \text{ is compatible.}
\]
Since $T \in V \cap \mathcal{PL}$, by Corollary 3.3 $T$ is embeddable in a complete infinitely distributive semigroup $W \in V \cap \mathcal{PL}$ where the structure maps of $W$ are retractions. For $v \in W$ note that $\{a \in S : a \leq v\}$ is a compatible subset of $W$ and of $S$. Let
\[ V = \{v \in W : v = \bigvee \{a \in S : a \leq v\}\}. \tag{9}\]
Since $W$ is infinitely distributive, and $v \in V$ implies $v^{-1} \in V$, it follows that $V$ is an inverse subsemigroup of $W$. Clearly $V$ is complete and infinitely distributive and embeds $S$. By (8), there is a map $\theta : V \rightarrow G^0$ given by
\[ \nu \theta = \begin{cases} a \phi, & \nu \geq a \text{ for some } a \in S \setminus S_\Delta, \\ 0, & \text{otherwise}. \end{cases} \]
We have $\nu \theta = 0$ if and only if $\{a \in S : a \leq \nu\} \subseteq S_\Delta$. Since $S_\Delta$ is an ideal and $S \setminus S_\Delta$ is a subsemigroup of $S$, it follows by (8), (9) and infinite distributivity that $\theta$ is a homomorphism that extends $\phi$. Clearly $V$ and $W$ satisfy (iii) and if $\theta$ extends to a homomorphism $\theta' : W \rightarrow G^0$, then the restriction $\psi$ of $\theta'$ to $T$ extends $\phi$.

Case (iii). $T \in \mathcal{PL}$, so for $t \in T_a$ and $e \in E(T_b)$, $\alpha, \beta \in \Omega$, we have $te = t\psi_{\alpha, \beta} = t\psi_{\alpha, \beta} = et$. Hence
\[ A = E(T) \cup E(T)S \]
is an inverse subsemigroup of $T$. By Remark 1.1, $T = T^0$ and $S = S^0$. We may assume that $S$ and $T$ have a common zero and $0\phi = 0$. Since $S$ is complete, we may define
\[ \alpha' = \bigvee \{ \gamma \in A : \gamma \leq \alpha \} \in A, \quad \alpha \in \Omega. \]
Note that if $\alpha, \beta \in \Omega$, then $\alpha' \geq (\alpha \beta)'$, but $\alpha \beta \geq \alpha' \beta' \in \Lambda$ so $\alpha' \beta' = (\alpha \beta)'$. If $f \in E(T_a)$, then \[ f\psi_{\alpha, \alpha'} = \bigvee \{a \in S : a \leq f\} \in E(S). \]
Also if $f \in E(T)$, $s \in S$ and $fs \in T_a$, then $(fs)\psi_{s, \alpha} = \bigvee \{a \in S : a \leq fs\}$. Define a map $\theta : A \rightarrow G^0$ by
\[ a \theta = a\psi_{\alpha, \alpha'} \phi, \quad a \in A \cap T_a. \]
Since $\gamma \delta' = (\gamma \delta)'$ for any $\gamma, \delta \in \Omega$, it follows that $\gamma', \delta' \in A \setminus A$ if and only if $(\gamma \delta)' \in A \setminus A$. So for $u \in A \cap T_y$, $v \in A \cap T_\delta$ we have
\[(uv)\theta = (uv)\psi_{\gamma \delta', (\gamma \delta)} \phi = (uv)\psi_{\gamma \delta', (\gamma \delta)}(uv\psi_{\delta', (\gamma \delta)}) \phi = u\theta(v\theta).\]
So $\theta$ is a homomorphism extending $\phi$. Our problem is now to extend $\theta$ to $\psi$ where $A$ and $T$ satisfy (iv).

Case (iv). The set $B = \bigcup \{T_a : \alpha \in A\} \cup S$ is an inverse subsemigroup of $T$ with ideal $J = \bigcup \{T_a : \alpha \in A\}$. The homomorphism $\theta : B \rightarrow G^0$ given by $J\theta = \{0\}$ and $s\theta = s\phi$ for $s \in S$ extends $\phi$. We must now extend $\theta$ to $\psi$; $B$ and $T$ satisfy (v).

