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Varieties of Commutative Residuated Integral Pomonoids and Their Residuation Subreducts

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INTRODUCTION

The notion of residuation can already be found in Dedekind's work on modules, and it has played an important role in ideal theory ever since. If R is a commutative ring with 1, and I and J are ideals of R then the residual of I with respect to J is the ideal commonly denoted $I{:}J$ and defined as $\{x \in R\colon\ xJ \subseteq I\}$. That is, $I{:}J$ is the ideal characterized by the condition $KJ \subseteq I$ iff $K \subseteq I{:}J$. This operation is well defined on the collection of all ideals of R and serves to capture the concept of division in the ring in terms of its ideals. Indeed, if R is an integral domain and $i, j \in R$ are such that $i = kj$ for some element $k \in R$, then $(i){:}(j) = (k)$; for $r \in R$, (r) stands here for the ideal generated by r. Krull [Kru24] and Ward and Dilworth [WD39] started a long line of investigation showing that much of the structure theory of Noetherian rings can be obtained in the abstract setting of lattices with a suitable multiplication operation (abstracted from ideal multiplication) and residuation (abstracted from ideal residuation as described above).

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Residuation also plays a central role in the algebraic study of logical systems. Typically the algebras that arise from logic are partially ordered by a relation reflecting deducibility, and endowed with operations realizing the connectives and quantifiers of the logical system. The fundamental connection between conjunction $\wedge$ (usually a semilattice operation) and $\rightarrow$ is given by

$$c \wedge b \leq a \qquad \text{iff } c \leq b \rightarrow a \tag{$*$}$$

for all elements a, b, c in the universe of the algebra. In the algebraic models of classical propositional logic, the Boolean algebras, the condition leads to the usual definition of $b \rightarrow a = (\neg b) \vee a$, while in the models of intuitionistic propositional logic the condition is used to define the implication in terms of the partial order and the conjunction. In the models of many valued logics and linear logic the conjunction is just a commutative operation that respects the partial order; here again the implication is defined in terms of the conjunction via $(*)$.

In the structures referred to above the crucial elements are the partial order and two binary operations, one of which is a commutative monoid operation $\oplus$ that respects the partial order, the other one the residuation $\dot{-}$; these three are related by

$$a \leq c \oplus b \qquad \text{iff } a \dot{-} b \leq c.$$

(For technical reasons we use a formulation dual to the one used in the two classes of example described above.) The aim of the present paper is to begin an algebraic investigation of classes of such structures $\langle A; \oplus, \dot{-}; \leq \rangle$. In doing this we will assume in addition that the structure is *integral*, i.e., the unit 0 of the monoid operation is the least element in the partial order. This assumption is satisfied in most examples from the literature. For instance, in the monoid of ideals of a commutative ring with 1, the largest ideal R plays the role of the unit, whereas in the algebras of logic an element T representing "the true" does the same. The assumption allows us to eliminate the partial order from the type of the structures (since now $a \leq b$ iff $a \dot{-} b = 0$) and to treat them as proper algebras, which will be termed *pocrims*.

In an attempt to characterize a suitable notion of residuation without reference to a monoid operation and partial order, Iséki proposed [Isé66] a set of axioms inspired by a little known logical system that had been introduced by Meredith. The models of the set of axioms are called *BCK*-algebras after the traditional labels of the combinators B, C, and K attached to the axioms of the logical system, and they have been the subject of intense investigation. It is easily verified that the residuation subreducts of pocrims are *BCK*-algebras. That Iséki's axioms do indeed characterize the notion of residuation was proved in the mid-1980s independently by Pałasinski [Pał82]. Ono and Komori [OK85], and Fleischer [Fle88], who showed that, conversely, every *BCK*-algebra is a subreduct of a pocrim. Knowledge of this fact has facilitated the solution of some previously open problems concerning *BCK*-algebras, e.g., it allows one to reduce them to problems concerning pocrims, which generally are better behaved and easier to handle.

The class $\mathscr{M}$ of all pocrims is a quasivariety which, by a result of Higgs, is not a variety. Our focus in this paper is mainly on varieties of pocrims. (These varieties form the algebraic counterparts of *strongly algebraizable* logics with conjunction and implication, in the sense of [BP89].) We establish some properties enjoyed by all such varieties, and we study in some detail special classes of varieties, in particular the varieties of n-potent pocrims (for some $n \in \omega$) and, at the other side of the spectrum, the varieties of cancellative pocrims. We pay special attention to the classes of *BCK*-algebras that arise as the subreducts of the varieties of pocrims considered.

We start by noting, in Sections 2 and 3, global properties of the quasivariety of all pocrims: it is relatively congruence distributive, has the relative congruence extension property, and is relatively point regular with respect to the monoid identity 0. Subvarieties of $\mathscr{M}$, being 0-regular, are congruence n-permutable for some n. We give an equational scheme characterizing pocrim varieties, in which the number of equations coincides with the degree of permutability. Nevertheless, it seems a hard problem to determine whether all such varieties are in fact 3-permutable. Whereas no nontrivial variety of *BCK*-algebras is congruence permutable, many (but not all) pocrim varieties are. A sufficient condition for permutability is given, which is applicable to all permutable pocrim varieties known to us. A related and apparently hard problem asks whether the quasivariety of residuation subreducts of a pocrim variety must always itself be a variety. We characterize pocrim varieties with this property syntactically, showing in particular that they are 3-permutable.

In Section 4, we give examples of pocrim varieties, indicating among them several well-known varieties of classical algebra or logic. We show that the pocrim varieties such that for some $n \in \omega$, all members have n-potent monoid operations are precisely the varieties of pocrims with equationally definable principal congruences (EDPC). This extends earlier results about varieties of hoops (in the sense of Büchi and Owens) and Wajsberg algebras. Similarly, a variety of *BCK*-algebras has EDPC iff for some n, it satisfies $x \dot{-} ny \approx x \dot{-} (n+1)y$. The variety of *BCK*-algebras defined by this identity is known as $\mathscr{E}_n$; we denote by $\mathscr{M}_n$ the variety of $(n+1)$-potent pocrims. The residuation subreduct class of $\mathscr{M}_n$ is a subvariety of $\mathscr{E}_n$. A plausible conjecture influencing some of our subsequent investigations is that it exhausts $\mathscr{E}_n$; we obtain partial confirmation of this in Section 6. We prove that all subquasivarieties of $\mathscr{E}_n$ are varieties; the same is not true of $\mathscr{M}_n$.

In Section 5, we examine the construction of ordinal sums of pocrims. Many subvarieties of $\mathscr{M}$ are closed under this construction. In several familiar varieties, such as the variety of Brouwerian semilattices and the variety of hoops, the finitely generated subdirectly irreducible algebras in the variety can be obtained from a finite number of simple algebras in the variety by a finite number of applications of the operations of variety generation and ordinal sum, and this fact has profound implications for the structure theory of these varieties. We show that within any variety of pocrims closed under ordinal sums, any variety $\mathscr{V}$ gives rise to a variety $\mathscr{V}^+$ that is also closed under ordinal sums, the structure of which is closely tied to that of $\mathscr{V}$. In particular, if $\mathscr{V}$ is locally finite or generated as a quasivariety by its finite members then so is $\mathscr{V}^+$. These facts generalize results from [BF] concerning hoops.

We give our construction new application in Section 6, by examining its effect on the varieties $\mathscr{M}_n^s$ and $\mathscr{E}_n^s$ generated, respectively, by all simple $(n+1)$-potent pocrims and all simple members of $\mathscr{E}_n$. Here, in addition to congruence distributivity and closure under ordinal sums, we have available the theory of varieties with EDPC, from which it follows that $\mathscr{M}_n^s$ and $\mathscr{E}_n^s$ are semisimple; in fact, we show that $\mathscr{M}_n^s$ is a discriminator variety. Thus, the subdirect irreducibles of $\mathscr{M}_n^{s+}$ or $\mathscr{E}_n^{s+}$ are ordinal sums of which the first summand is a simple algebra in $\mathscr{M}_n$ or $\mathscr{E}_n$. We use this fact to obtain equational axiomizations of $\mathscr{M}_n^{s+}$ and $\mathscr{E}_n^{s+}$, and to prove that $\mathscr{M}_n^{s+}$ is congruence permutable. By contrast, we show that $\mathscr{M}_n$ is not permutable unless $n = 1$. As a further application, we prove that $\mathscr{E}_n^{s+}$ is exactly the residuation subreduct class of $\mathscr{M}_n^{s+}$, evidence in favour of the conjectured corresponding relationship between $\mathscr{E}_n$ and $\mathscr{M}_n$.

In Section 7, we give special consideration to the case $n = 2$; we prove that $\mathscr{M}_2^{s+}$ is locally finite. The same is true of $\mathscr{E}_2^{s+}$, in view of the aforementioned relationship, and contrasts with Dyrda's result that neither $\mathscr{M}_2$ nor $\mathscr{E}_2$ is locally finite. We show that $\mathscr{M}_3^{s+}$ is not locally finite. Since $\mathscr{M}_n$ is a variety of finite type with EDPC, any finite subdirectly irreducible $(n+1)$-potent pocrim **A** is splitting in $\mathscr{M}_n$, i.e., $\mathscr{M}_n$ has a largest subvariety not containing **A**. Extending this property, we prove that $\mathscr{M}_2$ and $\mathscr{E}_2$ each have $2^{\aleph_0}$ semisimple subvarieties, whereas $\mathscr{M}_1$ and $\mathscr{E}_1$ (the varieties of Brouwerian semilattices and Hilbert algebras) each have just one semisimple subvariety (the varieties of generalized Boolean algebras and Tarski algebras).

Pocrims whose underlying monoids are cancellative are examined in Section 8 and lie at the opposite end of our topic's spectrum to n-potent pocrims, since for each n, only the trivial cancellative pocrim is n-potent.

Cancellative pocrims form a subquasivariety $\mathscr{C}$ of $\mathscr{M}$ which is not a variety and which lacks the congruence extension property (in the absolute sense). Nevertheless, $\mathscr{C}$ has a Mal'cev term and therefore generates a congruence permutable variety (not contained in $\mathscr{M}$), enabling us to prove that the subvarieties of $\mathscr{C}$ form a lattice $\mathbf{L}^v(\mathscr{C})$ (the corresponding question for $\mathscr{M}$ being open). Our focus in Section 8 is primarily on properties of this lattice, whose structure seems at present quite elusive. We show that $\mathbf{L}^v(\mathscr{C})$ has a unique atom: the smallest nontrivial variety of cancellative pocrims is generated (even as a quasivariety) by the pocrim of nonnegative integers, with the natural addition and order. We identify a denumerable strictly ascending chain in $\mathbf{L}^v(\mathscr{C})$, but we do not know whether $\mathbf{L}^v(\mathscr{C})$ is countable nor, indeed, whether it is linearly ordered. The main result of this section is that $\mathbf{L}^v(\mathscr{C})$ has no greatest element.

1. ALGEBRAIC PRELIMINARIES

For general universal algebraic background we refer the reader to [BS81], [Grä79], or [MMT87]. We denote algebras by boldface capitals $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ and their respective universes by $A, B, C, \ldots$. We use ω to denote the set of nonnegative integers. Let $\mathscr{K}$ be a class of algebras of a given similarity type and $\mathbf{A} = \langle A; \ldots \rangle$ a member of $\mathscr{K}$. We shall make standard use of the class operators $\mathbf{I}$, $\mathbf{H}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P_s}$ (for subdirect products), and $\mathbf{P_u}$ (for ultraproducts). A *tolerance* on $\mathbf{A}$ is a reflexive, symmetric binary relation τ on A which is also compatible with the fundamental operations of $\mathbf{A}$ (i.e., it is also a subuniverse of $\mathbf{A} \times \mathbf{A}$). The (algebraic) lattice of all tolerances on $\mathbf{A}$ is denoted by $\operatorname{Tol}\mathbf{A}$, while, as usual, $\operatorname{Con}\mathbf{A}$ is the congruence lattice of $\mathbf{A}$. For $X \subseteq A$, $T^{\mathbf{A}}(X)$ and $\Theta^{\mathbf{A}}(X)$ denote, respectively, the least tolerance and the least congruence on $\mathbf{A}$ that contain X; we abbreviate $T^{\mathbf{A}}(\{(a_1, b_1), \ldots, (a_n, b_n)\})$ by $T^{\mathbf{A}}((a_1, b_1), \ldots, (a_n, b_n))$ and $T^{\mathbf{A}}((a, b))$ by $T^{\mathbf{A}}(a, b)$, and we apply similar conventions to $\Theta^{\mathbf{A}}$. The following characterization of finitely generated tolerances will be needed.

LEMMA 1.1 [Cha81]. *Let $\mathbf{A}$ be an algebra and $a_1, \ldots, a_n, b_1, \ldots, b_n, c, d \in A$. Then $(c, d) \in T^{\mathbf{A}}((a_1, b_1), \ldots, (a_n, b_n))$ iff there is a $2n$-ary polynomial G on $\mathbf{A}$ such that*

$$c = G(a_1, \ldots, a_n, b_1, \ldots, b_n) \quad \text{and} \quad d = G(b_1, \ldots, b_n, a_1, \ldots, a_n).$$

For $\tau, \eta \subseteq A \times A$, we usually denote the relational product $\tau \circ \eta$ by $\tau\eta$, and we define $\tau^0 = \mathrm{id}_{\mathbf{A}} := \{(a, a): a \in A\}$ and $\tau^{n+1} = \tau^n\tau$ $(n \in \omega)$. The least positive $n \in \omega$, if it exists, such that τ^n is a congruence for every $\tau \in \operatorname{Tol}\mathbf{A}$, is called the *tolerance number* of $\mathbf{A}$ and is denoted by $tn(\mathbf{A})$. We also write $tn(\mathscr{K}) = n$ if n is the least positive integer such that $tn(\mathbf{B}) \leq n$

for all $B \in \mathscr{K}$. If $tn(\mathbf{A}) \leq n$ then $\mathbf{A}$ is congruence $(n+1)$-permutable [RS92], i.e., for any two congruences θ, φ of $\mathbf{A}$, the $(n+1)$-fold relational products $\theta\varphi\theta\varphi \cdots$ and $\varphi\theta\varphi\theta \cdots$ coincide. The converse fails (e.g., [Cha88]), but a variety $\mathscr{K}$ is congruence $(n+1)$-permutable iff $tn(\mathscr{K}) \leq n$ [RS92, Kea93]. We drop the prefix from "n-permutable" when $n = 2$.

A congruence θ of $\mathbf{A}$ is called a *$\mathscr{K}$-congruence* if $\mathbf{A}/\theta \in \mathscr{K}$; the set of all $\mathscr{K}$-congruences of $\mathbf{A}$ is denoted by $\mathrm{Con}_{\mathscr{K}}\mathbf{A}$. If $\mathscr{K}$ is a quasivariety, $\mathrm{Con}_{\mathscr{K}}\mathbf{A}$ is an algebraic lattice. If the quasivariety $\mathscr{K}$ is fixed and clear from the context, we shall refer to the $\mathscr{K}$-congruences as *relative congruences*. In this case we say that $\mathbf{A}$ is *relatively congruence distributive* if the lattice $\mathrm{Con}_{\mathscr{K}}\mathbf{A}$ is distributive, and *relatively* 0*-regular* if $0 \in A$ and for all relative congruences θ, θ' of $\mathbf{A}$, $\theta = \theta'$ iff $0/\theta = 0/\theta'$. We say that $\mathbf{A}$ has the *relative congruence extension property* if for any $\mathbf{B} \in \mathbf{S}(\mathbf{A})$ and any relative congruence θ of $\mathbf{B}$, there is a relative congruence θ' of $\mathbf{A}$ such that $\theta' \cap (B \times B) = \theta$. (We drop "relative(ly)" if $\mathscr{K}$ is known to be a variety.) We use $\mathbf{L}^q(\mathscr{K})$ (resp. $\mathbf{P}^v(\mathscr{K})$) to denote the lattice (resp. the poset) of subquasivarieties (resp. subvarieties) of a quasivariety $\mathscr{K}$, order by inclusion, and we replace $\mathbf{P}^v(\mathscr{K})$ by $\mathbf{L}^v(\mathscr{K})$ if $\mathbf{P}^v(\mathscr{K})$ is known to be a lattice, e.g., if $\mathscr{K}$ is a variety. The class of all subdirectly irreducible algebras in a variety $\mathscr{K}$ will be denoted by $\mathscr{K}_{\mathrm{SI}}$. We insist that subdirectly irreducible (in particular, simple) algebras be nontrivial. A congruence permutable, congruence distributive variety is called *arithmetical*.

2. POCRIMS AND THEIR RESIDUATION SUBREDUCTS

Let $\langle A; \oplus, 0; \leq \rangle$ be a *commutative* (*dually*) *integral pomonoid*, i.e, $\langle A; \leq \rangle$ is a partially ordered set, $\langle A; \oplus, 0\rangle$ is a commutative monoid whose identity 0 is the least element of $\langle A; \leq \rangle$, and $\leq$ is compatible with the monoid operation $\oplus$ in the sense that $a \oplus b \leq c \oplus d$ whenever $a, b, c, d \in A$ with $a \leq c$ and $b \leq d$. If for each $a, b \in A$ there is a least element c (denoted $a \dot{-} b$) of A such that $a \leq c \oplus b$, we say that $\langle A; \oplus, 0; \leq \rangle$ is (*dually*) *residuated* and we call $\dot{-}$ the *residuation operation*. The resulting structure $\mathbf{A} = \langle A; \oplus, \dot{-}, 0; \leq \rangle$ will then be called a *partially ordered commutative* (*dually*) *residuated* (*dually*) *integral monoid* or, briefly, a *pocrim*. In this case it follows that for any $a, b, d \in A$ we have $a \dot{-} b \leq d$ iff $a \leq d \oplus b$; thus $d \oplus b$ is the greatest element of $\{e \in A: e \dot{-} b \leq d\}$, while $a \leq b$ iff $a \dot{-} b = 0$. Consequently, the partial order $\leq$ is determined by the monoid operation $\oplus$ and the residuation operation $\dot{-}$, so that no harm comes of systematically confusing a pocrim $\mathbf{A}$ with the underlying algebra $\langle A; \oplus^{\mathbf{A}}, \dot{-}^{\mathbf{A}}, 0^{\mathbf{A}}\rangle$. (We drop the superscripts whenever there is no danger of confusion.) In this spirit, we may say that the class of all pocrims, which we denote by $\mathscr{M}$, is a quasivariety of algebras of type

$\langle 2, 2, 0 \rangle$, since Iséki [Isé80] has shown that pocrims may be axiomatized by the following set of four identities and a quasi-identity:

$$((x \dot{-} y) \dot{-} (x \dot{-} z)) \dot{-} (z \dot{-} y) \approx 0, \tag{1}$$

$$x \dot{-} 0 \approx x, \tag{2}$$

$$0 \dot{-} x \approx 0, \tag{3}$$

$$(x \dot{-} y) \dot{-} z \approx x \dot{-} (z \oplus y), \tag{4}$$

$$(x \dot{-} y \approx 0) \;\&\; (y \dot{-} x \approx 0) \to x \approx y. \tag{5}$$

It is easy to deduce that every pocrim satisfies the following identities:

$$x \dot{-} x \approx 0, \tag{6}$$

$$((x \oplus y) \dot{-} y) \oplus y \approx x \oplus y. \tag{7}$$

It also follows easily that in the poset $\langle A; \leq \rangle$ underlying a pocrim $\mathbf{A}$, any doubleton $\{a, b\}$ has the element $a \dot{-} (a \dot{-} b)$ as a lower bound and the element $(a \dot{-} b) \oplus b$ as an upper bound.

The set of ideals of a commutative ring $\mathbf{R}$ with 1 is a pocrim with respect to the monoid operation of ideal multiplication and the (lattice) order of *reversed* set inclusion. Residuation is defined by $I \dot{-} J = I{:}J = \{x \in R\colon xJ \subseteq I\}$. A further simple example of a pocrim is the set ω with natural addition and the natural linear order, where residuation is defined by $a \dot{-} b = \max\{0, a - b\}$. Another is the set of all subsets of a given set, with union as the monoid operation, set inclusion as the partial order, and set difference as residuation.