Case (v). Let
\[ C = S \cup \{t \in T : t \geq a \text{ for some } a \in S \setminus S_\Delta\}. \]
If $u, v \in T$ where $u \geq a$, $v \geq b$ for some $a, b \in S \setminus S_\Delta$, then $uv \geq ab \in S \setminus S_\Delta$. Furthermore, $u^{-1} \geq a^{-1} \in S \setminus S_\Delta$. Since $\bigcup \{T_a : \alpha \in A\}$ is an ideal of $T$, $C$ is an inverse sub-
semigroup of $T$. By (8), there is a map $\theta : C \to G^0$ given by

$$v\theta = \begin{cases} \alpha \phi, & v \geq \alpha \text{ for some } \alpha \in S \setminus S_A, \\ 0, & \text{otherwise}. \end{cases}$$

Since $v\theta = 0$ if and only if $v \in S_A$, and $S \setminus S_A$ is a semigroup, it follows that $\theta$ is a homomorphism extending $\phi$. So we must extend $\theta$ to $\psi : C$ and $T$ satisfy (vi).

Case (vi). Let $\eta = \sqrt{A}$. We may assume $S_{\eta} \phi_{\eta} \subseteq G$; otherwise $S\phi = \{0\}$ and we may choose $\psi$ such that $T\psi = \{0\}$. There is a homomorphism $\eta : T_0 \to G$ extending $\phi_\eta$ since $G$ is a $(\mathcal{V} \cap \mathcal{E})$-injective.

Case (vi). Let $a \in T_0$ and $a\psi_{\eta, a} = b\psi_{\eta, a}$ for some $\alpha \in A \setminus \Delta$. Then by (iv), $(ab^{-1})\psi_{\eta, a} \in E(T) = E(S)$, so $ab^{-1} \in S$ by property (vi). By (8), $(ab^{-1})\psi_{\eta, a} \phi_{\eta} = (ab^{-1})\phi_{\eta} = (ab^{-1})\psi_{\eta} \in G$, so $(ab^{-1})\psi_{\eta}$ is the identity of $G$. Thus $a\psi_{\eta} = b\psi_{\eta}$. Since the structure maps of $T$ are surjective we can now define a homomorphism $\eta : T_0 \to G^0$ by $\psi_{\eta, a} \phi_{\eta} = \psi_{\eta}$ if $\alpha \in A \setminus \Delta$ and $\psi_{\eta} = \phi_{\eta}$ if $\alpha \in \Delta$. Then for $\alpha \in A \setminus \Delta$, by Proposition 2.2, $\eta_0 \phi_{\eta} = \phi_{\eta}$. By property (vi), $\eta_0 \phi_{\eta}$ is a retraction, so $\psi_\eta$ extends $\phi_\eta$. Furthermore, for $\alpha \geq \beta$ in $A \setminus \Delta$ and $a \in T_0$, 

$$(a\psi_{\eta, a})\psi_{\eta, \beta} = a\psi_{\eta, \beta} \psi_{\eta} = a\psi_{\eta} = (a\psi_{\eta, \alpha})\psi_\eta.$$

If $\beta \in \Delta$, then $a\psi_{\eta, a} \psi_{\eta} \psi_{\eta} = 0$. Therefore, by Proposition 2.2 with $\psi : T \to G^0$ given by $t\psi = t\psi_0$ if $t \in T_0$, $\psi$ is a homomorphism. Clearly $\psi$ extends $\phi$. 

Now suppose $I = [\Gamma; I_\alpha, \psi_{\alpha, \beta}] \in \mathcal{P}_G\mathcal{B}$ is complete and infinitely distributive, $\psi_{\alpha, \beta}$ is a retraction and ker $\psi_{\alpha, \beta}$ is a $\mathcal{V}$-injective group for each $\alpha, \beta \in \Gamma$, $\alpha \geq \beta$. If $\varepsilon = \bigwedge \Gamma$, then $I_\varepsilon = \ker \psi_{\varepsilon, \varepsilon}$ so $I_\varepsilon$ is a $\mathcal{V}$-injective group. By Proposition 3.4, the direct product 

$$I = \bigoplus \{I_\alpha : \alpha \in \Gamma \} \in \mathcal{P}_G\mathcal{B}$$

is a $\mathcal{V}$-injective. Let $0_{\alpha}$ denote the $\mathcal{D}$-class of $I_\alpha^0$ containing only the zero element. Then a typical $\mathcal{D}$-class of $I$ is 

$$I_{\alpha} = \bigoplus \{I_\alpha : \alpha \in \Omega \} \times \bigoplus \{0_{\alpha} : \alpha \in \Gamma \setminus \Omega \}, \quad \Omega \subseteq \Gamma.$$  