If $\mathbf{A}$ is a pocrim and $a \in A$, we may define, inductively, elements $na \in A$ $(n \in \omega)$ by the following rules: $0a = 0$; $(n+1)a = (na) \oplus a$. It also follows from (4) that pocrims satisfy

$$(x \dot{-} y) \dot{-} z \approx (x \dot{-} z) \dot{-} y. \tag{8}$$

From (1) and (8), we obtain

$$((x \dot{-} y) \dot{-} (z \dot{-} y)) \dot{-} (x \dot{-} z) \approx 0. \tag{9}$$

If we abbreviate $(x \dot{-} y) \dot{-} z$ as $x \dot{-} y \dot{-} z$ then expressions such as $a \dot{-} b_1 \dot{-} \cdots \dot{-} b_n$ may be interpreted unambiguously, and for each $n \in \omega$, we use $x \dot{-} ny$ as an abbreviation of a $\langle \dot{-}, 0 \rangle$-term defined inductively by $x \dot{-} 0y = x$; $x \dot{-} (n+1)y = x \dot{-} ny \dot{-} y$. It is easy to deduce from the definitions that every pocrim satisfies

$$(x \oplus y) \dot{-} (z \oplus w) \leq (x \dot{-} z) \oplus (y \dot{-} w). \tag{10}$$

For any element c of a pocrim, the unary polynomial function determined by $x \dot{-} c$ (resp. $c \dot{-} x$) is isotone (resp. antitone).

It turns out that the class of all $\langle \dot{-}, 0\rangle$-subreducts of pocrims is precisely the quasivariety $\mathscr{BCK}$ of all *BCK*-algebras, which was introduced by Iséki in [Isé66]: see survey articles [IT78] and [Cor82a]. This relationship is discussed in detail by Wroński in [Wro85] and depends on an embedding theorem published by Pałasiński [Pał82], Ono and Komori [OK85], and Fleischer [Fle88]. The $\langle \dot{-}, 0\rangle$-reducts of pocrims have also been called *BCK-algebras with condition* (S) [Isé79]. Henceforth if $\mathscr{L}$ is either a class of algebras $\langle A; \oplus, \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$ or a class of algebras $\langle A; \dot{-}, 0\rangle$ of type $\langle 2, 0\rangle$, we denote the $\langle \dot{-}, 0\rangle$-reduct of $\mathbf{A} \in \mathscr{L}$ by $\mathbf{A}^-$ and we write $\mathscr{L}^-$ for $\{\mathbf{B}^-: \mathbf{B} \in \mathscr{L}\}$. We therefore have that $\mathscr{BCK} = \mathbf{S}(\mathscr{M}^-)$ and that a universal first-order sentence in the language $\langle \dot{-}, 0\rangle$ holds in $\mathscr{BCK}$ iff it holds in $\mathscr{M}$. We shall need to make use of the construction used by Fleischer, Ono, and Komori in the proof of the result $\mathscr{BCK} = \mathbf{S}(\mathscr{M}^-)$, so we present a version of it here.

Let $\mathbf{A} = \langle A; \dot{-}, 0\rangle$ be any *BCK*-algebra with associated poset $\langle A; \leq \rangle$. For each finite sequence $\vec{a} = a_1, \ldots, a_n$ of elements of A, let $J(\vec{a}) = J(a_1, \ldots, a_n) = \{b \in A: b \dot{-} a_1 \dot{-} \cdots \dot{-} a_n = 0\}$ and let $J(\mathbf{A})$ be the set of all such $J(\vec{a})$. Note that $\{0\} = J(0) \in J(\mathbf{A})$. Define a binary operation $+$ on $J(\mathbf{A})$ by

$$J(a_1, \ldots, a_n) + J(b_1, \ldots, b_m) = J(a_1, \ldots, a_n, b_1, \ldots, b_m).$$

Let $\mathbf{J}(\mathbf{A}) = \langle J(\mathbf{A}); +, \{0\}; \subseteq \rangle$.

For any commutative integral pomonoid $\mathbf{B} = \langle B; +, 0; \leq \rangle$, let $F(\mathbf{B})$ be the set of nonempty order filters of $\langle B; \leq \rangle$. For $F, G \in F(\mathbf{B})$, define $F \oplus G$ to be the order filter of $\langle B; \leq \rangle$ generated by $\{f + g: f \in F, g \in G\}$, i.e., $F \oplus G = \{h \in B: h \geq f + g \text{ for some } f \in F, g \in G\}$, and define $F \dot{-} G$ to be the filter of $\langle B; \leq \rangle$ generated by $\bigcup\{H \in F(\mathbf{B}): H \oplus G \subseteq F\}$. Let $\mathbf{F}(\mathbf{B}) = \langle F(\mathbf{B}); \oplus, \dot{-}, M; \supseteq \rangle$.

THEOREM 2.1 [Pał82, OK85, Fle88]. (i) $\mathbf{J}(\mathbf{A})$ *is a commutative integral pomonoid, the map* $\eta_{\mathbf{A}}$: $a \mapsto J(a)$ $(a \in A)$ *is an isotone embedding of* $\langle A; \leq \rangle$ *into* $\langle J(\mathbf{A}); \subseteq \rangle$ *and, for every* $a, b \in A$, *we have* $J(a \dot{-} b) = \bigcap\{I \in J(\mathbf{A}): I + J(b) \supseteq J(a)\}$.

(ii) $\mathbf{F}(\mathbf{B})$ *is a pocrim and the map* $\chi_{\mathbf{B}}$: $b \mapsto \{c \in B: b \leq c\}$ $(b \in B)$ *is an isotone integral monoid embedding of* $\mathbf{B}$ *into* $\langle F(\mathbf{B}); \oplus, M; \supseteq \rangle$ *which preserves existing residuals, i.e., whenever* $a, b \in B$ *and* c *is the least element of* B *for which* $c + b \geq a$, *we have* $\chi_{\mathbf{B}}(c) = \chi_{\mathbf{B}}(a) \dot{-} \chi_{\mathbf{B}}(b)$.

(iii) *The map* $\chi_{\mathbf{J}(\mathbf{A})} \circ \eta_{\mathbf{A}}$: $a \mapsto \{I \in J(\mathbf{A}): J(a) \subseteq I\}$ $(a \in A)$ *is a BCK-algebra embedding of* $\mathbf{A}$ *into the residuation reduct* $(\mathbf{F}(\mathbf{J}(\mathbf{A})))^-$ *of the pocrim* $\mathbf{F}(\mathbf{J}(\mathbf{A}))$.

Higgs [Hig84] observes that *BCK*-algebras may be defined as algebras of type $\langle 2, 0\rangle$ satisfying (1), (2), (3), and (5), and proves that the quasivariety $\mathscr{M}$ is not a variety. This extends Wroński's result [Wro83] that $\mathscr{BCK}$ is not a variety.

Let $\mathbf{A}$ be a pocrim or a *BCK*-algebra. By an *ideal* of $\mathbf{A}$, we mean a subset I of A with $0 \in I$, such that $a \in I$ whenever $a \in A$ and $a \dot{-} b$, $b \in I$. An ideal of $\mathbf{A}$ is a hereditary subset of $\langle A; \leq \rangle$. (In the case that $\mathbf{A}$ is a pocrim, it follows quite easily that the ideals of $\mathbf{A}$ are just the nonempty subsets of A which are hereditary and closed under the monoid operation $\oplus$.) For any congruence (or, indeed, tolerance) θ of $\mathbf{A}$, the 0-*class* $0/\theta = \{a \in A\colon (a, 0) \in \theta\}$ of θ is an ideal of $\mathbf{A}$. Conversely, given an ideal I of $\mathbf{A}$, there is at least one congruence whose 0-class is I, of which the largest, viz. $\varphi_I := \{(a, b) \in A \times A\colon a \dot{-} b,\ b \dot{-} a \in I\}$, is actually a relative congruence, i.e., $\mathbf{A}/\varphi_I$ (abbreviated $\mathbf{A}/I$) satisfies the quasi-identity (5), making $(\mathbf{A}/I)^-$ a *BCK*-algebra. A useful fact is that if $\tau \in \operatorname{Tol}\mathbf{A}$ then $0/\tau = 0/\tau^n$ for all positive $n \in \omega$; consequently $0/\tau = 0/\Theta(\tau)$, since $\Theta(\tau) = \bigcup_{n\in\omega} \tau^n$. These facts were proved in [RRS91, Theorem 2.2]. (They depend only on (2) and (6).)

The set of all ideals of $\mathbf{A}$, ordered by inclusion, is an algebraic lattice which we denote by $\operatorname{Id}\mathbf{A}$; $\operatorname{Id}\mathbf{A}$ is also distributive [Pał81]. We denote by $\langle X\rangle_{\mathbf{A}}$ the ideal of $\mathbf{A}$ generated by $X \subseteq A$, i.e., the intersection of all ideals of $\mathbf{A}$ containing X. Of course $\langle\varnothing\rangle_{\mathbf{A}} = \{0\}$ and it is well known and not difficult to check that if $\varnothing \neq X \subseteq A$ then

$$\langle X\rangle_{\mathbf{A}} = \big\{a \in A\colon (\exists n \in \omega)(\exists x_1, \ldots, x_n \in X)\big(a \dot{-} x_1 \dot{-} \cdots \dot{-} x_n = 0\big)\big\}.$$

We abbreviate $\langle X\rangle_{\mathbf{A}}$ by $\langle a_1, \ldots, a_m\rangle_{\mathbf{A}}$ when $X = \{a_1, \ldots, a_m\}$. The maps $I \mapsto \varphi_I$ and $\theta \mapsto 0/\theta$ are isotone functions between the lattices $\operatorname{Id}\mathbf{A}$ and $\operatorname{Con}\mathbf{A}$. Whereas $I = 0/\varphi_I$ for every ideal I, a congruence θ is generally smaller than $\varphi_{0/\theta}$. It is well known that $\mathbf{A}$ has the *ideal extension property*, i.e., whenever $\mathbf{B} \in \mathbf{S}(\mathbf{A})$ and I is an ideal of $\mathbf{B}$, there is an ideal J of $\mathbf{A}$ such that $J \cap B = I$. (We may simply take $J = \langle I\rangle_{\mathbf{A}}$.)

We shall need the following simple result which provides a useful method of constructing pocrims, and hence also *BCK*-algebras.

LEMMA 2.2. *Let* $\mathbf{A}$ *be a pocrim and let* $\langle B; \oplus, 0\rangle$ *be a submonoid of* $\langle A; \oplus, 0\rangle$ *such that for every* $a \in A$, *there is a least element* b *of* $\langle B; \leq \rangle$ *for which* $a \leq b$. *Then the monoid* $\langle B; \oplus, 0\rangle$ *is the* $\langle \oplus, 0\rangle$-*reduct of a pocrim* $\mathbf{B}$.

Proof. Let $\smallsetminus$ be the residuation operation on $\mathbf{A}$. For $b_1, b_2 \in B$, define $b_1 \dot{-} b_2$ to be the least element b of B such that $b_1 \smallsetminus b_2 \leq b$. Clearly, $b_1 \dot{-} b_2$ is also the least element b of B such that $b_1 \leq b \oplus b_2$. Thus, $\langle B; \oplus, 0; \leq \rangle$ is residuated, with residuation operation $\dot{-}$, as required.

Note that we do not assume B to be closed under $\setminus$, and the partial operation induced on B by $\setminus$ need not coincide with the restriction to its domain of the residuation operation $\dot{-}$ on $\mathbf{B}$.

3. PROPERTIES OF POCRIM VARIETIES

The following proposition is taken from [BR95], where it was stated only in the context of $\mathscr{BCK}$. The well-known arguments used there apply mutatis mutandis to the quasivariety $\mathscr{M}$.

PROPOSITION 3.1. *Let $\mathscr{K}$ be $\mathscr{M}$ or $\mathscr{BCK}$ and let $\mathbf{A} \in \mathscr{K}$.*

(i) *The maps $\theta \mapsto 0/\theta$ $(\theta \in \mathrm{Con}_{\mathscr{K}}\mathbf{A})$ and $I \mapsto \varphi_I$ $(I \in \mathrm{Id}\,\mathbf{A})$ are mutually inverse lattice isomorphisms between the relative congruence lattice of $\mathbf{A}$ and the ideal lattice of $\mathbf{A}$.*

(ii) *$\mathbf{A}$ is relatively 0-regular, is relatively congruence distributive, and has the relative congruence extension property.*

(iii) *$\mathbf{H}(\mathbf{A}) \subseteq \mathscr{K}$ iff $\mathbf{A}$ is 0-regular, in which case $\mathbf{A}$ is also congruence distributive. If $\mathbf{HS}(\mathbf{A}) \subseteq \mathscr{K}$ then $\mathbf{A}$ has the congruence extension property. In particular*:

(iv) *Every subvariety of $\mathscr{K}$ is 0-regular and congruence distributive, with the congruence extension property.*

The following easily checked facts (which are well known for *BCK*-algebras) go slightly further than the immediate consequences of the above proposition: *a pocrim or BCK-algebra is subdirectly irreducible* (*resp. simple*) *iff it has a smallest* (*resp. a unique*) *nonzero ideal*. Thus, by Proposition 3.1, the notion of relative subdirect irreducibility (resp. relative simplicity) with respect to $\mathscr{M}$ or $\mathscr{BCK}$ *does not differ from its absolute counterpart*. In particular, it follows from the relative congruence extension property that *nontrivial subalgebras of simple pocrims or BCK-algebras are simple*.

Neither the quasivariety of pocrims nor the quasivariety of *BCK* algebras has the congruence extension property in the absolute sense, however [BR93]. Any 0-regular variety is congruence n-permutable for some integer $n \geq 2$ [Hag73], so this is true of all pocrim varieties, by (iv). The degree n of permutability turns out to be recognizable from the length of a scheme of identities, the satisfaction of which characterizes certain classes as pocrim varieties. This is the content of the next result.

THEOREM 3.2. *Let $\mathscr{K}$ be a class of algebras of type $\langle 2, 2, 0\rangle$. Then $\mathbf{HSP}(\mathscr{K})$ is a variety of pocrims iff for some integer $n > 0$, there exist 6-ary*

$\langle \oplus, \dot{-} \rangle$*-terms* $t_1, \ldots, t_n$ *such that* $\mathscr{K}$ *satisfies* (1), (2), (3), (4), *and*

$$x \approx t_1(x, y, x \dot{-} y, y \dot{-} x, 0, 0), \tag{11$_0$}$$

$$t_i(x, y, 0, 0, x \dot{-} y, y \dot{-} x) \approx t_{i+1}(x, y, x \dot{-} y, y \dot{-} x, 0, 0) \quad (i = 1, \ldots, n-1), \tag{11$_i$}$$

$$t_n(x, y, 0, 0, x \dot{-} y, y \dot{-} x) \approx y. \tag{11$_n$}$$

In this case, $\mathbf{HSP}(\mathscr{K})$ *is congruence* $(n+1)$*-permutable. Conversely, if* $\mathbf{HSP}(\mathscr{K})$ *is a congruence* $(k+1)$*-permutable variety of pocrims then we may realise the above scheme of equations for some* $n \le k$. *In other words,* $tn(\mathbf{HSP}(\mathscr{K}))$ *is the minimum integer n for which* $\mathbf{HSP}(\mathscr{K})$ *satisfies a scheme of equations of the form* $(11)_{0-n}$.

Proof. $(\Rightarrow)$ Certainly $\mathbf{HSP}(\mathscr{K})$ satisfies (1), (2), (3), and (4). Let $\mathbf{F} = \langle F; \oplus, \dot{-}, 0\rangle$ be the $\mathbf{HSP}(\mathscr{K})$-free algebra on two free generators $\bar{x}, \bar{y}$ and let J be the ideal of $\mathbf{F}$ generated by $\{\bar{x} \dot{-} \bar{y}, \bar{y} \dot{-} \bar{x}\}$. Let $\tau = T^{\mathbf{F}}((\bar{x} \dot{-} \bar{y}, 0), (\bar{y} \dot{-} \bar{x}, 0))$. Then by Proposition 3.1(iv), since $\mathbf{HSP}(\mathscr{K})$ is a variety of pocrims, the relation φ_J is the unique congruence on $\mathbf{F}$ whose 0-class is J. Since $\tau \subseteq \varphi_J$ and J is an ideal of $\mathbf{F}$, we also have $0/\tau = J$. Let $\theta = \Theta^{\mathbf{F}}(\tau)$. We know that $0/\theta = 0/\tau = J$, so we have $\theta = \varphi_J$. Now $(\bar{x}, \bar{y}) \in \varphi_J$ so $(\bar{x}, \bar{y}) \in \tau^n$ for some positive $n \in \omega$, and if $\mathbf{HSP}(\mathscr{K})$ is $(k+1)$-permutable (i.e., $tn(\mathbf{HSP}(\mathscr{K})) \le k$), we may choose $n \le k$. By Lemma 1.1, there exist polynomials $G_1, \ldots, G_n$ on $\mathbf{F}$ with

$$\bar{x} = G_1(\bar{x} \dot{-} \bar{y}, \bar{y} \dot{-} \bar{x}, 0, 0), \tag{12$_0$}$$

$$G_i(0, 0, \bar{x} \dot{-} \bar{y}, \bar{y} \dot{-} \bar{x}) = G_{i+1}(\bar{x} \dot{-} \bar{y}, \bar{y} \dot{-} \bar{x}, 0, 0) \qquad (i = 1, \ldots, n-1), \tag{12$_i$}$$

$$G_n(0, 0, \bar{x} \dot{-} \bar{y}, \bar{y} \dot{-} \bar{x}) = \bar{y}. \tag{12$_n$}$$

Now for each $i \in \{1, \ldots, n\}$, there exist a $\langle \oplus, \dot{-}, 0\rangle$-term s_i and elements $\bar{u}_{i1}, \ldots, \bar{u}_{im_i} \in F$ with

$$G_i(\bar{a}, \bar{b}, \bar{c}, \bar{d}) = s_i^{\mathbf{F}}(\bar{u}_{i1}, \ldots, \bar{u}_{im_i}, \bar{a}, \bar{b}, \bar{c}, \bar{d}) \tag{13$_i$}$$

for all $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in F$. Find binary terms u_{ij} with $u_{ij}^{\mathbf{F}}(\bar{x}, \bar{y}) = \bar{u}_{ij}$ for each i, j and define

$$t_i(x, y, z_1, z_2, z_3, z_4) = s_i(u_{i1}(x, y), \ldots, u_{im_i}(x, y), z_1, z_2, z_3, z_4) \quad (i = 1, \ldots, n). \tag{14$_i$}$$

It follows from $(12)_{0-\mathrm{n}}$, $(13)_{0-\mathrm{n}}$, and $(14)_{0-\mathrm{n}}$ that $\mathbf{F}$ (and hence $\mathbf{HSP}(\mathscr{K})$) satisfies $(11)_{0-\mathrm{n}}$. We may clearly assume that the terms $t_1, \ldots, t_n$ are free of occurrences of 0, in view of the identities (2), (3), and (6) and the monoid identities $x \oplus 0 \approx x \approx 0 \oplus x$.

Now let τ be any tolerance on some $\mathbf{A} \in \mathbf{HSP}(\mathscr{K})$. We show that $\tau^{n+1} \subseteq \tau^n$. Let $(a, b) \in \tau^{n+1}$. Then $a \dot{-} b,\ b \dot{-} a \in 0/\tau^{n+1} = 0/\tau$ so

$$t_i^{\mathbf{A}}(a, b, a \dot{-} b, b \dot{-} a, 0, 0) \tau\, t_{i+1}^{\mathbf{A}}(a, b, 0, 0, a \dot{-} b, b \dot{-} a) \qquad (i = 1, \ldots, n-1);$$

from $(11)_{0-n}$ it follows that $(a, b) \in \tau^n$, as required. Thus $tn(\mathbf{A}) \leq n$, so $\mathbf{A}$ is congruence $(n+1)$-permutable.

$(\Leftarrow)$ Let $\mathscr{K}$ satisfy (1), (2), (3), (4), and $(11)_{0-n}$ and let $\mathbf{A} \in \mathbf{HSP}(\mathscr{K})$. Suppose $a, b \in A$ with $a \dot{-} b,\ b \dot{-} a = 0$. Since $\mathbf{A}$ satisfies all equations that hold in $\mathscr{K}$, we have $a = t_1^{\mathbf{A}}(a, b, 0, 0, 0, 0) = t_2^{\mathbf{A}}(a, b, 0, 0, 0, 0) = \cdots = t_n^{\mathbf{A}}(a, b, 0, 0, 0, 0) = b$. This means that $\mathbf{A}$ satisfies (5) and is therefore a pocrim, as required. ∎

Every variety of pocrims known to us is congruence 3-permutable but we do not know whether this is true of all pocrim varieties. By contrast, it is known that every variety of *BCK*-algebras is congruence 3-permutable (and no such nontrivial variety is congruence permutable) [Idz83]. An example showing that pocrim varieties need not be congruence permutable will be given in Section 6.

If $\mathscr{K}$ is a variety of *BCK*-algebras with an equational base Σ, it is easy to see that the class of all pocrims which are models of Σ is a variety $\mathscr{V}$ with $\mathbf{S}(\mathscr{V}^-) \subseteq \mathscr{K}$. Here, $\mathscr{V}$ does not depend on the choice of Σ. We need not have $\mathbf{S}(\mathscr{V}^-) = \mathscr{K}$ in general (see Section 4, Example VI below), but it turns out that $\mathbf{S}(\mathscr{V}^-)$ is variety of *BCK*-algebras. This is a consequence of a stronger result, the next theorem, which characterizes those pocrim varieties $\mathscr{V}$ for which $\mathbf{S}(\mathscr{V}^-)$ is a variety.