The structure maps are maps $\phi_{\Omega, \Lambda} : I_{\Omega} \to I_{\Lambda}, \Lambda \subseteq \Omega$, such that 

$$a\phi_{\Omega, \Lambda}(\alpha) = \begin{cases} a(\alpha) \in I_\alpha, & \alpha \in \Lambda, \\ 0(\alpha) \in 0_{\alpha}, & \alpha \in \Gamma \setminus \Lambda. \end{cases}$$

By Proposition 3.2, $I \cong \tilde{I}$. Since $\tilde{I}$ has projections for structure maps, it follows from (4)-(7) that the map 

$$\lambda : I \to \tilde{I}, \quad p \in I_\alpha \Rightarrow p\lambda \in \tilde{I}_\alpha, \quad p\lambda(\beta) = p\psi_{\alpha, \beta}$$

is a monomorphism embedding $I$ in $\tilde{I}$.

**Corollary 3.5.** Let $\mathcal{V} \in [\mathcal{P}, \mathcal{E}]$ and $I \in \mathcal{V} \cap \mathcal{P}_G\mathcal{B}$. Then $I$ is a $\mathcal{V}$-injective if and only if $I$ is a retract of a direct product $\bigoplus \{I_\alpha^0 : \alpha \in \Gamma \}$ where $I_\alpha$ is a $\mathcal{V}$-injective group for each $\alpha \in \Gamma$. 

Injectives in varieties

Proof. \( I^0 \) is a \( \mathcal{V} \)-injective by Proposition 3.4, so \( \prod \{ I^0_a : a \in \Gamma \} \) and its retracts are \( \mathcal{V} \)-injectives.

Conversely, suppose that \( I = [\Gamma; I_a, \psi_{a, \beta}] \) is a \( \mathcal{V} \)-injective. By Proposition 3.1 and (13), \( I \) is embedded in the \( \mathcal{V} \)-injective \( \hat{I} \). Since a \( \mathcal{V} \)-injective is an absolute retract, \( I \) is a retract of \( \hat{I} \). \( \square \)

Some information on injectives in varieties of groups is needed.

**Lemma 3.6.** Suppose \( \mathcal{V} \in \mathcal{L}(\mathfrak{G}) \), \( G \) and \( K \) are \( \mathcal{V} \)-injectives, \( K' \) is a subgroup of \( K \) and \( \psi : K' \to G \) is a homomorphism. Suppose for some integer \( n \) and \( k \in K \) that \( k^n \in K' \) and \( k^n \psi = g \). Then there exists \( f \in G \) such that \( g = f^n \).

**Proof.** If \( \mathcal{V} \) has infinite exponent, then \( G \) is divisible by Theorem 1.4, so \( f \) exists. Therefore assume \( \mathcal{V} \) has exponent \( p_1^{e_1} \cdots p_m^{e_m} \) for distinct prime numbers \( p_1, \ldots, p_m \). By Theorem 1.4, we may assume that \( G = \prod_{i=1}^{m} G_i^{S_i} \) and \( K = \prod_{i=1}^{m} G_i^{T_i} \) where \( G_i \) is a cyclic group of order \( p_i^{e_i} \) and \( S_i, T_i \) are sets, \( 1 \leq i \leq m \). Let \( g = (g_1, \ldots, g_m) \) and \( k = (k_1, \ldots, k_m) \) where \( g_i \in G_i^{S_i} \) and \( k_i \in G_i^{T_i} \). Let \( p_i^{S_i} \) be the greatest power of \( p_i \) dividing \( n \). Since \( g = k^n \psi \), \( k_i^n \) and \( g_i \) have order dividing \( p_i^{S_i-\delta_i} \). Let \( x_i \) be a generator of \( G_i^{S_i} \); it has order \( p_i^{S_i} \) so \( g_i = x_i^{q_i} p_i^{S_i} \), for some integer \( q_i \). Let \( n_i = n/p_i^{k_i} \). Since \( n_i \) is relatively prime to \( p_i^{S_i} \), there exist integers \( r_i, s_i \) such that \( r_i p_i^{S_i} + s_i n_i = 1 \), so \( g_i = g_i^{S_i n_i} = x_i^{S_i q_i n_i} \). We may choose \( f = (x_1^{q_1}, \ldots, x_m^{q_m}) \). \( \square \)

The converse of Proposition 3.1 can now be proved.

**Theorem 3.7.** Let \( \mathcal{V} \in [\mathcal{L}, \mathcal{R}] \) and \( I = [\Gamma; I_a, \psi_{a, \beta}] \in \mathcal{L}(\mathfrak{G}) \). Then \( I \) is a \( \mathcal{V} \)-injective if and only if \( I \) is complete and infinitely distributive, \( \psi_{a, \beta} \) is a retraction and \( \ker \psi_{a, \beta} \) is a \( \mathcal{V} \)-injective group for each \( a, \beta \in \Gamma \), \( \alpha \geq \beta \).

**Proof.** The necessity of the conditions follows from Proposition 3.1. So assume the conditions. We will prove that \( I \) is a retract of \( \hat{I} \) as defined in (10). The result will then be a consequence of Corollary 3.5. Recall that by (13), \( \lambda \) embeds \( I \) in \( \hat{I} \). Also note that by Theorem 1.4, \( \hat{I} \) and \( I \) are semilattices of abelian groups; that is, they are commutative.

If \( a \in \hat{I} \), then by (11), \( a \in I_\Omega \) for some \( \Omega \subseteq \Gamma \). Recall that \( \bar{a} \) denotes the ideal of \( \Gamma \) generated by \( a \in \Gamma \). Let \( \varepsilon = \wedge \Gamma \) and define

\[
\bar{a} = \bigvee \{ \alpha : \bar{a} \subseteq \Omega \cup \{ \varepsilon \} \} \subseteq \Gamma, \quad u \in I_\Omega.
\]

Define \((S, \theta_S)\) to be a retract pair modulo \( I \) if and only if \( S \) is a completely regular subsemigroup of \( \hat{I} \), \( S \supseteq \hat{I} \) and \( \theta_S : S \to I \) is a retraction such that \( \lambda \theta_S \) is the identity map on \( I \), and \( s \theta_S \in I_\Omega \) for each \( s \in S \). Let \( \mathcal{A} \) be the set of all retract pairs \((S, \theta_S)\) modulo \( I \). The aim is to find a retract pair \((\hat{I}, \theta)\) modulo \( I \). We may partially order \( \mathcal{A} \) by \((S, \theta_S) \preceq (R, \theta_R)\) if and only if \( S \subseteq R \) and \( \theta_R \) extends \( \theta_S \).

Let \( \mathcal{C} \) be a chain in \( \mathcal{A} \). Define \( A = \bigcup \{ S : (S, \theta_S) \in \mathcal{C} \} \) and \( \theta_A : A \to I \) by \( a \theta_A = a \theta_S \)
whenever \( a \in S \) and \((S, \theta_S) \in \mathcal{C}\) and is an upper bound for \( \mathcal{C} \). By Zorn’s lemma, \( \mathcal{A} \) has a maximal element \((M, \theta_M)\).

Let \( I_\alpha \) and \( I_\Omega \) respectively denote the identities of \( I_\alpha \) and \( I_\Omega \) for some \( \alpha \in \Gamma \) and \( \Omega \subseteq \Gamma \). We first check that \( I_\Omega \in M \). Let \( N = \langle M \cup \{I_\Omega\} \rangle \) be the least completely regular subsemigroup of \( I \) that contains \( M \cup \{I_\Omega\} \). Let \( I_\Omega = \delta \) and define \( \theta_N : N \rightarrow I \) by

\[
\theta_N = \theta_M, \quad \theta_N = \delta, \quad (I_\Omega u) \theta_N = \delta(u \theta_M), \quad u \in M.
\]

If \( I_\Omega \in M \), then \( \theta_M = \theta_N \). So assume \( I_\Omega \not\in M \). We must show \( \theta_N \) is a map. Note that \( I \) is commutative and if \( I_\Omega u = v \) for \( u, v \in M \), then \( I_\Omega u = I_\Omega v \). If \( I_\Omega u = I_\Omega v \), then for \( a \subseteq \Omega \), \( I_a u = I_a v \), so by infinite distributivity in \( I \), \( I_\delta(u \theta_M) = \bigvee \{(I_a u) \theta_M : a \subseteq \Omega \cup \{\varepsilon\}\} = I_\delta(v \theta_M) \). Hence \( \theta_N \) is a map; it is clearly a homomorphism. It follows easily that \((M, \theta_M) \leq (N, \theta_N) \in \mathcal{A} \), and by the maximality of \((M, \theta_M)\), \( N = M \). Hence \( I_\Omega \in M \) for all \( \Omega \subseteq \Gamma \).