THEOREM 3.3. (i) *Let $\mathscr{V}$ be a variety of pocrims. Then $\mathbf{S}(\mathscr{V}^-)$ is a variety (of BCK-algebras) iff there exist $n, m \in \omega$ and binary $\langle \dot{-} \rangle$-terms $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ such that $\mathscr{V}$ satisfies*

$$x \dot{-} \alpha_1(x, y) \dot{-} \cdots \dot{-} \alpha_n(x, y) \approx y \dot{-} \beta_1(x, y) \dot{-} \cdots \dot{-} \beta_m(x, y) \quad (15)$$

and $\mathscr{BCK}$ satisfies the identities $\alpha_i(x, x) \approx 0 \approx \beta_j(x, x)$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. In this case, $\mathbf{S}(\mathscr{V}^-)$ and $\mathscr{V}$ are congruence 3-permutable varieties.

(ii) *If $\mathscr{K}$ is any class of algebras of type $\langle 2, 2, 0 \rangle$ satisfying the identities* (1), (2), (3), *and* (4), *together with an identity of the form* (15) (*with the α_i and β_j as in* (i)) *then $\mathbf{HSP}(\mathscr{K})$ is a variety of pocrims* (*and is congruence* 3-*permutable, by* (i)).

Proof. (i) $(\Rightarrow)$ This follows from a result of Komori and Idziak (stated in [Idz83]) to the effect that every variety of *BCK*-algebras satisfies an identity of the form (15), with the α_i, β_j as described in the statement of the theorem, and is congruence 3-permutable. (Fuller proofs of these results may be found in [BR95] or [RS92].)

$(\Leftarrow)$ Let $\mathbf{A}$ be a subalgebra of $\mathbf{B}^-$, where $\mathbf{B} \in \mathscr{V}$. It suffices to show that $\mathbf{H}(\mathbf{A}) \subseteq \mathbf{S}(\mathscr{V}^-)$. Let $\theta \in \operatorname{Con} \mathbf{A}$ and let $I = 0/\theta$. Let J be the ideal of $\mathbf{B}$ generated by I. Clearly $\theta \subseteq \varphi_J \cap A^2$. On the other hand, if $(a, b) \in \varphi_J \cap A^2$ then $\alpha_i^{\mathbf{A}}(a, b) \varphi_J \alpha_i^{\mathbf{A}}(a, a) = 0$ so $\alpha_i^{\mathbf{A}}(a, b) \in 0/\varphi_J \cap A^2 = J \cap A^2 = I$ for $i = 1, \dots, n$. Similarly $\beta_j^{\mathbf{A}}(a, b) \in I$ for $j = 1, \dots, m$. By (13),

$$a = a \dot{-} 0 \dot{-} \cdots \dot{-} 0 \,\theta\, a \dot{-} \alpha_1^{\mathbf{A}}(a, b) \dot{-} \cdots \dot{-} \alpha_n^{\mathbf{A}}(a, b)$$

$$= b \dot{-} \beta_1^{\mathbf{A}}(a, b) \dot{-} \cdots \dot{-} \beta_m^{\mathbf{A}}(a, b) \,\theta\, b \dot{-} 0 \dot{-} \cdots \dot{-} 0 = b.$$

Thus, $\theta = \varphi_J \cap A^2$, hence $\mathbf{A}/\theta \in \mathbf{S}((\mathbf{B}/\varphi_J)^-)$, where $\mathbf{B}/\varphi_J \in \mathscr{V}$.

(ii) Certainly $\mathbf{HSP}(\mathscr{K})$ is a variety of type $\langle 2, 2, 0 \rangle$ satisfying all identities that hold in all members of $\mathscr{K}$. For $1 \le i \le n$ and $1 \le j \le m$, in view of the fact that $\mathscr{BCK}$ satisfies $\alpha_i(x, x) \approx 0 \approx \beta_j(x, x)$, it follows from [BR95, Lemma 7] that $\mathscr{BCK}$ satisfies $\alpha_i(x, y) \dot{-} p_i(x \dot{-} y) \dot{-} q_i(y \dot{-} x) \approx 0 \approx \beta_j(x, y) \dot{-} k_j(x \dot{-} y) \dot{-} l_j(y \dot{-} x)$ for some integers $p_i, q_i, k_j, l_j \in \omega$. If we define

$$\begin{aligned} &t_1(x, y, z, u, v, w) \\ &\qquad = x \dot{-} [\alpha_1(x, y) \dot{-} p_1 z \dot{-} q_1 u] \dot{-} \cdots \dot{-} [\alpha_n(x, y) \dot{-} p_n z \dot{-} q_n u], \\ &t_2(x, y, z, u, v, w) \\ &\qquad = y \dot{-} [\beta_1(x, y) \dot{-} k_1 v \dot{-} l_1 w] \dot{-} \cdots \dot{-} [\beta_m(x, y) \dot{-} k_m v \dot{-} l_m w], \end{aligned}$$

then equations $(11)_0$, $(11)_1$, and $(11)_2$ of Theorem 3.2 hold, by (15) and (2), and the result follows from Theorem 3.2. ∎

The next result offers a sufficient condition for a variety of pocrims to be congruence permutable. All congruence permutable pocrim varieties known to us satisfy this condition (the examples in the next section and Proposition 6.7 show that it is a useful test for permutability), but we do not know whether it characterizes congruence permutability for pocrim varieties.

THEOREM 3.4. *Let $k \in \omega$ and let $u_1, \dots, u_n, w_1, \dots, w_m$ be binary $\langle \dot{-} \rangle$-terms such that $\mathscr{BCK}$ satisfies $u_i(x, x) \approx 0 \approx w_j(x, x)$ for $1 \le i \le n$ and $1 \le j \le m$. If $\mathbf{A}$ is a pocrim satisfying*

$$\begin{aligned} &[y \dot{-} u_1(x, y) \dot{-} \cdots \dot{-} u_n(x, y)] \oplus w_1(x, y) \oplus \cdots \oplus w_m(x, y) \\ &\qquad \approx x \oplus k(y \dot{-} x) \end{aligned} \tag{16}$$

then **HSP(A)** *is a congruence permutable* (*hence arithmetical*) *variety of pocrims*.

Proof. An argument similar to that of Theorem 3.3(ii) shows that the variety **HSP(A)** consists of pocrims, hence this variety is congruence distributive, by Proposition 3.1. To prove congruence permutability, define

$$\begin{aligned} t(x, y, z) = \big[z &\dot{-} (y \dot{-} x) \dot{-} u_1(x, y \dot{-} (y \dot{-} x)) \\ &\dot{-} \cdots \dot{-} u_n(x, y \dot{-} (y \dot{-} x))\big] \\ &\oplus w_1(x, y \dot{-} (y \dot{-} x)) \oplus \cdots \oplus w_m(x, y \dot{-} (y \dot{-} x)). \end{aligned}$$

Then **A** satisfies $t(x, x, y) \approx [y \dot{-} u_1(x, x) \dot{-} \cdots \dot{-} u_n(x, x)] \oplus w_1(x, x) \oplus \cdots \oplus w_m(x, x) \approx y$ by (2), (6), and $y \oplus 0 \approx y$), as well as

$$\begin{aligned} t(x, y, y) \approx \big[y &\dot{-} (y \dot{-} x) \dot{-} u_1(x, y \dot{-} (y \dot{-} x)) \\ &\dot{-} \cdots \dot{-} u_n(x, y \dot{-} (y \dot{-} x))\big] \\ &\oplus w_1(x, y \dot{-} (y \dot{-} x)) \oplus \cdots \oplus w_m(x, y \dot{-} (y \dot{-} x)) \\ \approx x \oplus k&((y \dot{-} (y \dot{-} x)) \dot{-} x) \approx x \oplus k0 \approx x \end{aligned}$$

(by (16) and the fact that $y \dot{-} (y \dot{-} x)$ is always a lower bound of x in any pocrim). ∎

4. EXAMPLES OF POCRIM VARIETIES

Many well-known varieties from classical algebra and from logic consist of pocrims. We give examples of these and other pocrim varieties.

EXAMPLE I (The quasivariety generated by the natural numbers). We have already observed that the set ω of all nonnegative integers is the universe of a pocrim $\boldsymbol{\omega} = \langle \omega; +, \dot{-}, 0 \rangle$, where addition and the partial order are natural. It is known that the quasivariety $\mathbf{ISPP_u}(\boldsymbol{\omega})$ is a variety [BF93, Corollary 2.4] which is equationally complete [Ame84] and therefore an atom of $\mathbf{P}^{v}(\mathscr{M})$. For reasons to be discussed in the examples that follow and in Section 8, the members of this variety are called *cancellative hoops* in [BF93], the results of which imply that an equational base for this variety is provided by the identities (1), (2), (3), and (4) together with the two following identities:

$$x \oplus (y \dot{-} x) \approx y \oplus (x \dot{-} y), \tag{17}$$

$$(x \oplus y) \dot{-} y \approx x. \tag{18}$$

Let $\langle L; +, -, 0, \vee, \wedge \rangle$ be a lattice ordered Abelian group, i.e., $\langle L; +, -, 0\rangle$ is an Abelian group and there is a lattice order $\leq$ on L, compatible with $+$, whose supremum and infimum operations are $\vee$ and $\wedge$, respectively. For $a, b \in L$, define $a \dot{-} b = (a - b) \vee 0$. If L^+ is the positive cone $\{a \in L\colon 0 \leq a\}$ of L then $\mathbf{L}^+ = \langle L^+; +, \dot{-}, 0\rangle$ is a pocrim satisfying (17) and (18), hence a member of $\mathbf{ISPP_u}(\boldsymbol{\omega})$; conversely, any member of $\mathbf{ISSP_u}(\boldsymbol{\omega})$ is obtainable from the positive cone of a lattice ordered Abelian group in this way. This correspondence between $\mathbf{ISPP_u}(\boldsymbol{\omega})$ and lattice ordered Abelian groups arises in G. Birkhoff's work on "Abelian ℓ-groups" [Bir84] and is also stated in [Bos69], [Ame84], and [Pon86]. Moreover, it may be described as an equivalence of categories [Fer92, Theorem 4.7]. The variety $\mathbf{ISPP_u}(\omega)$ also satisfies

$$x \dot{-} (x \dot{-} y) \approx y \dot{-} (y \dot{-} x), \tag{T}$$

a special case of (15), thus illustrating Theorem 3.3.

EXAMPLE II (The variety of Brouwerian semilattices). A (*dual*) *Brouwerian semilattice* (see [Köh81] and [BP94]) is an algebra $\langle A; \vee, \dot{-}, 0\rangle$ of type $\langle 2, 2, 0\rangle$, where $\langle A; \vee \rangle$ is a join semilattice with least element 0, and for all $a, b \in A$, the condition

$$\forall x \, (a \leq b \vee x \leftrightarrow a \dot{-} b \leq x)$$

is satisfied. It follows immediately that a Brouwerian semilattice is a pocrim, with monoid operation $\vee$, satisfying the idempotent identity

$$x \oplus x \approx x. \tag{M_1}$$

Conversely, any pocrim satisfying (M_1) has the property that its underlying poset is an upper semilattice whose supremum operation coincides with the monoid operation. (This follows quite easily from (M_1) and (10).) We therefore denote the class of Brouwerian semilattices by $\mathscr{M}_1$; it is a variety. It is known that the class $\mathbf{S}(\mathscr{M}_1^-)$ of $\langle \dot{-}, 0\rangle$-subreducts of $\mathscr{M}_1$ is *just* the variety $\mathscr{E}_1$ of *Hilbert algebras* [Die66] (also known as *positive implicative BCK-algebras* (e.g., [Cor82a])). The members of $\mathscr{E}_1$ may be characterized as *BCK*-algebras satisfying the identity

$$(x \dot{-} y) \dot{-} y \approx x \dot{-} y. \tag{E_1}$$

This is equivalent, over $\mathscr{BCK}$, to the identity

$$x \dot{-} (x \dot{-} y) \dot{-} (y \dot{-} x) \approx y \dot{-} (x \dot{-} y) \dot{-} (y \dot{-} x). \tag{E_1}'$$

This fact illustrates Theorem 3.3, since $(E_1)'$ is clearly a special case of (15).

The lattice $\mathbf{L}^v(\mathscr{M}_1)$ of varieties of Brouwerian semilattices has a unique atom, viz., the variety of generalized Boolean algebras, which we denote by $\mathscr{G}$. Recall that a *generalized Boolean lattice* is just a distributive lattice $\langle L; \vee, \wedge \rangle$ with a least element 0, which is *relatively complemented* in the sense that whenever $a, b \in L$ with $b < a$, the quotient sublattice $a/b := \{c \in L: b \le c \le a\}$ is complemented. If we define $a \dot{-} b$ to be the complement of b in $(a \vee b)/0$ then the algebra $\langle L; \vee, \dot{-}, 0\rangle$ is called a *generalized Boolean algebra* and is a Brouwerian semilattice. The variety $\mathscr{G}$ of all generalized Boolean algebras may be characterized as the class of all pocrims satisfying

$$x \dot{-} (y \dot{-} x) \approx x. \tag{19}$$

Alternatively, $\mathscr{G} = \mathbf{ISP}(\mathbf{C}_2)$, where $\mathbf{C}_2$ is the unique two-element pocrim, i.e., the pocrim reduct of the two-element Boolean algebra. Also, $\mathscr{G}$ and $\mathbf{ISPP_u}(\omega)$ are the only atoms known to us in the poset $\mathbf{P}^v(\mathscr{M})$ of varieties of pocrims. The class $\mathbf{S}(\mathscr{G}^-)$ of $\langle \dot{-}, 0\rangle$-subreducts of members of $\mathscr{G}$ is the variety of *Tarski algebras* [Mon60] (alias *implication algebras* [Mit71] or *implicative BCK-algebras* [Cor82a]); this is the (unique) smallest nontrivial variety of *BCK*-algebras and is axiomatized by (1), (2), (3), and (19). Tarski algebras also satisfy (T).

The variety $\mathscr{M}$ of Brouwerian semilattices is known to have $2^{\aleph_0}$ subvarieties [Köh81, Owe74]. Thus, there are at least $2^{\aleph_0}$ varieties of pocrims and since pocrims have finite type, it follows that $|\mathbf{P}^v(\mathscr{M})| = 2^{\aleph_0}$.

EXAMPLE III (The variety of hoops). A *hoop* is a pocrim $\mathbf{A}$ which is naturally ordered in the sense that it satisfies the condition

$$(\forall a, b \in A)\ [b \le a \leftrightarrow (\exists c \in A)(a = b + c)].$$

These algebras were first investigated in a manuscript of J. R. Büchi and T. M. Owens, entitled "Complemented Monoids and Hoops," circa 1975. For more recent and comprehensive work on these algebras, see [Ame82], [Ame84], [BP94], [BF93], [BF] and [Fer92]. It is known (e.g., [BP94, Lemma 1.3]) that if $\mathbf{A}$ is a hoop then the underlying poset $\langle A; \le \rangle$ is an upper semilattice whose join operation may be defined by $x \vee y = x \oplus (y \dot{-} x)$. It follows easily that hoops are just those pocrims which satisfy the identity (17), and consequently that the class of all hoops (which we denote by $\mathscr{H}$) is a variety for which (1), (2), (3), (4), and (17) serve as an equational base. It is known that the lattice of subvarieties of $\mathscr{H}$ has exactly two atoms, viz., $\mathscr{G}$ and $\mathbf{ISPP_u}(\boldsymbol{\omega})$ [Ame84], and that the variety $\mathscr{M}_1$ of Brouwerian semilattices consists of hoops. Thus, *all examples of pocrim varieties mentioned thus far are varieties of hoops*. We have remarked that the ideal lattice of a commutative ring with 1 is a pocrim (with respect to ideal multiplication and reversed set inclusion); in particular, the ideal lattice of a Dedekind domain is a (cancellative) hoop, i.e., a member of $\mathbf{ISPP_u}(\omega)$. Hoops are congruence permutable (and therefore arithmetical, by Proposition 3.1) since the term $t(x, y, z) = (x \dot{-} (y \dot{-} z)) \vee (z \dot{-} (y \dot{-} x))$ is a Mal'cev term, i.e., $\mathscr{H}$ satisfies $t(x, x, y) \approx y$ and $t(x, y, y) \approx x$. Notice also that (17) is of the form of the identity (16) of Theorem 3.4 (take $n = 0$; $m = k = 1$ and $w_1(x, y) = x \dot{-} y$), giving an alternative proof that $\mathscr{H}$ is congruence permutable. Not every congruence permutable variety of pocrims consists of hoops (see the remarks following Theorem 6.1).

It is also known that hoops satisfy a useful identity introduced by Cornish [Cor81], viz.,

$$x \dot{-} (x \dot{-} (y \dot{-} (y \dot{-} x))) \approx y \dot{-} (y \dot{-} (x \dot{-} (x \dot{-} y))) \tag{J}$$

[BF93, Corollary 4.8]. Thus, by Theorem 3.3(i), the class $\mathbf{S}(\mathscr{V}^-)$ is a variety of BCK-algebras for every subvariety $\mathscr{V}$ of $\mathscr{H}$ (in particular, for all of the aforementioned pocrim varieties). The variety $\mathbf{S}(\mathscr{H}^-)$ is just the class of all BCK-algebras satisfying

$$(z \dot{-} x) \dot{-} (y \dot{-} x) \approx (z \dot{-} y) \dot{-} (x \dot{-} y) \tag{H}$$

[BF93, Corollary 4.5]. An equational base for this variety is therefore given by (1), (2), (3), (H), and (J). The residuation reducts of hoops have also been studied by Cornish [Cor82b, Cor84].

Notice that the identity (J) is a consequence of the identity (T) (over $\mathscr{BCK}$ and hence over $\mathscr{M}$). Pocrims or BCK-algebras satisfying (T) are called *commutative*. Such algebras were first studied by Tanaka [Tan75], who showed that their underlying posets are lower semilattices in which the infimum operation is definable by $x \wedge y = x \dot{-} (x \dot{-} y)$. We call a BCK-algebra $\mathbf{A}$ *directed* (resp. *bounded*) if the poset $\langle A; \leq \rangle$ is upward directed (resp. has a greatest element). Every directed commutative BCK-algebra satisfies $(x \dot{-} y) \wedge (y \dot{-} x) \approx 0$ [Tra79, Pał80]. This is equivalent, over $\mathscr{BCK}$, to the identity $z \dot{-} (z \dot{-} (x \dot{-} y)) \leq z \dot{-} (y \dot{-} x)$, the satisfaction of which characterizes a BCK-algebra as a subdirect product of ones with linearly ordered underlying posets [Pał80]. Thus in particular, *every subdirectly irreducible commutative pocrim is linearly ordered*. It is well known that every directed commutative BCK-algebra is embeddable into a bounded commutative BCK-algebra. More strongly, any directed BCK-algebra is embeddable into an ultraproduct of its bounded subalgebras: this is [BF93, Lemma 4.1] and was also discovered independently by T. Sturm [Stu]. Applying some of these results we obtain further examples of hoops:

PROPOSITION 4.1. *Every commutative pocrim is a hoop. Thus, the variety of commutative pocrims is a subvariety of* $\mathscr{H}$.

Proof. Let $\mathbf{A}$ be a commutative pocrim. Since $\mathbf{A}^-$ is directed, it is a subalgebra of a bounded commutative BCK-algebra $\langle B; \dot{-}, 0\rangle$ with greatest element 1, say. If we define $\oplus$ on B by $a \oplus b = 1 \dot{-} (1 \dot{-} a \dot{-} b)$ $(a, b \in B)$ then $\langle B; \oplus, \dot{-}, 0\rangle$ is a pocrim and, in fact, a hoop. (More strongly, if we define $\bar{a} = 1 \dot{-} a$ and $a \cdot b = a \dot{-} \bar{b}$ then $\langle B; \oplus, \cdot, \bar{\ }, 0, 1\rangle$ is an MV-algebra in the sense of Chang [Cha58]: see [Mun86].) Thus, $\mathbf{A}^- \in \mathbf{S}(\mathscr{H}^-)$ and so $\mathbf{A}$ satisfies (H). It follows from (4) that $\mathbf{A}$ satisfies $z \dot{-} [x \oplus (y \dot{-} x)] \approx z \dot{-} [y \oplus (x \dot{-} y)]$. Replacing z by $x \oplus (y \dot{-} x)$, we find that $\mathbf{A}$ satisfies $0 \approx [x \oplus (y \dot{-} x)] \dot{-} [y \oplus (x \dot{-} y)]$ and, by symmetry, $0 \approx [y \oplus (x \dot{-} y)] \dot{-} [x \oplus (y \dot{-} x)]$. By (5), $\mathbf{A}$ satisfies (17), i.e., $\mathbf{A}$ is a hoop. ∎

This means that commutative pocrims coincide with "Wajsberg hoops" in the sense of [BP94], i.e, with the hoop subreducts of MV-algebras. We shall now consider some pocrim varieties not consisting entirely of hoops.