Now suppose \( k \in I_\Omega \setminus M \) and \( N = \langle M \cup \{k\} \rangle \). Since \( I \) is commutative and \( k^0 \in M \), the elements of \( N \) are of the forms \( u \) or \( k^n u \) where \( u \in M \) and \( n \) is an integer. Assume that there is no integer \( r \neq 0 \) and \( \Lambda \subseteq \Omega \) such that \( k^r \phi_{\Omega A} \in M \) while \( k \phi_{\Omega A} \not\in M \). For some \( \delta \) let

\[
a = \bigvee \{k \phi_{\Omega A} \theta_M : k \phi_{\Omega A} \in M, \Lambda \subseteq \Omega \} \subseteq I_\delta.
\]

Since \( \psi_{k, \delta} \) is a retraction, there is a \( b \in I_k \) such that \( b \psi_{k, \delta} = a \). Define \( \theta_N : N \rightarrow I \) by

\[
\theta_N = \theta_M, \quad (k^n u) \theta_N = b^n(u \theta_M), \quad u \in M.
\]

If \( k^n u = k^m v \in I_A \) for some \( u, v \in M \), then \( k^{n-m} \phi_{\Omega A} = k^{n-m} u^0 \in M \). Hence either \( k \phi_{\Omega A} \in M \) or \( n = m \). If \( k \phi_{\Omega A} \in M \), then \( k^n u = k^m v \in M \) and \( \delta \geq \gamma \) where we set \( \gamma = k^n u \). Then

\[
(k^n u) \theta_N = b^n(u \theta_M) = b^n \psi_{k, \gamma}(u \theta_M) = a^n \psi_{k, \gamma}(u \theta_M) = (k^n \phi_{\Omega A} \theta_M) \theta_M = (k^n u) \theta_M.
\]

Likewise \((k^n v) \theta_N = (k^m u) \theta_N = (k^n u) \theta_N\). Alternatively, if \( n = m \), then \( k^0 u = k^0 v \) so \( b^0(u \theta_M) = (k^0 u) \theta_M = (k^0 u) \theta_M = b^0(v \theta_M) \) whence \( b^0(u \theta_M) - b^n(v \theta_M) \) and \( (k^n u) \theta_N = (k^m v) \theta_N \). Hence \( \theta_N \) is a map. It is clear that \( \theta_N \) is a homomorphism and \((M, \theta_M) \leq (N, \theta_N) \in \mathcal{A} \). This contradicts the maximality of \((M, \theta_M)\).

Therefore assume that there is a least integer \( r > 1 \) such that for some \( \Lambda \subseteq \Omega \), \( k^r \phi_{\Omega A} \in M \) and \( k \phi_{\Omega A} \not\in M \). We may also assume that \( A = \Omega \); otherwise replace \( k \) by \( k \phi_{\Omega A} \). If also \( k^s \in M \) and \( q \) is the greatest common divisor of \( r \) and \( s \), then there exist integers \( x, y \) such that \( xr + ys = q \). Then \( k^q \in M \), and since \( r \) is least with this property, \( r = q \) and \( r \) divides \( s \). Let \( k^r \theta_M = c \). With \( a \) and \( \delta \) given by (14) and since \( \psi_{k, \delta} \) is a retraction, there exists \( d \in I_k \) such that \( d \psi_{k, \delta} = a \). So \( c = d'g \) for some \( g \in \ker \psi_{k, \delta} \). Since \( I_k \) is an abelian group and \( \psi_{k, \delta} \) is a retraction, \( I_k \cong I_\delta \times \ker \psi_{k, \delta} \) so there is a projection \( \tau : I_k \rightarrow \ker \psi_{k, \delta} \) such that \( k^r \theta_M \tau = g \). But \( \ker \psi_{k, \delta} \) is a \( \mathcal{V} \)-injective group and therefore a \( \mathcal{V} \cap \mathcal{W} \)-injective, so by Lemma 3.6 there exists \( h \in \ker \psi_{k, \delta} \) such that \( g = h^r \). So let \( b = dh \). Define \( \theta_N : N \rightarrow I \) by

\[
\theta_N = \theta_M, \quad (k^n u) \theta_N = b^n(u \theta_M), \quad u \in M.
\]
If $k^n u = k^m v \in I_\lambda$ for some $u, v \in M$, then, as above, $k^{n-m} \phi_{\Omega \lambda} \in M$. Either $k \phi_{\Omega \lambda} \in M$ or $r$ divides $n - m$. If $k \phi_{\Omega \lambda} \in M$, then, as in the last paragraph, $(k^n u) \theta_N = (k^m v) \theta_N$. If $r$ divides $n - m$, we have $k^{n-m} u = k^0 v$ so $b^{n-m}(u \theta_M) = (k^{n-m} u) \theta_M = (k^0 v) \theta_M = b^0 (v \theta_M)$. Hence $(k^n u) \theta_N = b^n (u \theta_M) = b^m (v \theta_M) = (k^m v) \theta_N$. So $\theta_N$ is a map. It is also a homomorphism and again we get $(M, \theta_N) \preceq (N, \theta_N) \in \mathcal{A}$, which is a contradiction. So we must have $M \leq I$.

4. Normal bands of groups

By Theorem 1.3, $\mathcal{V}$-injectives are known when $\mathcal{V} \in \mathcal{L}(\mathcal{G})$. The next result characterizes $\mathcal{V}$-injectives when $\mathcal{V} \in \mathcal{L}(\mathcal{B}) \setminus \mathcal{L}(\mathcal{P})$.

**Theorem 4.1.** Suppose $\mathcal{V} \in \mathcal{L}(\mathcal{B}, \mathcal{G})$. Then $I$ is a $\mathcal{V}$-injective if and only if

(i) $I = \{ [I_\alpha, \psi_{\alpha, \beta}] \in \mathcal{V} \cap \mathcal{L} \cap \mathcal{ABB} : \text{is complete and infinitely distributive, the structure maps are retractions and } \ker (\psi_{\alpha, \beta} |_{H_\alpha}) \text{ is a } \mathcal{V}$-injective group for each $\alpha, \beta \in \Gamma$, $\alpha \geq \beta$, and any $\mathcal{H}$-class $H_\alpha$ of $I_\alpha$;

(ii) either $I \in \mathcal{P} \mathcal{D} \mathcal{G}$ or $\mathcal{V} \setminus \{ \mathcal{L} \mathcal{A} \mathcal{B}, \mathcal{R} \mathcal{A} \mathcal{B}, \mathcal{A} \mathcal{B} \}$; and

(iii) if $\mathcal{B}$ or $\mathcal{R}$ is not a congruence on some member of $\mathcal{V}$, then $I \in \mathcal{L} \mathcal{A} \mathcal{B} \mathcal{B}$ or $I \in \mathcal{R} \mathcal{A} \mathcal{B} \mathcal{B}$ respectively.

**Proof.** Suppose $I$ is a $\mathcal{V}$-injective. By Proposition 2.1, $I \in \mathcal{V} \cap \mathcal{L} \cap \mathcal{ABB}$ and (ii) holds. By Propositions 2.3 and 2.5, the structure maps are retractions and (iii) holds. By Proposition 1.5 and Theorem 1.2, $E(I)$ is complete and infinitely distributive.

Let $\eta = \bigvee \Gamma$ and $I = \{ [I_\alpha, \psi_{\alpha, \beta}] \in \mathcal{V} \cap \mathcal{L} \cap \mathcal{ABB} : \text{ is complete and infinitely distributive, the structure maps are retractions and } \ker (\psi_{\alpha, \beta} |_{H_\alpha}) \text{ is a } \mathcal{V}$-injective group for each $\alpha, \beta \in \Gamma$, $\alpha \geq \beta$, and any $\mathcal{H}$-class $H_\alpha$ of $I_\alpha$;

Let $e \in E(I_\eta)$ and for $\alpha \in \Gamma$, define

$$H = e e \mathcal{L} e, \quad H_\alpha = H \cap I_\alpha = e e \mathcal{L} e, \quad \alpha \in \Gamma.$$ (16)

Clearly $H_\alpha$ is an $\mathcal{H}$-class of $I_\alpha$ and $I$, $H = \{ [I; H_\alpha, \psi_{\alpha, \beta} |_{H_\alpha}] \in \mathcal{V} \cap \mathcal{L} \mathcal{G} : \text{ the structure maps are surjections } H_\alpha \psi_{\alpha, \beta} = H_\beta \}$. By (15), the map

$$\theta : I \rightarrow H, \quad a \theta = e e a e, \quad a \in I,$$ (17)

is a retraction. Hence $H$ is a $\mathcal{V}$-injective. By Theorem 3.7, $\ker (\psi_{\alpha, \beta} |_{H_\alpha})$ is a $\mathcal{V}$-injective group and $H$ is complete.