EXAMPLE IV (The varieties of n-potent pocrims). Given $n \in \omega$, a pocrim is said to be $(n+1)$-*potent* if it satisfies any one of the following equivalent identities, whose equivalence is proved in [BR95, Proposition 13 and Lemma 14]:

$$(n+1)x \approx nx; \tag{M_n}$$

$$x \dot{-} (n+1)y \approx x \dot{-} ny, \tag{E_n}$$

$$x \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x) \approx y \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x). \tag{$(E_n)'$}$$

Note that for any integers $i, j, m, n \in \omega$, the identity

$$x \dot{-} i(x \dot{-} y) \dot{-} j(y \dot{-} x) \approx y \dot{-} m(x \dot{-} y) \dot{-} n(y \dot{-} x) \tag{C^{ij}_{mn}}$$

is of the form of (15). Since $(E_n)'$ is a special case of this, it follows from Theorem 3.3 that for each $n \in \omega$, the class $\mathscr{M}_n$ of all $(n+1)$-potent pocrims is a congruence 3-permutable variety for which the identities (1), (2), (3), (4), and any one of (M_n), (E_n), and $(E_n)'$ serve as an equational base, and that the class $\mathbf{S}(\mathscr{M}_n^-)$ is also a variety (of BCK-algebras). The variety of BCK-algebras axiomatized by the identities (1), (2), (3), and (E_n) (or equivalently $(E_n)'$) is usually denoted by $\mathscr{E}_n$. Of course, $\mathscr{M}_1$ and $\mathscr{E}_1$ are the aforementioned varieties of Brouwerian semilattices and of Hilbert algebras; we have mentioned that $\mathbf{S}(\mathscr{M}_n^-) \subseteq \mathscr{E}_n$; whether or not equality holds for $n > 1$ is an open problem. We have also mentioned that Brouwerian semilattices are congruence permutable but we shall show in Section 6 that for each $n > 1$, the variety $\mathscr{M}_n$ is not congruence permutable (hence $\mathscr{M}_n \nsubseteq \mathscr{H}$).

It is known and quite easy to see that every finite BCK-algebra (resp. pocrim) satisfies (E_n) for some $n \in \omega$ and therefore generates a variety of

BCK-algebras (resp. pocrims). Also, if $\mathbf{A}$ is a subdirectly irreducible algebra in any of the varieties $\mathscr{E}_n$ then the poset $\langle A; \leq \rangle$ has a unique atom [Pała]. Of course, $\mathscr{E}_0$ is the trivial variety; the varieties $\mathscr{E}_n$ and $\mathscr{M}_n$ are locally finite iff $n \leq 1$ [Die66, Dyr87]. The chains of varieties $\mathscr{M}_n$, $n \in \omega$, and $\mathscr{E}_n$, $n \in \omega$, are strictly increasing [Cor82a]. It is easy to see that the equationally complete variety $\mathbf{ISPP_u}(\boldsymbol{\omega})$ meets each of the varieties $\mathscr{M}_n$ trivially.

Recall that a variety $\mathscr{V}$ is said to have *definable principal congruences* (DPC) if there is a 4-ary first-order formula φ in the language of $\mathscr{V}$ such that for all $\mathbf{A} \in \mathscr{V}$ and all $a, b, c, d \in A$,

$$c, d \in \Theta^{\mathbf{A}}(a, b) \quad \Leftrightarrow \quad \mathbf{A} \vDash \varphi(a, b, c, d).$$

If, moreover, φ may be chosen to be a (finite) conjunction of equations then we say that $\mathscr{V}$ has *equationally definable principal congruences* (EDPC). All varieties with a ternary deductive term have EDPC [BP94, Corollary 2.5]. Here, a *ternary deductive term* (briefly, a *TD term*) for a variety $\mathscr{V}$ means a ternary term p of the language of $\mathscr{V}$ such that $\mathscr{V}$ satisfies $p(x, x, z) \approx z$ and for all $\mathbf{A} \in \mathscr{V}$ and $a, b, c, d \in A$, if $(c, d) \in \Theta^{\mathbf{A}}(a, b)$ then $p^{\mathbf{A}}(a, b, c) = p^{\mathbf{A}}(a, b, d)$. Such a term p is called *commutative* if $\mathscr{V}$ also satisfies $p(x, y, p(x', y', z)) \approx p(x', y', p(x, y, z))$; it is called *regular* if for some constant term e of $\mathscr{V}$ and all $\mathbf{A} \in \mathscr{V}$, we have $(a, b) \in \Theta^{\mathbf{A}}(p^{\mathbf{A}}(a, b, \mathrm{e}^{\mathbf{A}}), \mathrm{e}^{\mathbf{A}})$ whenever $a, b \in A$. The varieties $\mathscr{M}_n$ and $\mathscr{E}_n$ have EDPC; in fact, the term $p(x, y, z) = z \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x)$ is a commutative (but not regular) TD term for both of them [Pałb] and it is not hard to show (e.g., using Lemma 4.3 below) that $q(x, y, z) = z \oplus n(x \dot{-} y) \oplus n(y \dot{-} x)$ is a commutative regular TD term for $\mathscr{M}_n$, with respect to the constant 0. The following is a converse of these results:

THEOREM 4.2. *The following conditions on a variety $\mathscr{V}$ of pocrims* (*resp. of BCK-algebras*) *are equivalent*:

1. *$\mathscr{V}$ has EDPC*;
2. *$\mathscr{V}$ has DPC*;
3. *$\mathscr{V}$ is a subvariety of $\mathscr{M}_n$* (*resp. of $\mathscr{E}_n$*) *for some $n \in \omega$.*

Proof. Statements 1 and 2 are equivalent for any congruence distributive variety with the congruence extension property [BP88, Corollary 4.8], so their equivalence in the present context follows from Proposition 3.1. It remains only to prove that statement 2 implies statement 3.

Suppose $\mathscr{V}$ has DPC. In view of the isomorphism between the congruence and ideal lattices of members of $\mathscr{V}$ (Proposition 3.1), this implies the existence of a binary first-order formula φ in the language of $\mathscr{V}$ such that for all $\mathbf{A} \in \mathscr{V}$ and all $a, b \in A$,

$$a \in \langle b \rangle_{\mathbf{A}} \quad \Leftrightarrow \quad \mathbf{A} \vDash \varphi(a, b).$$

By the characterization of ideal generation given earlier, this clearly means that

$$\mathbf{A} \vDash \varphi(a, b) \quad \Leftrightarrow \quad (\exists n \in \omega)(a \dot{-} nb = 0).$$

We claim, however, that there is a fixed $n \in \omega$ such that

$$a \in \langle b \rangle_{\mathbf{A}} \quad \Leftrightarrow \quad a \dot{-} nb = 0.$$

For if not, then for every $n \in \omega$, there exist $\mathbf{A}_n \in \mathscr{V}$ and $a_n, b_n \in A$ such that $a_n \in \langle b_n \rangle_{\mathbf{A}_n}$ but $a_n \dot{-} nb_n \neq 0$. Let U be a free ultrafilter over ω, let $\mathbf{A}$ be the ultraproduct $\prod_{n \in \omega} \mathbf{A}_n / U$ and consider $\bar{a} = (a_0, a_1, \ldots)$, $\bar{b} = (b_0, b_1, \ldots) \in \Pi_{n \in \omega} A_n$. Since $\mathbf{A}_n \vDash \varphi(a_n, b_n)$ for each $n \in \omega$, we have that $\mathbf{A} \vDash \varphi(\bar{a}/U, \bar{b}/U)$ and so $\bar{a}/U \in \langle \bar{b}/U \rangle_{\mathbf{A}}$. But for each $n \in \omega$, $\{i \in \omega\colon \bar{a}(i) \dot{-} n\bar{b}(i) \neq 0\} \supseteq \{i \in \omega\colon i \geq n\} \in U$, so $\bar{a}/U \dot{-} n\bar{b}/U \neq 0$, contradicting $\bar{a}/U \in \langle \bar{b}/U \rangle_{\mathbf{A}}$. This vindicates the above claim. If n is as in the claim, then $\mathscr{V} \subseteq \mathscr{M}_n$ (resp. $\mathscr{E}_n$). For otherwise, we may choose $\mathbf{B} \in \mathscr{V}$ and $a, b \in B$ such that $a \dot{-} nb \neq a \dot{-} (n+1)b$, whence

$$[a \dot{-} (a \dot{-} (n+1)b)] \dot{-} nb = (a \dot{-} nb) \dot{-} (a \dot{-} (n+1)b) \neq 0.$$

Thus, if $c = a \dot{-} (a \dot{-} (n+1)b)$ then $c \notin \langle b \rangle_{\mathbf{B}}$. But this contradicts the fact that $c \dot{-} (n+1)b = 0$, completing the proof. ∎

LEMMA 4.3. *Let* $\mathbf{A}$ *be a BCK-algebra in* $\mathscr{E}_n$ $(n \in \omega)$ *and let I be the ideal of* $\mathbf{A}$ *generated by* $b_1, \ldots, b_m \in A$.

(i) $I = \{a \in A\colon a \dot{-} nb_1 \dot{-} \cdots \dot{-} nb_m = 0\}$. *In particular, if* $\mathbf{A}$ *is the* $\langle \dot{-}, 0 \rangle$*-reduct* $\mathbf{B}^-$ *of a pocrim* $\mathbf{B}$ (*e.g., if* $\mathbf{B} \in \mathscr{M}_n$) *then* $I = \{a \in B\colon a \dot{-} nb = 0\} = \{a \in B\colon a \leq nb\}$ *is the principal ideal of* $\mathbf{B}$ *generated by* $b = b_1 \oplus \cdots \oplus b_m$. *Thus, every finitely generated congruence of an algebra in* $\mathscr{M}_n$ *is principal.*

(*ii*) *The map* $f\colon A \to A$ *defined by* $f(a) = a \dot{-} nb_1 \dot{-} \cdots \dot{-} nb_m$ *is a homomorphism of BCK-algebras and* $0/\ker f = I$.

Proof. (i) This follows from the characterization of ideals generated by subsets given in Section 2 and the identities (E_n), (4) and (8).

(ii) Let $a, c \in A$. Abbreviating $a \dot{-} nb_1 \dot{-} \cdots \dot{-} nb_m$ as $a \dot{-} \Sigma_i nb_i$, we have $c \geq c \dot{-} \Sigma_i nb_i$, hence

$$\begin{aligned}(a \dot{-} c) \dot{-} \sum_i nb_i &= a \dot{-} \sum_i nb_i \dot{-} c \qquad \text{(by (8))} \\ &\leq \Big(a \dot{-} \sum_i nb_i\Big) \dot{-} \Big(c \dot{-} \sum_i nb_i\Big).\end{aligned}$$

Conversely,

$$\begin{aligned}
&\Big(a \dot{-} \sum_i nb_i\Big) \dot{-} \Big(c \dot{-} \sum_i nb_i\Big) \\
&\qquad = a \dot{-} \sum_i nb_i \dot{-} \sum_i nb_i \dot{-} \Big(c \dot{-} \sum_i nb_i\Big) \qquad (\text{by } (E_n) \text{ and } (8)) \\
&\qquad = a \dot{-} \sum_i nb_i \dot{-} \Big(c \dot{-} \sum_i nb_i\Big) \dot{-} \sum_i nb_i \qquad (\text{by } (8)) \\
&\qquad \leq (a \dot{-} c) \dot{-} \sum_i nb_i \\
&\qquad\qquad \Big(\text{by isotonicity of } x \dot{-} \sum_i nb_i \text{ and repeated application of } (9)\Big),
\end{aligned}$$

as required. By (i) and Proposition 3.1, $0/\ker f = I$. ∎

THEOREM 4.4. *For each $n \in \omega$, every subquasivariety of $\mathscr{E}_n$ is a variety.*

Proof. Let $\mathscr{Q}$ be a subquasivariety of $\mathscr{E}_n$. We have to show that $\mathbf{H}(\mathscr{Q}) \subseteq \mathscr{Q}$. Let $\mathbf{A} \in \mathscr{Q}$ and let $g\colon \mathbf{A} \to \mathbf{B}$ be a surjective homomorphism. Suppose $\mathbf{B} \notin \mathscr{Q}$. Then there is a quasi-identity

$$\mathop{\&}_{i=1}^{k} s_i(\vec{x}) \approx t_i(\vec{x}) \quad \to \quad s(\vec{x}) \approx t(\vec{x}) \tag{α}$$

in the language of *BCK*-algebras such that $\mathscr{Q} \vDash (\alpha)$ and $\mathbf{B} \nvDash (\alpha)$. Choose $\vec{c}$ (abbreviating $c_1, \ldots, c_n \in B$) such that $s_i^{\mathbf{B}}(\vec{c}) = t_i^{\mathbf{B}}(\vec{c})$ for $i = 1, \ldots, k$, but $s^{\mathbf{B}}(\vec{c}) \neq t^{\mathbf{B}}(\vec{c})$. Let $\vec{a}$ abbreviate a sequence of preimages $a_1, \ldots, a_n$ of $c_1, \ldots, c_n$ under g. Set $\theta = \Theta^{\mathbf{A}}((s_1^{\mathbf{A}}(\vec{a}), t_1^{\mathbf{A}}(\vec{a})), \ldots, (s_k^{\mathbf{A}}(\vec{a}), t_k^{\mathbf{A}}(\vec{a})))$. Then $\theta \subseteq \ker g$ and $(s^{\mathbf{A}}(\vec{a}), t^{\mathbf{A}}(\vec{a})) \notin \theta$, so $\mathbf{A}/\theta \nvDash (\alpha)$.

Since $\mathbf{A}$ belongs to the (0-regular) subvariety $\mathscr{E}_n$ of $\mathscr{BCK}$, it follows that $\theta = \varphi_I$, where $I = 0/\theta \in \operatorname{Id}\mathbf{A}$ (Proposition 3.1). Thus, $\mathbf{A}/\theta = \mathbf{A}/I$, where I is finitely generated, say $I = \langle b_1, \ldots, b_m \rangle_{\mathbf{A}}$. By the previous lemma, the map $f\colon \mathbf{A} \to \mathbf{A}$ defined by $f(d) = d \dot{-} nb_1 \dot{-} \cdots \dot{-} nb_m$ is a homomorphism and $0/\ker f = I$, so $\ker f = \theta$, by 0-regularity. By the homomorphism theorem, $\mathbf{A}/\theta = \mathbf{A}/\ker f \cong f[\mathbf{A}] \in \mathbf{S}(\mathbf{A}) \subseteq \mathscr{Q}$, so $\mathbf{A}/\theta \vDash (\alpha)$, a contradiction. ∎

Our argument does not extend to a proof that every subquasivariety of $\mathscr{M}_n$ is a variety, since it is easy to find examples of algebras in $\mathscr{M}_2$ for which maps f as in Lemma 4.3(ii) are not $\oplus$-homomorphisms. In fact, a subquasivariety of $\mathscr{M}_3$ which is not a variety may be constructed as follows. Let $A = \{(n, i) \in \omega \times \omega\colon\ n \leq 2,\ i \leq 1\}$ and define a linear order on A by

$(n,i) < (m,j)$ iff either $n < m$ or both $n = m$ and $i < j$. Define a binary operation on A as follows, where $x, y \in A$ and $a \in \{0,1,2\}$: $x \oplus y = y \oplus x$, $x \oplus (0,0) = x$, $(0,1) \oplus (a,1) = (a,1) = (0,1) \oplus (a,0)$, $(1,0) \oplus (1,0) = (2,0)$, and $x \oplus y = (2,1)$ in all remaining cases. Then $\langle A; \oplus, (0,0); \leq \rangle$ is a commutative integral pomonoid which must be residuated, since it is well ordered. Let **A** be the resultant pocrim, which is clearly in $\mathscr{M}_3$. Note that **A** has an ideal $I = \{(0,0),(0,1)\}$ such that $\mathbf{A}/I$ is isomorphic to the unique pocrim $\mathbf{B}_3$ on $\{0,1,2\}$ (ordered naturally) in which $1 \oplus 1 = 2$. Let $\mathscr{Q} \subseteq \mathscr{M}_3$ be the quasivariety generated by **A**. Since **A** is finite, $\mathscr{Q}_{\mathrm{SI}} \subseteq \mathbf{IS}(\mathbf{A})$. But $\mathbf{B}_3$ is simple and it is easily checked that $\mathbf{B}_3$ is not isomorphic to any subalgebra of **A**, so $\mathscr{Q}$ is not a variety.

Our investigation of the varieties $\mathscr{M}_n$ and $\mathscr{E}_n$ will continue in Sections 6 and 7. Here we give further examples of pocrim varieties which are not contained in the variety of hoops, nor in any of the varieties $\mathscr{M}_n$.

EXAMPLE V. For any $i, j, m, n \in \omega$, the class $\mathscr{K}_{mn}^{ij}$ of all pocrims satisfying

$$x \oplus i(x \dot{-} y) \oplus j(y \dot{-} x) \approx y \oplus m(x \dot{-} y) \oplus n(y \dot{-} x) \qquad \left(K_{mn}^{ij}\right)$$

is a congruence 3-permutable variety. This follows by setting $n = 2$, $t_1(x,y,z,u,v,w) = x \oplus iv \oplus jw$, and $t_2(x,y,z,u,v,w) = y \oplus mz \oplus nu$ in Theorem 3.2. It would be interesting to know whether all of these varieties satisfy the conditions of Theorem 3.3(i). It can be shown that $\mathscr{K}_{21}^{12}$ is not congruence permutable.

EXAMPLE VI. For each $n \in \{-1\} \cup \omega$, we define, inductively, a $\langle \dot{-}, 0\rangle$-term $j_n(x,y)$ by

$$\begin{aligned} j_{-1}(x,y) &= x, \\ j_{2n}(x,y) &= y \dot{-} (y \dot{-} (j_{2n-1}(x,y))), \\ j_{2n+1}(x,y) &= x \dot{-} (x \dot{-} (j_{2n}(x,y))), \end{aligned}$$

and an identity:

$$j_n(x,y) \approx j_n(y,x). \qquad (J_n)$$

These identities were introduced in [BR95]. As an easily application of Theorem 3.3, the class $\mathscr{J}_n$ of all pocrims satisfying (J_n) is a (congruence 3-permutable) variety for each $n \in \omega$. Moreover, it follows from [BR95, Lemma 17] that every finite pocrim lies in some $\mathscr{J}_n$ and that the sequence of varieties $\mathscr{J}_n$, $n \in \omega$, is an ascending chain. We shall need these varieties in Section 8, where we shall show in particular that this chain is strictly ascending (Corollary 8.11). The reader should be warned that in [BR95], $\mathscr{J}_n$ denotes the variety of *BCK*-algebras (rather than pocrims) satisfying (J_n).

The identity (J_0) is just the identity (T) defining commutative pocrims and *BCK*-algebras. We have mentioned that subdirectly irreducible commutative pocrims are linearly ordered. Since there exist simple commutative *BCK*-algebras which are not linearly ordered (e.g., [Stu82, Example 2e]), it follows that $\mathbf{S}(\mathscr{J}_0^-)$ is properly contained in the variety of all commutative *BCK*-algebras. Proposition 4.1 says that $\mathscr{J}_0 \subseteq \mathscr{H}$. Note that (J_1) is just Cornish's identity (J), mentioned in Example III above, and so $\mathscr{H} \subseteq \mathscr{J}_1$. Since the pocrim $\boldsymbol{\omega}$ satisfies (T), we have $\mathbf{ISPP_u}(\boldsymbol{\omega}) \subseteq \mathscr{J}_0$. The last three inclusions are all sharp, as can be deduced by considering, respectively, the example in [Tra88, **3**, Remark 3], the four-element pocrim $\mathbf{A}$ to be discussed in the sequel to Theorem 6.1 and the two-element pocrim $\mathbf{C}_2$.

5. ORDINAL SUMS

A natural technique for generating pocrims (and hence *BCK*-algebras) is the "ordinal sum" construction. Given pocrims $\mathbf{A} = \langle A; \oplus, \dot{-}, 0^{\mathbf{A}} \rangle$ and $\mathbf{B} = \langle B; \oplus, \dot{-}, 0^{\mathbf{B}} \rangle$ with $A \cap B \subseteq \{0^{\mathbf{A}}\}$, we define the *ordinal sum* $\mathbf{A} + \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ to be the pocrim with universe $A + B = A \cup (B \setminus \{0^{\mathbf{B}}\})$, the restrictions of whose partial order $\leq$ (resp. monoid operation $\oplus$) to A and to $B \setminus \{0^{\mathbf{B}}\}$ coincide with the original orders (resp. monoid operations) on $\mathbf{A}$ and $\mathbf{B}$, having the additional property that $a < b$ (resp. $a \oplus b = b$) whenever $a \in A$, $b \in B \setminus \{0^{\mathbf{B}}\}$. Consequently $0^{\mathbf{A}}$ is the least element of $\mathbf{A} + \mathbf{B}$ and the restriction to A of the residuation operation $\dot{-}$ on $\mathbf{A} + \mathbf{B}$ coincides with residuation in $\mathbf{A}$. For $a \in A$ and $b, b' \in B \setminus \{0^{\mathbf{B}}\}$, we have $a \dot{-} b = 0^{\mathbf{A}}$ and $b \dot{-} a = b$, while $b \dot{-} b'$ is as in $\mathbf{B}$ unless $b \leq b'$, in which case, $b \dot{-} b' = 0^{\mathbf{A}}$.