Let $P$ be a compatible subset of $I$ and $E = \{ pp^{-1} : p \in P \}$, then $E$ is compatible in $E(I)$, and since $E(I)$ is complete, $\bigvee E$ exists. Let $f = \bigvee E \in E(I) \cap I_\alpha$ for some $\alpha \in \Gamma$, and $H_f$ be the $\mathcal{H}$-class of $f$. If $p \in P \cap I_\beta$, then $\alpha \geq \beta$ and $f \psi_{\alpha, \beta} = pp^{-1}$, so $H_f \psi_{\alpha, \beta} = H_p$. Since structure maps are retractions, we may assume that $e$ has been chosen such that $H_f \subseteq H$. Then $P \subseteq H$ and $P$ has a least upper bound $\bigvee P$ in $H$. If $k \in I_\gamma$ is also an upper bound for $P$, then by the completeness of $E$ in $E(I)$, $\gamma \geq \alpha$.
and \( k k^{-1} \psi_{\gamma, a} = f \). Then \( k \psi_{\gamma, a} \) is an upper bound for \( P \), so we must have \( k \psi_{\gamma, a} = \sqrt{P} \), the unique upper bound of \( P \) in \( H_f \). Thus \( \sqrt{P} \) is the least upper bound of \( P \) in \( I \), and \( I \) is complete.

If \( P \) and \( Q \) are compatible sets in \( I \), then by the infinite distributivity of \( E(I) \), \( \sqrt{(PQ)} \) and \( (\sqrt{P})(\sqrt{Q}) \) lie in the same \( \mathcal{K} \)-class. Clearly \( (\sqrt{P})(\sqrt{Q}) \) is an upper bound of every element of \( PQ \) so \( \sqrt{(PQ)} = (\sqrt{P})(\sqrt{Q}) \).

Conversely, suppose \( I \) satisfies (i), (ii) and (iii). Assume \( I \in \mathcal{H}_B \); otherwise the result follows from Theorem 3.7. By (iii), either \( \mathcal{L} \) or \( \mathcal{R} \) is a congruence for every member of \( \mathcal{V} \) and (ii) the comments and construction associated with (15), (16) and (17) apply for \( I \). The subsemigroup \( H \) satisfies the conditions of Theorem 3.7, so \( H \) is a \( \mathcal{V} \)-injective.

Let \( S, T \in \mathcal{V} \) where \( S \) is a subsemigroup of \( T \) and let \( \phi : S \rightarrow I \) be a homomorphism. Since \( H \) is a \( \mathcal{V} \)-injective, there is a homomorphism \( \psi_H : T \rightarrow H \) that extends \( \phi \).

Suppose the \( \mathcal{L} \)-relations on \( S \) and \( T \) respectively are \( \mathcal{L}_S \) and \( \mathcal{L}_T \) and that \( \mathcal{L}_T \) is a congruence. Since \( T \in \mathcal{B} \), \( \mathcal{L}_S \) is the restriction of \( \mathcal{L}_T \) to \( S \). Hence we may regard \( S/\mathcal{L}_S \) as being a subsemigroup of \( T/\mathcal{L}_T \). If \( a \in I \), then by (15) \( a\mathcal{L}_T^{-1}a\mathcal{L}_T^{-1}a \). Also, by the structure map definition of multiplication in a normal band of groups, if \( a\mathcal{R}_b \) in \( I \), then \( ea^{-1}a = eb^{-1}b \). Since \( \phi \) maps \( \mathcal{L} \)-classes to \( \mathcal{L} \)-classes, we can define a map

\[
\phi_L : S/\mathcal{L}_S \rightarrow E(I), \quad s\mathcal{L}_S \phi_L = e((s^{-1}s)\phi).
\]