If $A \cap B \not\subseteq \{0^{\mathbf{A}}\}$, we abuse notation by writing $\mathbf{A} + \mathbf{B}$ for the ordinal sum $\mathbf{A} + \mathbf{B}'$, where $\mathbf{B}'$ is any algebra with $\mathbf{B}' \cong \mathbf{B}$ and $A \cap B' \subseteq \{0^{\mathbf{A}}\}$.

If $\mathbf{A}$ and $\mathbf{B}$ are *BCK*-algebras, we define the ordinal sum $\mathbf{A} + \mathbf{B}$ similarly, omitting the specification of $\oplus$ and using the above characterization of residuation as the definition of $\dot{-}$. It is well known that $\mathbf{A} + \mathbf{B} \in \mathscr{BCK}$ in this case. Moreover, for pocrims $\mathbf{A}$ and $\mathbf{B}$, we have $(\mathbf{A} + \mathbf{B})^- = \mathbf{A}^- + \mathbf{B}^-$.

If $\mathbf{A}$ and $\mathbf{B}$ are both pocrims or both *BCK*-algebras, then the ideals of $\mathbf{A} + \mathbf{B}$ are just the ideals of $\mathbf{A}$, together with all sets of the form $A \cup (J \setminus \{0^{\mathbf{B}}\})$, where J is an ideal of $\mathbf{B}$. Note also that both $\mathbf{A}$ and $\mathbf{B}$ are isomorphic to subalgebras of $\mathbf{A} + \mathbf{B}$, and that $(\mathbf{A} + \mathbf{B})/A \cong \mathbf{B}$.

A variety $\mathscr{V}$ of pocrims or of *BCK*-algebras is said to be *closed under ordinal sums* if $\mathbf{A} + \mathbf{B} \in \mathscr{V}$ whenever $\mathbf{A}, \mathbf{B} \in \mathscr{V}$ and $A \cap B \subseteq \{0^{\mathbf{A}}\}$. The following proposition is easy to prove and shows that closure under ordinal sums is a very common property.

PROPOSITION 5.1. *The following varieties are closed under ordinal sums*: $\mathscr{H}$, $\mathscr{M}_n$ $(n \in \omega)$, $\mathscr{K}_{(m+1)n}^{m(n+1)}$ $(m, n \in \omega)$, $\mathscr{E}_n$ $(n \in \omega)$,

$$\left\{\mathbf{A} \in \mathscr{M} \ (\textit{resp}.\ \mathscr{BCK})\colon \mathbf{A} \vDash \left(C_{mn}^{ij}\right)\right\} \qquad (1 \leq i, j, m, n \in \omega),$$

$$\{\mathbf{A} \in \mathscr{M} \ (\textit{resp}.\ \mathscr{BCK})\colon \mathbf{A} \vDash (J_n)\} \qquad (1 \leq n \in \omega),$$

and any variety of the form $\mathbf{S}(\mathscr{V}^-)$, *where* $\mathscr{V}$ *is a variety of pocrims that is closed under ordinal sums*.

On the other hand, $\mathscr{G}$, $\mathbf{ISPP_u}(\omega)$, $\mathscr{J}_0$, and the variety of commutative *BCK*-algebras are not closed under ordinal sums.

The wide applicability of this closure property leads us to consider the following recursive construction. For the rest of this section, $\mathscr{W}$ shall denote *any variety of pocrims or of BCK-algebras that is closed under ordinal sums* and $\mathscr{V}$ shall denote *an arbitrary subvariety of* $\mathscr{W}$.

We define classes $\mathscr{K}^i = \mathscr{K}^i(\mathscr{V})$ $(i \in \omega)$ and varieties $\mathscr{V}^i$ and $\mathscr{V}^+$ $(i \in \omega \cup \{-1\})$ as follows:

$$\mathscr{V}^{-1} \text{ is the trivial subvariety of } \mathscr{V}; \qquad \mathscr{K}^0 = \mathscr{V}_{\mathrm{SI}}, \qquad \mathscr{V}^0 = \mathscr{V};$$

$$\mathscr{K}^m = \left\{\mathbf{A} + \mathbf{B}\colon \mathbf{A} \in \mathscr{V}_{\mathrm{SI}} \text{ and } \mathbf{B} \in \mathscr{V}^{m-1}\right\}, \qquad \mathscr{V}^m = \mathbf{HSP}(\mathscr{K}^m), \qquad 0 < m \in \omega;$$

$$\mathscr{V}^+ \text{ is the supremum in } \mathbf{L}^v(\mathscr{W}) \text{ of the set } \{\mathscr{V}^k\colon k \in \omega\}, \text{ i.e., } \mathscr{V}^+ = \mathbf{HSP}\Bigl(\bigcup_{k \in \omega} \mathscr{V}^k\Bigr).$$

Of course, $\mathscr{V}^{i-1} \subseteq \mathscr{V}^i \subseteq \mathscr{V}^+ \subseteq \mathscr{W}$ for all $i \in \omega$. The only role that $\mathscr{W}$ will play is that its existence ensures that $\mathscr{V}^+$ consists of pocrims or *BCK*-algebras and is therefore congruence distributive; in particular, Jónsson's theorem is applicable to $\mathscr{V}^+$. Actually, every candidate for $\mathscr{V}$ known to us is contained in some variety of pocrims or *BCK*-algebras that is closed under ordinal sums. Consequently, although we have not proved the existence of $\mathscr{W}$ to be a universally satisfied requirement, we regard it as a weak assumption.

LEMMA 5.2. *Let* $\mathscr{V}'$ *be a subvariety of* $\mathscr{W}$ *with* $\mathscr{V} \subseteq \mathscr{V}'$ *and let* $\{\mathbf{A}_i\colon i \in I\}$ *and* $\{\mathbf{B}_i\colon i \in I\}$ *be families of algebras in* $\mathscr{V}$ *and* $\mathscr{V}'$, *respectively*. *Then any* $\mathbf{C} \in \mathbf{P_u}\{\mathbf{A}_i + \mathbf{B}_i\colon i \in I\}$ *is an ordinal sum* $\mathbf{A} + \mathbf{B}$, *with* $\mathbf{A} \in \mathbf{P_u}\{\mathbf{A}_i\colon i \in I\} \subseteq \mathscr{V}$ *and* $\mathbf{B} \in \mathscr{V}'$.

Proof. Assume first that $\mathscr{W} \subseteq \mathscr{M}$. Consider the enlargement $\langle \oplus, \dot{-}, 0; P \rangle$ of the language of pocrims by a unary predicate symbol P. Let Σ

be a set of identities axiomatizing $\mathscr{V}'$. Let Γ be the union of Σ and the set consisting of the following first-order sentences ($x \leq y$ abbreviates $x \dot{-} y \approx 0$):

$$\forall x\, \forall y \qquad [(P(x) \,\&\, P(y)) \rightarrow P(x \oplus y)]$$
$$\forall x\, \forall y \qquad [(P(x) \,\&\, P(y)) \rightarrow P(x \dot{-} y)]$$
$$P(0)$$
$$\forall x\, \forall y \qquad [((\neg P(x)) \,\&\, \neg P(y)) \rightarrow \neg P(x \oplus y)]$$
$$\forall x\, \forall y \qquad [((\neg P(x)) \,\&\, (\neg P(y)) \,\&\, x \not\leq y) \rightarrow \neg P(x \dot{-} y)]$$
$$\forall x\, \forall y \qquad [(P(x) \,\&\, \neg P(y)) \rightarrow (x \leq y \,\&\, y \dot{-} x \approx y)]$$
$$\forall x\, \forall y \qquad [(P(x) \,\&\, \neg P(y)) \rightarrow x \oplus y \approx y].$$

Let $\mathbf{C}_i = \mathbf{A}_i + \mathbf{B}_i$ for each $i \in I$. Each $\mathbf{C}_i$ becomes a model of Γ if we interpret $P^{\mathbf{C}_i}$ as A_i. If $J \subseteq I$ and $\mathbf{C} = \prod_{i \in J} \mathbf{C}_i / U$, where U is an ultrafilter over J then $\mathbf{C} \models \Gamma$. If we define $A = \{a \in C\colon P^{\mathbf{C}}(a)\}$ and $B = (C \setminus A) \cup \{0^{\mathbf{C}}\}$ then A and B are universes of subalgebras $\mathbf{A}, \mathbf{B}$ of $\mathbf{C} \in \mathscr{V}'$ and $\mathbf{C} = \mathbf{A} + \mathbf{B}$. If $c/U \in C$, where $c \in \prod_{i \in J} C_i$ then either $K = \{i \in J\colon c(i) \in A_i\}$ or $J \setminus K$ is an element of U, so $\mathbf{A} = \prod_{i \in J} \mathbf{A}_i / U \in \mathscr{V}$.

If $\mathscr{W} \subseteq \mathscr{BCK}$, we delete all sentences involving $\oplus$ from Γ and use the same argument. ∎

PROPOSITION 5.3. *For each $m \in \omega$, the subdirectly irreducible algebras in the variety $\mathscr{V}^m$ are just those algebras isomorphic to ordinal sums $\mathbf{A} + \mathbf{B}$, where $\mathbf{A} \in \mathscr{V}_{\mathrm{SI}}$ and $\mathbf{B} \in \mathscr{V}^{m-1}$.*

Proof. Clearly any ordinal sum $\mathbf{A} + \mathbf{B}$ with $\mathbf{A} \in \mathscr{V}_{\mathrm{SI}}$ is subdirectly irreducible; its least nonzero ideal coincides with that of $\mathbf{A}$. The converse is proved by induction on m and is clearly true when $m = 0$. Let $m > 0$. Recall that $\mathscr{W}$ is congruence distributive (Proposition 3.1) so by Jónsson's theorem, $\mathscr{V}^m_{\mathrm{SI}} \subseteq \mathbf{HSP_u}(\mathscr{K}^m)$. By the previous lemma, $\mathbf{P_u}(\mathscr{K}^m)$ consists of ordinal sums $\mathbf{A} + \mathbf{B}$ with $\mathbf{A} \in \mathscr{V}$ and $\mathbf{B} \in \mathscr{V}^{m-1}$, where $\mathbf{A}$ is nontrivial. Since $\mathscr{V}$ and $\mathscr{V}^{m-1}$ are varieties, every member of $\mathbf{SP_u}(\mathscr{K}^m)$ is either in $\mathscr{V}^{m-1}$ or is also an ordinal sum $\mathbf{A} + \mathbf{B}$ with $\mathbf{A} \in \mathscr{V}$, $\mathbf{B} \in \mathscr{V}^{m-1}$ ($\mathbf{A}$ nontrivial). Likewise, the same is true of members of $\mathbf{HSP_u}(\mathscr{K}^m)$. Thus, if $\mathbf{C} \in \mathscr{V}^m_{\mathrm{SI}}$ then either $\mathbf{C} \in \mathscr{V}^{m-1}_{\mathrm{SI}}$ or $\mathbf{C} = \mathbf{D} + \mathbf{E}$ with $\mathbf{D} \in \mathscr{V}$ (hence $\mathbf{D} \in \mathscr{V}_{\mathrm{SI}}$) and $\mathbf{E} \in \mathscr{V}^{m-1}$. In the latter case, $\mathbf{C}$ has the desired form. The same conclusion follows in the former case, by the induction hypothesis. ∎

Recall that a variety is *semisimple* if all of its subdirectly irreducible members are simple.

COROLLARY 5.4. *If $\mathscr{V}$ is a semisimple variety then for each $m \in \omega$,*

$$\mathscr{V}_{\mathrm{SI}}^{m} = \{\mathbf{A} + \mathbf{B} \colon \mathbf{A} \in \mathscr{V} \text{ is simple and } \mathbf{B} \in \mathscr{V}^{m-1}\}.$$

PROPOSITION 5.5. *For each $r \in \omega$, every r-generated algebra in $\mathscr{V}^{+}$ is a member of $\mathscr{V}^{r-1}$.*

Proof. The proof is by induction on r. The result is clearly true when $r = 0$, so assume $r > 0$ and consider a subdirectly irreducible r-generated algebra $\mathbf{C} \in \mathscr{V}^{m}$, where $m \in \omega$. By Proposition 5.3, $\mathbf{C} = \mathbf{A} + \mathbf{B}$, where $\mathbf{A} \in \mathscr{V}_{\mathrm{SI}}$ and $\mathbf{B} \in \mathscr{V}^{m-1}$. Since $\mathbf{A}$ is nontrivial and $B \cup \{0^{\mathbf{A}}\}$ is a subuniverse of $\mathbf{C}$, at least one generator of $\mathbf{C}$ lies in $A \setminus \{0^{\mathbf{A}}\}$, so the algebra $\mathbf{B} \cong \mathbf{C}/A$ is $(r-1)$-generated. By the induction hypothesis, $\mathbf{B} \in \mathscr{V}^{r-2}$ so $\mathbf{C} \in \mathscr{V}^{r-1}$. Now the $\mathscr{V}^{+}$-free algebra on r free generators is a subdirect product of subdirectly irreducible algebras in $\bigcup_{k \in \omega} \mathscr{V}^{k}$, each on at most r generators, and this completes the proof. ∎

COROLLARY 5.6. *For any variety $\mathscr{V}$ of pocrims* (*resp. BCK-algebras*), *if there is a variety $\mathscr{U}$ of pocrims* (*resp. BCK-algebras*) *such that $\mathscr{V}$ is a subvariety of $\mathscr{U}$ and $\mathscr{U}$ is closed under ordinal sums, then the smallest such variety $\mathscr{U}$ is $\mathscr{V}^{+}$.*

Proof. Any variety $\mathscr{U}$ as described in the statement of the corollary must clearly contain $\mathscr{V}^{+}$. It remains to show that $\mathscr{V}^{+}$ is closed under ordinal sums. Let $\mathbf{A}, \mathbf{B} \in \mathscr{V}^{+}$. In showing that $\mathbf{A} + \mathbf{B} \in \mathscr{V}^{+}$, we may assume without loss of generality that $\mathbf{A}$ and $\mathbf{B}$ are finitely generated (since a finitely generated subalgebra of an ordinal sum is an ordinal sum of finitely generated subalgebras of its summands). By the previous proposition, it suffices to show that for any $k, m \in \omega$, if $\mathbf{A} \in \mathscr{V}^{k}$ and $\mathbf{B} \in \mathscr{V}^{m}$ then $\mathbf{A} + \mathbf{B} \in \mathscr{V}^{n}$ for some $n \in \omega$; the proof is by induction on k. Observe that for any k, the claim is true if all subdirectly irreducible homomorphic images of $\mathbf{A} + \mathbf{B}$ lie in some $\mathscr{V}^{n}$. But such images either belong to $\mathscr{V}^{m}$ or are themselves ordinal sums with lower summand in $\mathscr{V}_{\mathrm{SI}}^{k}$, so we need only show that if $\mathbf{A} \in \mathscr{V}_{\mathrm{SI}}^{k}$ and $\mathbf{B} \in \mathscr{V}^{m}$ then some $\mathscr{V}^{n}$ contains $\mathbf{A} + \mathbf{B}$. This is true when $k = 0$ (take $n = m + 1$). For $k > 0$, by Proposition 5.3, we have $\mathbf{A} = \mathbf{C} + \mathbf{D}$ for some $\mathbf{C} \in \mathscr{V}_{\mathrm{SI}}$ and $\mathbf{D} \in \mathscr{V}^{k-1}$, and $\mathbf{A} + \mathbf{B} \cong \mathbf{C} + (\mathbf{D} + \mathbf{B})$. By the induction hypothesis, $\mathbf{D} + \mathbf{B} \in \mathscr{V}^{n}$ for some $n \in \omega$, so $\mathbf{A} + \mathbf{B} \in \mathscr{V}^{n+1}$. ∎

We show that certain important varietal properties are preserved in the passage from $\mathscr{V}$ to $\mathscr{V}^{+}$. The next two theorems generalize results obtained for hoops in [BF].

THEOREM 5.7. *Let $\mathscr{V}$ be a locally finite subvariety of some variety of pocrims or BCK-algebras that is closed under ordinal sums. Then $\mathscr{V}^{+}$ is also locally finite.*

Proof. By assumption, $\mathscr{V}$ contains, for each $r \in \omega$, only finitely many subdirectly irreducible algebras on at most $r+1$ generators, all of which are finite. Suppose $k > 0$ and $r \in \omega$. An $(r+1)$-generated algebra $\mathbf{E} \in \mathscr{V}^k$ is a subdirect product of algebras $\mathbf{C}_i \in \mathscr{V}^k_{\mathrm{SI}}$, each on at most $r+1$ generators. By Proposition 5.3, these $\mathbf{C}_i$ have the form $\mathbf{A}_i + \mathbf{B}_i$, where $\mathbf{A}_i \in \mathscr{K}^0 = \mathscr{V}_{\mathrm{SI}}$ and $\mathbf{B}_i \in \mathscr{V}^{k-1}$. If $\mathbf{B}_i$ is trivial then $\mathbf{C}_i \in \mathscr{K}^0$ is a finite (subdirectly irreducible) algebra and there are at most finitely many (nonisomorphic) $\mathbf{C}_i$ of this form. Otherwise, $\mathbf{B}_i$ contains at least one and at most r of the generators of $\mathbf{C}_i$ and is a subdirect product of algebras $\mathbf{D}_{ij} \in \mathscr{V}^{k-1}_{\mathrm{SI}}$, each on at most r generators. By the induction hypothesis, these algebras are all finite and there are only finitely many such (nonisomorphic) algebras $\mathbf{D}_{ij}$. Notice that those of the $r+1$ generators of $\mathbf{C}_i$ that lie in $\mathbf{B}_i$ form a generating set S_i for the algebra $\mathbf{B}_i$, with $|S_i| \le r$. Since there are only finitely many maps from S_i to (nonisomorphic) $\mathbf{D}_{ij}$'s, each $\mathbf{B}_i$ is isomorphic to a subdirect product of finitely many finite algebras, and is therefore finite. In particular, there are only finitely many (nonisomorphic) $\mathbf{B}_i$ as described above, hence there are only finitely many (nonisomorphic) $(r+1)$-generated $\mathbf{C}_i \in \mathscr{V}^k_{\mathrm{SI}}$. Now the same argument shows that $\mathbf{E}$ is finite. Consequently, by induction, $\mathscr{V}^k$ contains only finitely many $(r+1)$-generated algebras, all of which are finite, for all $k, r \in \omega$. In particular, the $\mathscr{V}^+$-free algebra on $r+1$ free generators ($r \in \omega$) is in $\mathscr{V}^r$ (Proposition 5.5), and is therefore finite. ∎

The class of finite algebras in a quasivariety $\mathscr{Q}$ shall be denoted by $\mathscr{Q}_{\mathrm{fin}}$. If $\mathscr{Q} = \mathbf{ISPP_u}(\mathscr{Q}_{\mathrm{fin}})$, we say that $\mathscr{Q}$ is *generated* (*as a quasivariety*) *by its finite members*. A partial algebra $\mathbf{P} = \langle P; \oplus^{\mathbf{P}}, \dot{-}^{\mathbf{P}}, 0^{\mathbf{P}} \rangle$ of type $\langle 2,2,0 \rangle$ is called a *partial pocrim* if there is a pocrim $\mathbf{A} = \langle A; \oplus, \dot{-}, 0 \rangle$ with $P \subseteq A$ such that $\oplus^{\mathbf{P}}, \dot{-}^{\mathbf{P}}, 0^{\mathbf{P}}$ are the restrictions to P of $\oplus, \dot{-}, 0$, respectively, wherever the former are defined. We then call $\mathbf{P}$ a *partial subpocrim* of $\mathbf{A}$. A class $\mathscr{K}$ of pocrims has the *finite embeddability property* (FEP) if for every finite partial subpocrim $\mathbf{P}$ of any $\mathbf{A} \in \mathscr{K}$, there exists a finite $\mathbf{B} \in \mathscr{K}$ such that $\mathbf{P}$ is a partial subpocrim of $\mathbf{B}$. We implicitly adopt analogous terminology for *BCK*-algebras. By a result of Evans [Eva69], a quasivariety has the FEP if and only if it is generated by its finite members.