By (15) and since \( x^{-1}xy^{-1}y = (xy)^{-1}xy \) is a law in \( \mathcal{G} \cap \mathcal{N} \mathcal{B} \mathcal{G} \) it follows that \( \phi_L \) is a homomorphism. Since \( E(I) \) is a \( (\mathcal{G} \cap \mathcal{B}) \)-injective by (i) and Theorem 1.2, and \( S/\mathcal{L}_S \) is a subsemigroup of \( T/\mathcal{L}_T \) in \( \mathcal{G} \cap \mathcal{B} \), there is a homomorphism \( \psi_L : T/\mathcal{L}_T \rightarrow E(I) \) that extends \( \phi_L \). Clearly \( \mathcal{L}_S \phi_L \) extends to \( \mathcal{L}_E \psi_L \).

If \( \mathcal{R}_T \) is a congruence on \( T \), we may dually define a homomorphism \( \phi_R : S/\mathcal{R}_S \rightarrow E(I) \) by \( s\mathcal{R}_S \phi_R = (ss^{-1})\phi_e \), and extend it to \( \psi_R : T/\mathcal{R}_T \rightarrow E(I) \). Also, \( \mathcal{R}_E \psi_R \) extends \( \mathcal{R}_S \phi_R \).

By our assumptions, \( I \in \mathcal{H}_B \) and either \( \mathcal{L} \) is a congruence on every member of \( \mathcal{V} \), or \( \mathcal{R} \) is. Suppose \( I \in \mathcal{N} \mathcal{B} \mathcal{G} \); then \( \mathcal{L}_T \) is a congruence by (iii). Since \( I \in \mathcal{N} \mathcal{B} \mathcal{G} \), \( e \) is a left identity of \( I \). So for \( s \in S \)

\[
s\phi = s\phi((s^{-1}s)\phi) = e(s\phi)ee((s^{-1}s)\phi) = (s\phi)(s\mathcal{L}_S \phi_L).
\]

Define \( \psi : T \rightarrow I \) by \( t\psi = t\psi_H(t\mathcal{L}_T \psi_L) \). Then for \( r, t \in T \), by (15), since \( r\mathcal{L}_T \psi_L \in F(I) \),

\[
(rt)\psi = (r\psi_H(t\psi_H)(r\mathcal{L}_T \psi_L))(t\mathcal{L}_T \psi_L)
= r\psi_H(r\mathcal{L}_T \psi_L)(t\psi_H)(r\mathcal{L}_T \psi_L)(t\mathcal{L}_T \psi_L)
- r\psi_H((r\mathcal{L}_T \psi_L)(t\psi_H)(t\mathcal{L}_T \psi_L)) - r\psi(t\psi).
\]

Hence \( \psi \) is a homomorphism. Since \( \psi \) extends \( \phi \), \( I \) is a \( \mathcal{V} \)-injective. A dual result applies if \( I \in \mathcal{N} \mathcal{B} \mathcal{G} \).
Finally, suppose that \( I \in \mathcal{NBG} \) and \( L_T \) and \( R_T \) are both congruences. By a similar argument to the above, \( s\phi = (sR_S^# \phi_R)(s\phi \theta)(sL_S^# \phi_L) \), and the map \( \psi : T \to I \) given by \( t\psi = (tR_T^# \psi_R)(t\psi \theta)(tL_T^# \psi_L) \) is a homomorphism extending \( \phi \). □

**Remark.** If \( S \in \mathcal{HR} \), then \( S \in \mathcal{BG} \) if and only if \( H \) is a congruence on \( S \). So by Theorem 4.1(ii) and (iii), the \( \mathcal{Y} \)-injective \( I \) is in \( \mathcal{NBG} \backslash (\mathcal{LNBG} \cup \mathcal{RNBG}) \) only if \( \mathcal{Y} \cap \mathcal{B} = \mathcal{NB} \) and \( H \) is a congruence on each member of \( \mathcal{Y} \); that is, only if \( \mathcal{Y} \in [\mathcal{NB}, \mathcal{NBG}] \). Also, \( I \in \mathcal{LNBG} \backslash \mathcal{JBG} \) only if \( \mathcal{Y} \cap \mathcal{B} \in [\mathcal{LNB}, \mathcal{NB}] \) and \( H \) is a congruence on each member of \( \mathcal{Y} \).

It should be noted that the description would be greatly simplified if all injectives in varieties of groups were trivial. By [1] and [6] there are many varieties of groups with non-trivial injectives.

**References**