THEOREM 5.8. *Let $\mathscr{V}$ be a subvariety of some variety of pocrims or BCK-algebras that is closed under ordinal sums. If $\mathscr{V}$ is generated as a quasivariety by its finite members then so is $\mathscr{V}^+$.*

Proof. Assume $\mathscr{V} = \mathbf{ISPP_u}(\mathscr{V}_{\mathrm{fin}})$. It suffices to show that $\mathscr{V}^+$ has the FEP, and for this it is enough to prove that $\mathscr{V}^+_{\mathrm{SI}}$ has the FEP (see [BF, Lemma 3.7]). If $\mathbf{P}$ is a finite partial subalgebra of $\mathbf{A} \in \mathscr{V}^+_{\mathrm{SI}}$ then $\mathbf{A}$ may be chosen finitely generated and hence in $\mathscr{V}^m_{\mathrm{SI}}$ for some $m \in \omega$, by Theorem 5.5, so we need only prove (by induction on m) that each $\mathscr{V}^m_{\mathrm{SI}}$ has the

FEP. By Theorem 5.3, any $\mathbf{A} \in \mathscr{V}_{\mathrm{SI}}^{m}$ is $\mathbf{B} + \mathbf{C}$ for some $\mathbf{B} \in \mathscr{V}_{\mathrm{SI}}$ and some $\mathbf{C} \in \mathscr{V}^{m-1}$. If $m = 0$, the result follows from the assumption on $\mathscr{V}$. For $m > 0$ and $\mathbf{P}$ as above, $P \cap B$ and $(P \setminus \{0^{\mathbf{P}}\}) \cap C$ are universes of finite partial subalgebras of $\mathbf{B}$ and $\mathbf{C}$, respectively, so $\mathbf{P} \cap \mathbf{B}$ embeds in a finite $\mathbf{B}' \in \mathscr{V}_{\mathrm{SI}}$ and, by the induction hypothesis, $(\mathbf{P} \setminus \{0^{\mathbf{P}}\}) \cap \mathbf{C}$ embeds in a finite $\mathbf{C}' \in \mathscr{V}^{m-1}$. Then $\mathbf{P}$ is embeddable in the finite algebra $\mathbf{B}' + \mathbf{C}' \in \mathscr{V}_{\mathrm{SI}}^{m}$, as required. ∎

6. n-POTENT POCRIMS; THE VARIETIES $\mathscr{M}_n^{s+}$ AND $\mathscr{E}_n^{s+}$

We have seen that for each $n \in \omega$, the varieties $\mathscr{M}_n$ and $\mathscr{E}_n$ are closed under ordinal sums. In this section, we show how the results of Section 5 may be sharpened when we take $\mathscr{W}$ to be $\mathscr{M}_n$ or $\mathscr{E}_n$ and $\mathscr{V}$ to be the variety generated by all simple algebras in $\mathscr{W}$.

A variety $\mathscr{V}$ is called a *fixedpoint discriminator variety* if for some subclass $\mathscr{X}$ of $\mathscr{V}$ with $\mathscr{V} = \mathbf{HSP}(\mathscr{X})$ and some ternary term p of $\mathscr{V}$, the following is true: $\mathscr{V}$ satisfies $p(x, x, y) \approx y$ and for each $\mathbf{A} \in \mathscr{X}$, there exists $d \in A$ such that for any $a, b, c \in A$ with $a \neq b$, $p^{\mathbf{A}}(a, b, c) = d$. In this case, p is called a *fixedpoint discriminator term* for $\mathscr{V}$. By [BP94, Theorem 3.4], a variety $\mathscr{V}$ generated by simple algebras which has a commutative TD term (see Section 4, Example IV) is a fixedpoint discriminator variety for which the TD term is a fixedpoint discriminator term. Every fixedpoint discriminator variety is semisimple, by [BP94, Theorem 3.5]. Recall also that a *discriminator variety* is a variety $\mathscr{V}$ generated by a class $\mathscr{X}$ of algebras, having a ternary term p such that $\mathscr{V}$ satisfies $p(x, x, y) \approx y$ and $p^{\mathbf{A}}(a, b, c) = a$ whenever $\mathbf{A} \in \mathscr{X}$, $a, b, c \in A$, and $a \neq b$. In this case $\mathscr{V}$ is semisimple and arithmetical [BS81, Theorem IV.9.4].

Let $n \in \omega$ be given. The following result may be considered an extension of Theorem 3.4.

THEOREM 6.1. (i) *If $\mathscr{K}$ is any class of simple $(n+1)$-potent pocrims then $\mathbf{HSP}(\mathscr{K})$ is a discriminator variety of pocrims.*

(ii) *If $\mathscr{K}$ is any class of simple BCK-algebras in $\mathscr{E}_n$ then $\mathbf{HSP}(\mathscr{K})$ is a semisimple variety of BCK-algebras.*

In particular, every finite set of finite simple pocrims (resp. BCK-algebras) generates a discriminator (resp. semisimple) variety of pocrims (resp. BCK-algebras).

Proof. (i) Define $t(x, y, z) = [z \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x)] \oplus [x \dot{-} (x \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x))]$. Let $\mathbf{A} \in \mathscr{K}$. Since $\mathbf{A}$ is a pocrim, $\mathbf{A}$ satisfies $t(x, x, z) \approx z \oplus (x \dot{-} x) \approx z$ (using (2), (6), and $z \oplus 0 \approx z$). Since $\mathbf{A}$ is simple, it has no nontrivial ideals, so if $a, b, c \in A$ with $a \neq b$ then $\langle a \dot{-} b, b \dot{-} a \rangle_{\mathbf{A}} = A$, so that $c \dot{-} k(a \dot{-} b) \dot{-} l(b \dot{-} a) = 0$ for some $k, l \in \omega$ (see the characterization in Section 2 of ideals generated by subsets). But this implies that $c \dot{-} n(a \dot{-} b) \dot{-} n(b \dot{-} a) = 0$, since $\mathbf{A}$ is $(n+1)$-potent. Thus, $\mathbf{A}$ satisfies $x \not\approx y \to t(x, y, z) \approx 0 \oplus (x \dot{-} 0) \approx x$. This means that t is a discriminator term for $\mathbf{HSP}(\mathscr{K})$ and of course, $\mathbf{HSP}(\mathscr{K}) \subseteq \mathscr{M}_n \subseteq \mathscr{M}$.

(ii) Recall that $p(x, y, z) = z \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x)$ is a commutative TD term for $\mathscr{E}_n$, and therefore for $\mathbf{HSP}(\mathscr{K})$, so (ii) is a special case of the remarks preceding this theorem. (In this case, we may take $d = 0$ throughout $\mathscr{K}$.)

The last statement follows because every finite pocrim (resp. *BCK*-algebra) satisfies (E_n) for some $n \in \omega$ and the varieties $\mathscr{M}_n$ (resp. $\mathscr{E}_n$), $n \in \omega$, form an ascending chain. ∎

Inasmuch as the argument of (ii) also proves the semisimplicity of $\mathbf{HSP}(\mathscr{K})$ in the case where $\mathscr{K}$ is a class of simple $(n+1)$-potent pocrims, Theorem 6.1(i) illustrates [BP94, Theorem 3.8], which says that the discriminator varieties are just the congruence permutable semisimple varieties with EDPC. Of course, we cannot extend Theorem 6.1(i) to the case where $\mathscr{K}$ is a class of simple algebras in $\mathscr{E}_n$, since no nontrivial variety of *BCK*-algebras is congruence permutable. Theorem 6.1(i) may be used to show that *not every congruence permutable variety of pocrims consists of hoops*. For example, consider the (unique) pocrim $\mathbf{A} = \langle A; \oplus, \dot{-}, 0 \rangle$, with $A = \{0, 1, a, 2\}$ satisfying $0 < 1 < a < 2$ and $x \oplus y = 2$ for all $x, y \geq 1$. Since $\mathbf{A}$ is finite and simple, $\mathbf{HSP}(\mathbf{A})$ is a discriminator variety (and therefore congruence permutable) but $\mathbf{A}$ itself is not a hoop, since $1 < a$ but there is no $b \in A$ such that $1 \oplus b = a$. Theorem 6.1 and its proof also suggest that simple algebras in $\mathscr{M}_n$ or $\mathscr{E}_n$ are "nicely structured." Of course, a minimal (nonzero) ideal of an algebra in $\mathscr{M}_n$ or in $\mathscr{E}_n$ is just such a simple algebra (by the congruence extension property). The following result will be useful:

LEMMA 6.2. *Let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathscr{M}_n$ or in $\mathscr{E}_n$ and let I be its (unique) minimal nonzero ideal. Then $\langle A; \leq \rangle$ has a unique atom a, and $I = \{c \in A\colon c \dot{-} na = 0\}$. In the case that $\mathbf{A} \in \mathscr{M}_n$, I is the principal order ideal $(b] = \{c \in A\colon c \leq b\}$ of $\langle A; \leq \rangle$, where $b = na$, and for any $c \in A$, we have $c \notin I$ iff $n(c \dot{-} b) \geq b$.*

Proof. Pałasinski [Pał a, Theorem 2] has shown that if $\mathbf{B}$ is any subdirectly irreducible *BCK*-algebra in the variety $\mathscr{E}_n$ then the poset $\langle B; \leq \rangle$ has a unique atom. In particular, therefore, $\langle A; \leq \rangle$ has a unique atom a, and $a \in I$. Since I is the universe of a simple subalgebra of $\mathbf{A}$, the ideal I is generated by a, and since $\mathbf{A}$ is $(n+1)$-potent, I must have a greatest element, viz., $b = na$. If $c \in A \setminus I$ then $c \dot{-} b \neq 0$, hence $c \dot{-} b \geq a$ so that $n(c \dot{-} b) \geq na = b$. If $c \in I$ then $c \dot{-} b = 0$ so $n(c \dot{-} b) = 0 \ngeq b$. ∎

We define $\mathscr{M}_n^s$ and $\mathscr{E}_n^s$ to be the varieties generated, respectively, by the class of all simple $(n+1)$-potent pocrims and the class of all simple algebras in $\mathscr{E}_n$. For the remainder of this section, *$\mathscr{V}$ and $\mathscr{W}$ shall denote either $\mathscr{M}_n^s$ and $\mathscr{M}_n$, or $\mathscr{E}_n^s$ and $\mathscr{E}_n$* (for a fixed but arbitrary $n \in \omega$). By Theorem 6.1, $\mathscr{V}$ is semisimple, so $\mathscr{V}_{\mathrm{SI}} = \mathscr{K}^0$ is the class of all simple algebras in $\mathscr{V}$, and $\mathscr{K}^m = \{\mathbf{A} + \mathbf{B}\colon \mathbf{A} \in \mathscr{K}^0 \text{ and } \mathbf{B} \in \mathscr{V}^{m-1}\} = \mathscr{V}_{\mathrm{SI}}^m$ for all $m \in \omega$ (Corollary 5.4). Also, $\mathscr{V}^+$ is the smallest variety $\mathscr{U}$ such that for every *simple* $\mathbf{A} \in \mathscr{V}$, we have $\mathbf{A} + \mathbf{B} \in \mathscr{U}$ whenever $\mathbf{B} \in \mathscr{U}$ (Corollary 5.6). These results and Proposition 5.5 offer model theoretic descriptions of the variety $\mathscr{V}^+$. We shall now obtain equational characterizations of $\mathscr{V}^+$ (in both of the cases $\mathscr{V} \subseteq \mathscr{M}_n$ and $\mathscr{V} \subseteq \mathscr{E}_n$).

Consider the following identities (recall that an inequality $t(x,y) \leq s(x,y)$ abbreviates the identity $t(x,y) \dot{-} s(x,y) \approx 0$):

$$\sigma(x,y)\colon \qquad (x \oplus ny) \dot{-} x \leq (ny) \dot{-} n(x \dot{-} ny);$$

$$\beta(x,y)\colon \qquad x \dot{-} (x \dot{-} y) \dot{-} n[x \dot{-} y \dot{-} (x \dot{-} ny)] \leq y \dot{-} n(x \dot{-} y);$$

$$\gamma(x,y)\colon \quad y \dot{-} x \dot{-} n\big[x \dot{-} y \dot{-} (x \dot{-} ny)\big] \leq y \dot{-} n(x \dot{-} y).$$

By way of motivation, note that $\sigma(x,y)$ implies (over $\mathscr{M}$) the quasi-identity

$$\sigma'(x,y)\colon \qquad ny \leq n(x \dot{-} ny) \quad \rightarrow \quad x \oplus ny \leq x,$$

which is equivalent (over $\mathscr{M}_n$) to the quasi-identity

$$\sigma''(x,y)\colon \ [y \oplus y \approx y \ \& \ y \leq n(x \dot{-} y)] \quad \rightarrow \quad x \oplus y \approx x,$$

since $\mathscr{M}_n$ satisfies $y \oplus y \approx y \Leftrightarrow y \approx ny$. Also, $\beta(x,y)$ & $\gamma(x,y)$ implies (over $\mathscr{BCK}$)

$$\alpha(x,y)\colon \qquad \big[x \dot{-} y \approx x \dot{-} 2y \ \& \ y \dot{-} n(x \dot{-} y) \approx 0\big] \rightarrow \big[x \dot{-} y \approx x \ \& \ y \dot{-} x \approx 0\big]$$

(which is equivalent to the conjunction of two quasi-identities). We shall see presently that $\sigma(x,y)$ and $\sigma''(x,y)$ are equivalent over $\mathscr{M}_n$, and that $\beta(x,y)$ & $\gamma(x,y)$ is equivalent to $\alpha(x,y)$ over $\mathscr{E}_n$.

Note that every pocrim satisfies both $\sigma(x,0)$ and $x \leq y \rightarrow \sigma(x,y)$ (by definition of residuation). Also every BCK-algebra satisfies both $\beta(x,0)$ & $\gamma(x,0)$ and $x \leq y \rightarrow [\beta(x,y) \ \& \ \gamma(x,y)]$.

PROPOSITION 6.3. *$\mathscr{V}^+$ satisfies $\beta(x,y)$ and $\gamma(x,y)$ (hence $\alpha(x,y)$), and $\mathscr{M}_n^{s+}$ satisfies $\sigma(x,y)$ (hence $\sigma''(x,y)$).*

Proof. For the first assertion, it suffices to show that every subdirectly irreducible 2-generated member of $\mathscr{V}^+$ satisfies $\beta(x,y)$ & $\gamma(x,y)$. By Proposition 5.5, we need only show that $\mathscr{V}_{\mathrm{SI}}^1$ satisfies $\beta(x,y)$ & $\gamma(x,y)$. Moreover, by the remarks preceding this proposition, we need only show that $\mathscr{V}_{\mathrm{SI}}^1$ satisfies $[y \not\approx 0$ & $x \not\leq y] \to [\beta(x,y)$ & $\gamma(x,y)]$. Similarly, for the second assertion, it suffices to show that $(\mathscr{M}_n^s)_{\mathrm{SI}}^1$ satisfies $[y \not\approx 0$ & $x \not\leq y] \to \sigma(x,y)$.

Let $\mathbf{C} \in \mathscr{V}_{\mathrm{SI}}^1$ and $a, b \in C$ with $b \neq 0$ and $a \not\leq b$. First suppose that $\mathbf{C} \in \mathscr{V}_{\mathrm{SI}} = \mathscr{K}^0(\mathscr{V})$. By simplicity of $\mathbf{C}$, $a \dot{-} nb = 0$ and $c \dot{-} n(a \dot{-} b) = 0$ for all $c \in C$. Thus,

$$a \dot{-} (a \dot{-} b) \dot{-} n\big[a \dot{-} b \dot{-} (a \dot{-} nb)\big] = 0 = b \dot{-} a \dot{-} n\big[a \dot{-} b \dot{-} (a \dot{-} nb)\big]$$

so $\beta(a,b)$ and $\gamma(a,b)$ are true. If in addition, C is a pocrim (e.g., if $\mathscr{V} = \mathscr{M}_n^s$) then

$$(a \oplus nb) \dot{-} a \leq nb = nb \dot{-} 0 = (nb) \dot{-} n(a \dot{-} nb),$$

so $\sigma(a,b)$ is true. This shows that $\mathscr{V}$ satisfies $\beta(x,y)$ & $\gamma(x,y)$ and that $\mathscr{M}_n^s$ satisfies $\sigma(x,y)$.

Now suppose that $\mathbf{C} \notin \mathscr{V}_{\mathrm{SI}}$. By Corollary 5.4, $\mathbf{C} = \mathbf{A} + \mathbf{B}$, where $\mathbf{A} \in \mathscr{K}^0(\mathscr{V})$, $\mathbf{B} \in \mathscr{V}$, and both $\mathbf{A}$ and $\mathbf{B}$ are nontrivial. In view of the foregoing arguments, we may assume that neither A nor B contains $\{a, b\}$. It follows that $b \in A$ and $a \in B$, hence $b \leq a$ (i.e., $b \dot{-} a = 0$) and $a \dot{-} b = a$ (so $a \dot{-} (a \dot{-} b) = 0$). The left-hand sides of $\beta(a,b)$ and $\gamma(a,b)$ are therefore both 0, so $\beta(a,b)$ and $\gamma(a,b)$ are true. If $\mathbf{C}$ is a pocrim then $nb \in A$ (since A is an ideal of $\mathbf{C}$), so $a \oplus nb = a$, hence $(a \oplus nb) \dot{-} a = 0$. This is the left-hand side of $\sigma(a,b)$, so $\sigma(a,b)$ is true. ∎

PROPOSITION 6.4. *Let $\mathbf{A}$ be a subdirectly irreducible pocrim or BCK-algebra with $\mathbf{A}^- \in \mathscr{E}_n$. If $\mathbf{A}$ satisfies $\alpha(x,y)$ or $\mathbf{A}$ is a pocrim satisfying $\sigma''(x,y)$ then $\mathbf{A}$ is isomorphic to an ordinal sum $\mathbf{I} + \mathbf{D}$, where I is the least nonzero ideal of $\mathbf{A}$.*

Proof. We may assume that $n > 0$. Suppose first that $\mathbf{A}$ satisfies $\alpha(x,y)$. Let I be the least nonzero ideal of $\mathbf{A}$ and let $D = (A \setminus I) \cup \{0\}$. Let $b \in I$ and $a, c \in D$. We know that I is a subuniverse of $\mathbf{A}$. We want to show that $b \leq a$, that $a \dot{-} b = a$, and that $a \dot{-} c \in D$ unless $a \leq c$. Using (E_n), we have

$$a \dot{-} (n-1)b \dot{-} b = a \dot{-} nb = a \dot{-} (n+1)b = a \dot{-} (n-1)b \dot{-} b \dot{-} b.$$

Also, $a \dot{-} kb \notin I$ (because $a \notin I$) for all $k \geq 0$, so the ideal of **A** generated by $a \dot{-} kb$ contains I, whence $b \dot{-} n(a \dot{-} kb) = 0$ (see Lemma 6.2). So

$$b \dot{-} n[a \dot{-}(n-1)b \dot{-} b] = b \dot{-} n[a \dot{-} nb] = 0.$$

By $\alpha(a \dot{-}(n-1)b, b)$,

$$a \dot{-}(n-1)b \dot{-} b = a \dot{-}(n-1)b \qquad (\text{i.e., } a \dot{-} nb = a \dot{-}(n-1)b)$$

and $b \dot{-}[a \dot{-}(n-1)b] = 0$, i.e., $b \leq a \dot{-}(n-1)b$. If $n = 1$, these are the required results about a and b.

If $n \geq 2$, we repeat this argument, replacing $a \dot{-}(n-1)b$ by $a \dot{-}(n-2)b$, and obtain $a \dot{-} nb = a \dot{-}(n-2)b$ and, eventually, $a \dot{-} nb = a \dot{-} b$.

In particular, $a \dot{-} b = a \dot{-} b \dot{-} b$ and $b \dot{-} n(a \dot{-} b) = 0$, so by $\alpha(a, b)$, $a \dot{-} b = a$ and $b \dot{-} a = 0$, i.e., $b \leq a$. Now if $a \nleq c$, we claim that $a \dot{-} c \in A \setminus I$. For if $a \dot{-} c \in I$ then $a = a \dot{-}(a \dot{-} c)$ (by the above), implying $a \leq c$! This shows that **A** is the ordinal sum of **I** and **D**.

Now suppose **A** is a pocrim (hence $\mathbf{A} \in \mathscr{M}_n$) satisfying $\sigma''(x, y)$. By Lemma 6.2, $I = (b]$, with $b = na$, where a is the unique atom of $\langle A; \leq \rangle$. Let $[b) = \{c \in A: b \leq c\}$. Notice that if $c \in A$ and $c \nleq b$ then $n(c \dot{-} b) \geq na = b$, whence $c \oplus b = c$, by $\sigma''(x, y)$, from which $b \leq c$ follows. Thus, $A = (b] \cup [b)$, and **A** is the ordinal sum of the subalgebras with universes $(b]$ and $[b) \cup \{0\}$. ∎

THEOREM 6.5. (i) *The identities* (1), (2), (3), (E_n), $\beta(x, y)$, *and* $\gamma(x, y)$ *form an equational base for* $\mathscr{E}_n^{s+}$.

(ii) *The identities* (1), (2), (3), (4), (M_n), *and* $\sigma(x, y)$ *constitute an equational base* $\mathscr{M}_n^{s+}$.

Proof. The following argument proves both (i) and (ii): let $\mathscr{V}$ and $\mathscr{W}$ be $\mathscr{E}_n^s$ and $\mathscr{E}_n$ in (i), and $\mathscr{M}_n^s$ and $\mathscr{M}_n$ in (ii).

Let $\mathscr{X}$ be the variety axiomatized by the identities in the statement of the theorem. Then $\mathscr{V}^+ \subseteq \mathscr{X}$, by Proposition 6.3 and the fact that $\mathscr{V}^+ \subseteq \mathscr{W}$. Conversely, let **A** be a finitely generated member of $\mathscr{X}$. In showing that $\mathbf{A} \in \mathscr{V}^+$, we may clearly assume without loss of generality that **A** is subdirectly irreducible. By Proposition 6.4, we may assume that **A** is isomorphic to an ordinal sum $\mathbf{I} + \mathbf{D}$, where I is the least nonzero ideal of **A**, and $\mathbf{D} \in \mathscr{X}$. At least one generator of **A** lies in I, by the definition of ordinal sums. If **A** is 1-generated then **D** is trivial and $\mathbf{A} \in \mathscr{K}^0(\mathscr{V}) \subseteq \mathscr{V}^+$. Suppose **A** is r-generated, where $r > 1$. Then $\mathbf{D} \cong \mathbf{A}/I$ is $(r-1)$-generated and is a subdirect product of a family $\{\mathbf{B}_j: j \in J\}$ of subdirectly irreducible members of $\mathscr{X}$, each of which is $(r-1)$-generated. Applying Proposition 6.4 and an induction hypothesis to the $\mathbf{B}_j$, we deduce that $\mathbf{D} \in \mathscr{V}^+$, hence $\mathbf{A} \in \mathscr{V}^+$, by Corollary 5.6. We have shown that $\mathscr{V}^+$ contains all finitely generated members of $\mathscr{X}$. Since every algebra is embeddable into an ultraproduct of its finitely generated subalgebras, it follows that $\mathscr{X} \subseteq \mathscr{V}^{+}$. ∎

The next result is an immediate corollary but would appear to be difficult to derive directly.

COROLLARY 6.6. (i) $\mathscr{E}_n \models [\beta(x, y) \ \& \ \gamma(x, y)] \leftrightarrow \alpha(x, y)$.
(ii) $\mathscr{M}_n \models \sigma(x, y) \leftrightarrow \sigma''(x, y)$.
(iii) $\mathscr{M}_n \models [\beta(x, y) \ \& \ \gamma(x, y)] \leftrightarrow \sigma(x, y)$.

The preceding construction and results extend earlier work on Brouwerian semilattices and hoops. It is known and easy to see that $\mathscr{M}_1^{s+} = \mathscr{M}_1$ and $\mathscr{E}_1^{s+} = \mathscr{E}_1$. The variety $\mathscr{H}$ of hoops coincides with $\mathscr{J}_0^{+}$ (recall that $\mathscr{J}_0$ is the variety of all commutative pocrims, i.e, the variety generated by all linearly ordered commutative pocrims): this is [BF93, Theorem 3.3] combined with our Proposition 4.1 and it yields a proof of the fact that $\mathscr{H}$ is generated by its finite members [BF93, Corollary 3.6]. (The result illustrates Theorem 5.8.) Recall that hoops (and, in particular, Brouwerian semilattices) are congruence permutable. Similarly, we have:

PROPOSITION 6.7. *The variety $\mathscr{M}_n^{s+}$ is congruence permutable* (*hence arithmetical*).

Proof. In view of Theorem 3.4, it suffices to show that $\mathscr{M}_n^{s+}$ satisfies the identity

$$\begin{aligned}\delta(x, y): \quad & [y \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x)] \\ & \oplus [x \dot{-} (x \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x))] \approx x,\end{aligned}$$

and for this purpose it suffices to consider subdirectly irreducible 2-generated algebras. It is easy to see that every pocrim satisfies $\delta(x, x)$. Since $\mathscr{M}_n^{s}$ is generated by simple $(n+1)$-potent pocrims, it satisfies

$$x \not\approx y \to z \dot{-} n(x \dot{-} y) \dot{-} n(y \dot{-} x) \approx 0,$$

and therefore also $\delta(x, y)$. Now suppose that $\mathbf{C} = \mathbf{A} + \mathbf{B}$, where $\mathbf{A} \in \mathscr{K}^{0}(\mathscr{M}_n^{s})$ and $\mathbf{B} \in \mathscr{M}_n^{s}$, and let $a, b \in C$ with $a \neq b$. If both a and b are in A or both are in B then $\delta(a, b)$ follows from the case just considered. If $a \in A$ and $b \in B$ then $a \leq b$ and $b \dot{-} a = b$, so the left-hand side of $\delta(a, b)$ evaluates to $[b \dot{-} nb] \oplus [a \dot{-} (a \dot{-} nb)] = 0 \oplus (a \dot{-} 0) = a$, as required. If $a \in B$ and $b \in A$ then $b \leq a$ and $a \dot{-} b = a$ so the left-hand side of $\delta(a, b)$ becomes $[b \dot{-} na] \oplus [a \dot{-} (a \dot{-} na)] = 0 \oplus (a \dot{-} 0) = a$. This completes the proof, in view of Propositions 5.3 and 5.5. ∎

On the other hand, we have:

PROPOSITION 6.8. *For each* $n \geq 2$, *the variety* $\mathscr{M}_n$ *is not congruence permutable, hence* $\mathscr{M}_n^{s+} \neq \mathscr{M}_n$.

Proof. It suffices to show that $\mathscr{M}_2$ is not congruence permutable.

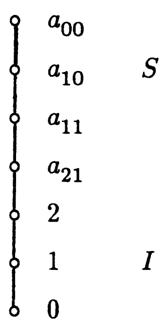
Let $\langle A; \leq \rangle$ be the poset depicted in Figure 1; $A = I \cup S$, where $I = \{0, 1, 2\}$ and $S = \{a_{00}, a_{10}, a_{11}, a_{21}\}$. Define a binary $\oplus$ on A by

$$\begin{aligned} x \oplus y &= y \oplus x \qquad (x, y \in A), \\ i \oplus j &= \min\{i + j, 2\} \qquad (i, j \in I), \\ a_{ij} \oplus k &= a_{i \dot{-} k, j \dot{-} k} \qquad \bigl(a_{ij} \in S,\ k \in I\bigr), \\ a_{ij} \oplus a_{kl} &= a_{00} \qquad \bigl(a_{ij}, a_{kl} \in S\bigr). \end{aligned}$$

Then $\langle A; \oplus, 0; \leq \rangle$ is a commutative integral pomonoid, which is residuated as follows:

$$\begin{aligned} x \leq y \quad \Rightarrow \quad & x \dot{-} y = 0, \quad a_{00} \dot{-} a_{21} = 2, \\ & x \dot{-} 0 = x, \qquad x \dot{-} y = 1 \text{ for all other pairs } x, y \in S \\ & \qquad\qquad\qquad \text{with } x > y, \\ & 2 \dot{-} 1 = 1, \qquad a_{ij} \dot{-} k = a_{\min\{i+k, 2\}, \min\{j+k, 1\}} \\ & \qquad\qquad\qquad (a_{ij} \in S,\ k \in I). \end{aligned}$$

The resultant pocrim $\mathbf{A} = \langle A; \oplus, \dot{-}, 0 \rangle$ clearly satisfies $2x \approx 3x$, so $\mathbf{A} \in \mathscr{M}_2$. The relation $\tau = (I \times I) \cup ((S \times S) \setminus \{(a_{11}, a_{10}), (a_{10}, a_{11})\})$ is a toler-



$\langle A; \leq \rangle$

FIGURE 1

ance on $\mathbf{A}$ which is not transitive, i.e., $\tau \notin \operatorname{Con}\mathbf{A}$. Thus, $tn(\mathbf{A}) > 1$. In view of the remarks following Lemma 1.1 and the fact that $\mathscr{M}_2$ is congruence 3-permutable, we conclude that $tn(\mathbf{A}) = 2 = tn(\mathscr{M}_2)$, so $\mathscr{M}_2$ is not congruence permutable. (Ironically, $\mathbf{A}$ *is* congruence permutable.) ∎

We remark that $\mathscr{M}_2^{s+}$ is not the largest congruence permutable subvariety of $\mathscr{M}_2$. Let $\mathbf{A}$ be the unique pocrim whose underlying poset is the chain $\{0, 1, a, 2\}$ with $0 < 1 < a < 2$ such that $1 \oplus 1 = 1$ and $x \oplus y = 2$ for all $x > 1$ and $y \geq 1$. It is easily checked that

$$t(x, y, z) = [z \dot{-} 2(x \dot{-} y) \dot{-} 2(y \dot{-} x)] \oplus [x \dot{-} (x \dot{-} 2(x \dot{-} y) \dot{-} 2(y \dot{-} x))]$$

is a Mal'cev term for $\mathbf{HSP}(\mathbf{A})$, i.e., $\mathbf{A}$ satisfies $t(x, x, y) \approx y \approx t(y, x, x)$, so $\mathbf{HSP}(\mathbf{A})$ is a congruence permutable subvariety of $\mathscr{M}_2$. But $\mathbf{HSP}(\mathbf{A}) \not\subseteq \mathscr{M}_2^{s+}$; in fact, $\mathbf{A} \notin \mathscr{M}_2^{s+}$. To see this, it suffices to show that $\mathbf{A} \not\models \sigma''(x, y)$. Indeed, $1 \oplus 1 = 1$ and $2(a \dot{-} 1) = 2a = 2 > 1$, but $a \oplus 1 = 2 \neq a$.

The fact that $\mathscr{M}_2$ is not congruence permutable brings clarity to a question concerning varieties with EDPC. Consider a variety $\mathscr{K}$ with a constant term 0. We say that $\mathscr{K}$ is *Fregean with respect to* 0 (briefly, 0-*Fregean*) if $\mathscr{K}$ is 0-regular and

$$\Theta^{\mathbf{A}}(a, 0) = \Theta^{\mathbf{A}}(b, 0) \quad \Rightarrow \quad a = b$$

holds for all $\mathbf{A} \in \mathscr{K}$ and all $a, b \in A$. A binary term $t(x, y)$ is called a *weak meet with respect to* 0 (briefly a *weak* 0-*meet*) if

$$\Theta^{\mathbf{A}}\big(t^{\mathbf{A}}(a, b), 0\big) = \Theta^{\mathbf{A}}((a, 0), (b, 0))$$

for all $\mathbf{A} \in \mathscr{K}$ and all $a, b \in A$. These definitions are taken from [BKP84], whose Corollary 3.10 shows that a 0-Fregean variety $\mathscr{K}$ with EDPC has a weak 0-meet iff $\mathscr{K}$ is congruence permutable. In the remarks that follow this corollary, it is claimed that the implication from left to right would fail were the requirement dropped that $\mathscr{K}$ be 0-Fregean. P. Idziak pointed out that the justification given in [BKP84] for this claim is flawed. The claim is vindicated, however, by a consideration of the variety $\mathscr{M}_2$. Indeed, we have noted that each of the varieties $\mathscr{M}_n$ has EDPC; as pocrim varieties, they are 0-regular and $t(x, y) = x \oplus y$ is clearly a weak 0-meet, but $\mathscr{M}_2$ is not congruence permutable.

We have made repeated use of the fact that $\mathbf{S}(\mathscr{M}_n^-) \subseteq \mathscr{E}_n$. It is an open problem whether every *BCK*-algebra in $\mathscr{E}_n$ is a residuation subreduct of an $(n + 1)$-potent pocrim, i.e,. whether $\mathbf{S}(\mathscr{M}_n^-) = \mathscr{E}_n$. A partial result in this direction is the following.

THEOREM 6.9. *For each $n \in \omega$, the variety $\mathscr{E}_n^{s+}$ is just the class of residuation subreducts of the variety $\mathscr{M}_n^{s+}$, i.e.,* $\mathbf{S}((\mathscr{M}_n^{s+})^-) = \mathscr{E}_n^{s+}$.

Proof. Since $\mathbf{S}(\mathscr{M}_n^-) \subseteq \mathscr{E}_n$ and nontrivial subalgebras of simple *BCK*-algebras are simple, we have $\mathbf{S}((\mathscr{K}^0(\mathscr{M}_n^s))^-) \subseteq \mathscr{K}^0(\mathscr{E}_n^{s+})$. Since $\mathscr{M}_n^s$ and $\mathscr{E}_n^s$ are semisimple and $\mathbf{S}((\mathbf{P_s}(\mathscr{L}))^-) = \mathbf{P_s}(\mathbf{S}(\mathscr{L}^-))$ for any class $\mathscr{L}$ of pocrims, it follows that $\mathbf{S}((\mathscr{M}_n^s)^-) \subseteq \mathscr{E}_n^s$.

Let $\mathbf{A} \in \mathscr{K}^0(\mathscr{E}_n^s)$, i.e., $\mathbf{A}$ is a simple algebra in $\mathscr{E}_n$. We show that $\mathbf{A} \in \mathbf{S}(\mathbf{B}^-)$ for some simple algebra $\mathbf{B} \in \mathscr{M}_n$. By Lemma 6.2, $\langle A; \le \rangle$ has a unique atom, a. By simplicity, we have $b \dot{-} na = 0$ for all $b \in A$. Since a is an atom, this means that $b \dot{-} a_1 \dot{-} \cdots \dot{-} a_n = 0$ for all $b \in A$ and all nonzero $a_1, \dots, a_n \in A$. We shall use the notational conventions of Theorem 2.1. Observe that $\langle J(\mathbf{A}); \subseteq \rangle$ also has a unique atom, viz., $I = J(a) = \{0, a\}$, and $nI = A$, where nI is $I + \cdots + I$ (n summands). This means that for any $J \in J(\mathbf{A})$, if $J \neq \{0\}$ then $nJ = A$. It follows that for any proper order filter of $\langle J(\mathbf{A}); \subseteq \rangle$, say $F \in F(\mathbf{J}(\mathbf{A}))$, we have $nF = J(\mathbf{A})$, whence $nF = (n+1)F$. Of course $nJ(\mathbf{A}) = J(\mathbf{A}) = (n+1)J(\mathbf{A})$ also, so the pocrim $\mathbf{C} = \mathbf{F}(\mathbf{J}(\mathbf{A}))$ is $(n+1)$-potent. We also have that $G = \{J \in J(\mathbf{A}) \colon J(a) \subseteq J\}$ is the unique atom of $\langle C; \supseteq \rangle$, so $\mathbf{C}$ is a simple algebra. By Theorem 2.1, $\mathbf{A}$ is isomorphic to a subalgebra of $\mathbf{C}^-$, as required. Since $\mathscr{E}_n^s$ is semisimple and $\mathbf{P_s}(\mathbf{S}((\mathscr{K}^0(\mathscr{M}_n^s))^-)) \subseteq \mathbf{S}((\mathbf{P_s}(\mathscr{K}^0(\mathscr{M}_n^s)))^-)$, we obtain $\mathscr{E}_n^s \subseteq \mathbf{S}((\mathscr{M}_n^s)^-)$, and so $\mathscr{E}_n^s = \mathbf{S}((\mathscr{M}_n^s)^-)$.

We claim that $(\mathscr{E}_n^s)^m = \mathbf{S}(((\mathscr{M}_n^s)^m)^-)$ for any $m \in \omega$. Since the residuation subreducts of the ordinal sum of a pair $\mathbf{A}, \mathbf{B}$ of pocrims are just the ordinal sums of pairs of residuation subreducts of $\mathbf{A}, \mathbf{B}$, it follows from Corollary 5.4 and the induction hypothesis that $(\mathscr{E}_n^s)^m_{\mathrm{SI}} = \mathbf{S}(((\mathscr{M}_n^s)^m_{\mathrm{SI}})^-)$. The subdirect product argument used in the case $m = 0$ therefore shows that $(\mathscr{E}_n^s)^m$ and $\mathbf{S}(((\mathscr{M}_n^s)^m)^-)$ have the same subdirectly irreducible members, completing the proof of our claim. By Proposition 5.5, $\mathscr{E}_n^{s+}$ and $\mathbf{S}((\mathscr{M}_n^{s+})^-)$ have the same finitely generated algebras, so we must have $\mathscr{E}_n^{s+} = \mathbf{S}((\mathscr{M}_n^{s+})^-)$. ∎

7. THE VARIETIES $\mathscr{M}_2^{s+}$ AND $\mathscr{E}_2^{s+}$; LOCAL FINITENESS

We have mentioned that $\mathscr{M}_1$ $(= \mathscr{M}_1^{s+})$ and $\mathscr{E}_1$ $(= \mathscr{E}_1^{s+})$ are locally finite varieties and that the varieties $\mathscr{M}_n$ and $\mathscr{E}_n$ are not locally finite for any $n \ge 2$. In this section, we show that $\mathscr{M}_2^{s+}$ and $\mathscr{E}_2^{s+}$ are also locally finite and that $\mathscr{M}_2^s$ and $\mathscr{E}_2^s$ have continuously many subvarieties.

LEMMA 7.1. *The varieties $\mathscr{M}_2^s$ and $\mathscr{E}_2^s$ are locally finite. The variety $\mathscr{M}_3^s$ is not locally finite.*

Proof. Let $\mathbf{A}$ be a finitely generated simple pocrim in $\mathscr{M}_2$, let 1 be the (unique) atom of $\langle A; \leq \rangle$ and let $2 = 1 \oplus 1$. Suppose that the generators of $\mathbf{A}$ other than $0, 1, 2$ are $x_1, \ldots, x_r$. By simplicity, we must have $1 < x_i < 2$ for all i, and by compatibility of the partial order with the monoid operation, also $x_i \oplus x_j = 1 \oplus x_i = 2$ for all i, j. Thus, $x_i \dot{-} x_j = 1$ whenever $x_i \nleq x_j$. This means that $|A| \leq r + 3$. It also follows that for each $m \in \omega$, the set W of all (nonisomorphic) simple algebras in $\mathscr{M}_2$ on at most m generators is a finite set of finite algebras. If $\mathbf{F}$ is the $\mathscr{M}_2^s$-free algebra on m free generators $\bar{x}_1, \ldots, \bar{x}_m$, then there are only finitely many maps from $\{\bar{x}_1, \ldots, \bar{x}_m\}$ to members of W, so that $\mathbf{F}$ is isomorphic to a subdirect product of finitely many finite algebras, hence $\mathbf{F}$ is a finite algebra. Thus $\mathscr{M}_2^s$ is a locally finite variety.

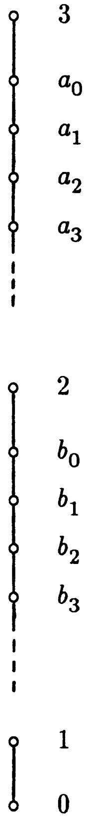
By the proof of Theorem 6.9, $\mathscr{E}_2^s = \mathbf{S}((\mathscr{M}_2^s)^-)$. If a finite set X generates $\mathbf{A} \in \mathscr{E}_2^s$, where $\mathbf{A}$ is a residuation subdirect of $\mathbf{B} \in \mathscr{M}_2^s$, then X generates a finite subpocrim $\mathbf{C}$ of $\mathbf{B}$, by the local finiteness of $\mathscr{M}_2^s$. Since $\mathbf{A}$ is a subreduct of $\mathbf{C}$, it follows that $\mathbf{A}$ is finite. Thus, $\mathscr{E}_2^s$ is also locally finite.

It remains to show that $\mathscr{M}_3^s$ is not locally finite. We exhibit an infinite 1-generated pocrim $\mathbf{A} \in \mathscr{K}^0(\mathscr{M}_3^s)$. Let $\langle A; \leq \rangle$ be the chain depicted in Figure 2. For a rational nonnegative number k, we denote by $\lfloor k \rfloor$ the greatest $m \in \omega$ such that $m \leq k$. Define $\oplus$ on A by the following rules, where $i, j \in \omega$ and $x, y \in A$:

$$x \oplus y = y \oplus x,$$
$$x \oplus 0 = x,$$
$$1 \oplus 1 = b_i \oplus 1 = 2,$$
$$b_i \oplus b_j = a_{\lfloor (i+j)/2 \rfloor},$$
$$x \geq 1,\ y \geq 2 \quad \Rightarrow \quad x \oplus y = 3.$$

It is not hard to see that $\langle A; \oplus, 0; \leq \rangle$ is a commutative integral pomonoid. It is also residuated as follows:

$$x \leq y \quad \Rightarrow \quad x \dot{-} y = 0,$$
$$x \dot{-} 0 = x,$$
$$b_i \dot{-} 1 = 2 \dot{-} 1 = 2 \dot{-} b_i = a_i \dot{-} 2 = 3 \dot{-} 2 = 3 \dot{-} a_i = 1,$$
$$a_i \dot{-} 1 = 3 \dot{-} 1 = 3 \dot{-} b_i = 2,$$
$$a_i \dot{-} a_j = b_i \dot{-} b_j = \begin{cases} 0, & \text{if } i \geq j, \\ 1, & \text{if } i < j, \end{cases}$$
$$a_i \dot{-} b_j = \begin{cases} 2, & \text{if } \left\lfloor \tfrac{1}{2} j \right\rfloor > i, \\ b_k, & \text{where } k = \max\left\{k' \in \omega : \left\lfloor \tfrac{1}{2}(j + k') \right\rfloor \leq i\right\}, \text{ otherwise.} \end{cases}$$



$\langle A; \leq \rangle$

FIGURE 2

Thus, $\mathbf{A} = \langle A; \oplus, \dot{-}, 0\rangle$ is a pocrim. Clearly, $\mathbf{A}$ satisfies $3x \approx 4x$, since $3c = 3$ whenever $0 \neq c \in A$. Also, $\mathbf{A}$ is simple, since $1 \oplus 1 \oplus 1 = 3$. We therefore have $\mathbf{A} \in \mathscr{K}^0(\mathscr{M}_3^s)$ and $\mathbf{A}$ is generated by $\{b_0\}$, since $b_i \oplus b_i = a_i$, $a_i \dot{-} b_i = b_{i+1}$, $b_i \dot{-} b_{i+1} = 1$, $1 \oplus 1 = 2$, and $2 \oplus 1 = 3$. ∎

THEOREM 7.2. *The varieties $\mathscr{M}_2^{s+}$ and $\mathscr{E}_2^{s+}$ are locally finite.*

Proof. This follows from the previous result and Theorem 5.7. ∎

By Lemma 7.1, $\mathscr{M}_n^{s+}$ is not locally finite for any $n > 2$. We do not know whether any of the varieties $\mathscr{E}_n^{s+}$ $(n > 2)$ are locally finite. It clearly suffices to settle this question for the varieties $\mathscr{E}_n^s$.

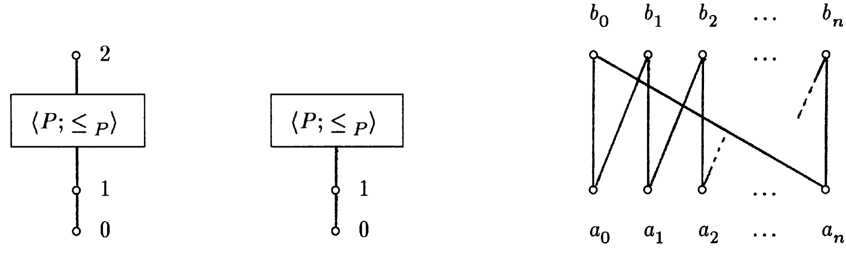
THEOREM 7.3. *Each of the varieties $\mathscr{M}_2^s$ and $\mathscr{E}_2^s$ has $2^{\aleph_0}$ distinct subvarieties.*

Proof. By the proof of Lemma 7.1, the subdirectly irreducible (i.e., simple) algebras in $\mathscr{M}_2^s$ are, up to isomorphism, just the pocrims $\mathbf{A}_P$, where $A_P = \{0,1,2\} \cup P$, $\langle P; \leq_P \rangle$ is any poset, the pocrim order $\leq$ contains $\leq_P$, $0 < 1 < p < 2$ for all $p \in P$, and $a \oplus b = 2$ whenever $1 \leq a, b \in A$. (We allow $P = \emptyset$. (See Fig. 3)). It is easy to see that $\mathbf{A}_P \in \mathbf{IS}(\mathbf{A}_Q)$ iff $\langle P; \leq_P \rangle$ is (order-) embeddable in $\langle Q; \leq_Q \rangle$.

Now consider the sequence of posets $K_1, K_2, \ldots,$ where K_n is the $2(n+1)$-element "crown" depicted in Figure 3. It is easy to see that for $n, m \geq 1$, K_n is a subposet of K_m iff $n = m$. Writing $\mathbf{A}_n$ for $\mathbf{A}_{K_n}$, we have $\mathbf{A}_n \notin \mathbf{IS}(\mathbf{A}_m)$ whenever $n \neq m$ $(n, m \geq 1)$. For each $n \geq 1$, the assertion "there is no subalgebra isomorphic to $\mathbf{A}_n$" is equivalent to a first-order sentence in the language $\langle \oplus, \dot{-}, 0 \rangle$, which is valid in every $\mathbf{A}_m$ $(n \neq m \geq 1)$. Thus if $\mathscr{B} \subseteq \{\mathbf{A}_m : m \geq 1\}$ and $\mathbf{A}_n \notin \mathscr{B}$, we have $\mathbf{A}_n \notin \mathbf{ISP_u}(\mathscr{B}) = \mathbf{HSP_u}(\mathscr{B})$ (by semisimplicity and the fact that nontrivial subalgebras and ultraproducts of simple $(n+1)$-potent pocrims are simple). By Jónsson's theorem, $\mathbf{A}_n \notin \mathbf{HSP}(\mathscr{B})$. Thus, if $\mathscr{B}$ and $\mathscr{C}$ are distinct subsets of $\{\mathbf{A}_m : m \geq 1\}$ then $\mathbf{HSP}(\mathscr{B}) \neq \mathbf{HSP}(\mathscr{C})$. The fact that $\{\mathbf{A}_m : m \geq 1\}$ has $2^{\aleph_0}$ subsets completes the proof, as far as $\mathscr{M}_2^s$ is concerned.

For $\mathscr{E}_2^s$, the argument requires only minor modification, since $\mathscr{E}_2^s$ is also semisimple and congruence distributive, nontrivial subalgebras of simple *BCK*-algebras are simple, and the subdirectly irreducible members of $\mathscr{E}_2^s$ are just the residuation reducts of those of $\mathscr{M}_2^s$, together with the subalgebras of these that are obtained by removing the top element. (See Fig. 3.) ∎

This means that $\mathscr{M}_2$ and $\mathscr{E}_2$ each have $2^{\aleph_0}$ distinct semisimple subvarieties, contrasting with the fact that $\mathscr{M}_1$ and $\mathscr{E}_1$ each have just one semisimple subvariety (viz., the varieties $\mathscr{G}$ and $\mathbf{S}(\mathscr{G}^-)$).



MEMBER OF $(\mathscr{M}_2^s)_{SI}$ MEMBER OF $(\mathscr{E}_2^s)_{SI} \setminus (\mathscr{M}_2^s)^-$ THE CROWN K_n

FIGURE 3

In the previous proof, the fact that $\mathbf{A}_n \notin \mathbf{HSP}(\mathscr{B})$ whenever $\mathbf{A}_n \notin \mathscr{B} \subseteq \{\mathbf{A}_i\colon i = 1, 2, \dots\}$ is actually an instance of a more general phenomenon. By [BP82, Corollary 3.2], in a variety $\mathscr{V}$ of finite type which has EDPC, every finite subdirectly irreducible algebra $\mathbf{A}$ is splitting (i.e., there is a largest subvariety of $\mathscr{V}$ not containing $\mathbf{HSP}(\mathbf{A})$). Since $\mathscr{M}_2$ and $\mathscr{E}_2$ have EDPC and $\mathbf{A}_n \notin \mathbf{IS}(\mathbf{A}_m) = \mathbf{HS}(\mathbf{A}_m) \supseteq (\mathbf{HSP}(\mathbf{A}_m))_{\mathrm{SI}}$ (Jónsson's theorem), i.e., $\mathbf{A}_n \notin \mathbf{HSP}(\mathbf{A}_m)$ $(n \neq m)$, we may infer $\mathbf{A}_n \notin \mathbf{HSP}(\mathscr{B})$ from the fact that $\mathbf{A}_n$ is splitting.

8. VARIETIES OF CANCELLATIVE POCRIMS

It is natural to consider pocrims $\mathbf{A}$ whose underlying monoids are cancellative. In this section we investigate varieties of such "cancellative" pocrims. First recall the identity

$$(x \oplus y) \mathbin{\dot{-}} y \approx x \tag{18}$$

and consider the quasi-identities

$$x \oplus y \leq x \oplus z \quad \Rightarrow \quad y \leq z \tag{20}$$

$$x \oplus y \approx x \oplus z \quad \Rightarrow \quad y \approx z. \tag{21}$$

LEMMA 8.1. *Conditions* (18), (20), *and* (21) *are equivalent in any pocrim.*

Proof. Since $\leq$ is a partial order with respect to which a map $a \mapsto a \mathbin{\dot{-}} x$ is isotone for any x, we clearly have that (18) $\Rightarrow$ (20) $\Rightarrow$ (21). It is an easy consequence of (7) that (21) $\Rightarrow$ (18). ∎

We shall say that a pocrim $\mathbf{A}$ is *cancellative* if it satisfies the equivalent conditions (18), (20), and (21), i.e., if the underlying monoid $\langle A; \oplus, 0\rangle$ is cancellative. We denote by $\mathscr{C}$ the class of all cancellative pocrims, which is clearly a quasivariety. Since a finite pocrim must have a greatest element, it is easy to deduce from (18) that every nontrivial cancellative pocrim has an unbounded underlying poset and is therefore infinite. More generally, $\mathscr{C} \cap \mathscr{M}_n$ is the trivial variety, for each $n \in \omega$. Observe that if $\mathbf{A} \in \mathscr{C}$, then the term $t(x, y, z) = (x \oplus z) \mathbin{\dot{-}} y$ is a Mal'cev term for $\mathbf{A}$, i.e., $\mathbf{A}$ satisfies $t(x, x, y) \approx y$ and $t(x, y, y) \approx x$. We therefore have:

PROPOSITION 8.2. *If* $\mathbf{A}$ *is a cancellative pocrim then the tolerance number of* $\mathbf{A}$ *is* 1, *hence* $\mathbf{A}$ *is congruence permutable. Every variety of cancellative pocrims is arithmetical and* 0-*regular, with the congruence extension property.*

Proof. Let $\tau \in \operatorname{Tol} \mathbf{A}$ and $(a, b), (b, c) \in \tau$. Then $(t^{\mathbf{A}}(a, b, b), t^{\mathbf{A}}(a, a, c)) \in \tau$, i.e, $(a, c) \in \tau$. Thus, $tn(\mathbf{A}) = 1$, so $\mathbf{A}$ is congruence permutable. Now Proposition 3.1 remains true if we set $\mathscr{K} = \mathscr{C}$. This establishes the remainder of the result. ∎

In [BR93], we constructed a pocrim $\mathbf{A}$ and a subalgebra $\mathbf{B}$ of $\mathbf{A}$ such that $\langle A; \oplus, 0\rangle$ is a cancellative monoid and there is a congruence η on $\mathbf{B}$ such that (i) $\mathbf{B}/\eta$ is not a pocrim, and (ii) there is no congruence θ on $\mathbf{A}$ such that $\eta = \theta \cap (B \times B)$. Thus, in contrast to the previous result, we have:

PROPOSITION 8.3. *The quasivariety $\mathscr{C}$ of all cancellative pocrims is not a variety and does not enjoy the congruence extension property.*

Varieties of cancellative pocrims are characterized by the next result, which is an immediate corollary of Theorem 3.2 and Proposition 8.2:

PROPOSITION 8.4. *Let $\mathscr{K}$ be a class of algebras of type $\langle 2, 2, 0\rangle$. Then $\mathbf{HSP}(\mathscr{K})$ is a variety of cancellative pocrims iff there exists a* 6*-ary* $\langle \oplus, \dot{-} \rangle$*-term such that $\mathscr{K}$ satisfies the identities* (1), (2), (3), (4), (18), $x \approx t(x, y, x \dot{-} y, y \dot{-} x, 0, 0)$, *and* $t(x, y, 0, 0, x \dot{-} y, y \dot{-} x) \approx y$.

For an arbitrary variety $\mathscr{V}$ of pocrims, it is not clear whether we necessarily have $\mathbf{H}(\mathscr{V}^{-}) \subseteq \mathscr{BCK}$. The situation for varieties of cancellative pocrims is clearer:

PROPOSITION 8.5. *Let $\mathbf{A}$ be a cancellative pocrim. Then* $\operatorname{Con} \mathbf{A} = \operatorname{Con} \mathbf{A}^{-}$. *Consequently, if $\mathbf{H}(\mathbf{A}) \subseteq \mathscr{M}$, e.g., if $\mathbf{A}$ generates a variety of (cancellative) pocrims, then $\mathbf{H}(\mathbf{A}^{-}) \subseteq \mathscr{BCK}$.*

Proof. Let $\theta \in \operatorname{Con} \mathbf{A}^{-}$ and let $a, b, c, d \in A$ such that $a \,\theta\, b$ and $c \,\theta\, d$. Then by (18), $a \oplus c = ((a \oplus c \oplus b) \dot{-} b) \,\theta\, ((a \oplus c \oplus b) \dot{-} a) = c \oplus b$. Similarly $(c \oplus b) \,\theta\, (b \oplus d)$ and the result follows from the transitivity of θ. ∎

The poset $\mathbf{P}^{v}(\mathscr{M})$ of all varieties of pocrims (ordered by inclusion) is a *nearlattice*, i.e., a lower semilattice in which any finite subset which is bounded above has a supremum. In addition, it is clearly closed under arbitrary *nonempty* intersections, but it has no greatest element [BR93, Theorem 3]. Also, since pocrim varieties are congruence distributive, it follows easily that $\mathbf{P}^{v}(\mathscr{M})$ is a *distributive* nearlattice, i.e., one in which the meet operation distributes over all existent finite suprema. It is not clear, however, whether $\mathbf{P}^{v}(\mathscr{M})$ is a lattice. (It clearly suffices to establish whether the union of any pair of pocrim varieties is contained in a pocrim variety.) The analogous question for varieties of *BCK*-algebras was answered affirmatively in [BR95, Theorem 11], but the methods used there do not generalize readily to the case of pocrim varieties. Nevertheless, for different reasons, the corresponding result about cancellative pocrims is true:

THEOREM 8.6. *The poset $\mathbf{P}^{v}(\mathscr{C})$ of all varieties of cancellative pocrims is a (distributive) lattice. In fact, it is a sublattice of the lattice $\mathbf{L}^{q}(\mathscr{C})$ of all subquasivarieties of $\mathscr{C}$.*

Proof. We have observed that $t(x, y, z) = (x \oplus z) \dot{-} y$ is a Mal'cev term for all cancellative pocrims, and therefore also for the variety $\mathbf{H}(\mathscr{C})$ generated by $\mathscr{C}$. It follows that $\mathbf{H}(\mathscr{C})$ is congruence permutable, hence congruence modular. Let $\mathscr{K}, \mathscr{L} \in \mathbf{P}^v(\mathscr{C})$. Hagemann and Hermann [HH79, Corollary 4.3] proved that in the subvariety lattice of a congruence modular variety, the join of two congruence distributive subvarieties is congruence distributive. In $\mathbf{L}^v(\mathbf{H}(\mathscr{C}))$, therefore, the join $\mathscr{N} = \mathscr{K} \vee \mathscr{L} = \mathbf{HSP}(\mathscr{K} \cup \mathscr{L})$ is a congruence distributive variety. By a well-known consequence of Jónsson's theorem, $\mathscr{N}_{\mathrm{SI}} = \mathscr{K}_{\mathrm{SI}} \cup \mathscr{L}_{\mathrm{SI}}$. Since $\mathscr{N} = \mathbf{IP_s}(\mathscr{N}_{\mathrm{SI}})$, we have $\mathscr{N} \subseteq \mathbf{ISPP_u}(\mathscr{K} \cup \mathscr{L}) \subseteq \mathscr{C}$. It follows immediately that $\mathscr{N}$ is the join of $\mathscr{K}$ and $\mathscr{L}$ in $\mathbf{P}^v(\mathscr{C})$. Of course, the join $\mathbf{ISPP_u}(\mathscr{K} \cup \mathscr{L})$ of $\mathscr{K}$ and $\mathscr{L}$ in $\mathbf{L}^q(\mathscr{C})$ must be contained in $\mathscr{N}$, so $\mathscr{N} = \mathbf{ISPP_u}(\mathscr{K} \cup \mathscr{L})$. Since $\mathbf{P}^v(\mathscr{C})$ is also closed under intersections, this shows that $\mathbf{P}^v(\mathscr{C})$ is a sublattice both of $\mathbf{L}^v(\mathbf{H}(\mathscr{C}))$ and of $\mathbf{L}^q(\mathscr{C})$. ∎

Henceforth, we shall write $\mathbf{L}^v(\mathscr{C})$ in place of $\mathbf{P}^v(\mathscr{C})$.

The most obvious example of a cancellative pocrim is the algebra of natural numbers $\boldsymbol{\omega}$. In fact the variety $\mathbf{ISPP_u}(\boldsymbol{\omega})$ is just the class of all cancellative hoops [BF93, Corollary 2.4]. As an immediate consequence of Theorem 4.1, we therefore have:

PROPOSITION 8.7. *The variety* $\mathbf{ISPP_u}(\boldsymbol{\omega})$ *is just the class of all commutative cancellative pocrims, i.e., it is axiomatized by* (1), (2), (3), (4), (18), *and* (T).

We have already mentioned that $\mathbf{ISPP_u}(\boldsymbol{\omega})$ is an atom of the poset $\mathbf{P}^v(\mathscr{M})$ of varieties of pocrims, so it is also an atom of $\mathbf{L}^v(\mathscr{C})$. Its position in $\mathbf{L}^v(\mathscr{C})$ may be described more precisely with the aid of a technical lemma involving the following weaker variant of the cancellative law (18):

$$(x \oplus x) \dot{-} x \approx x. \tag{22}$$

LEMMA 8.8. *A nontrivial quasivariety* $\mathscr{V}$ *of pocrims satisfies the identity* $(x \oplus x) \dot{-} x \approx x$ *iff the* $\mathscr{V}$*-free algebra on one free generator is isomorphic to* $\boldsymbol{\omega}$.

Proof. Sufficiency is obvious since $\boldsymbol{\omega}$ satisfies (22). Conversely, suppose that $\mathscr{V}$ satisfies (22). Let $n, m \in \omega$ with $m < n$. We know that $\mathscr{M}$, and hence $\mathscr{V}$, satisfies the identities $mx \le nx$ and $nx \oplus mx \approx (n + m)x$. Also, $\mathscr{V}$ cannot satisfy $mx \approx nx$. Indeed, if $\mathscr{V}$ satisfies $mx \approx (m + 1)x$ then $m > 0$ and $\mathscr{V}$ must satisfy $mx \approx kx$ for all $k \ge m$, whence, by (22), $\mathscr{V}$ also satisfies $mx \approx (2mx) \dot{-} (mx) \approx 0$, contradicting nontriviality. It therefore suffices to check that $\mathscr{V}$ satisfies $(nx) \dot{-} (mx) \approx (n - m)x$. We may assume $m > 0$. From elementary properties of pocrims it follows that $\mathscr{M}$ satisfies

$$(nx) \dot{-} (mx) \le ((n - l)x) \dot{-} ((m - l)x) \tag{23}$$

whenever $m \geq l \in \omega$. In particular, $\mathscr{M}$ satisfies $(nx) \dot{-} (mx) \leq (n-m)x$. For the reverse inequality, consider two cases. If $n \geq 2m$ then for $l = n - 2m$, $\mathscr{V}$ satisfies

$$\begin{aligned}(n-m)x &\approx ((2(n-m))x) \dot{-} ((n-m)x) \\ &\leq ((2(n-m)-l)x) \dot{-} ((n-m-l)x) \qquad \text{(by (23))} \\ &\approx (nx) \dot{-} (mx).\end{aligned}$$

If $n < 2m$, choose k to be the first positive integer such that $2^k(n-m) \geq n$. Set $l = 2^k(n-m) - n \leq (2^k - 1)(n-m)$. By repeated application of (22), $\mathscr{V}$ satisfies

$$\begin{aligned}(n-m)x &\approx \left((2^k(n-m))x\right) \dot{-} \left((2^{k-1}(n-m))x\right) \\ &\quad \dot{-} \cdots \dot{-} ((2(n-m))x) \dot{-} ((n-m)x) \\ &\approx \left((2^k(n-m))x\right) \dot{-} \left(\left(\sum_{i=0}^{k-1} 2^i(n-m)\right)x\right) \\ &\approx \left((2^k(n-m))x\right) \dot{-} \left((2^k-1)(n-m)x\right) \\ &\leq \left((2^k(n-m)-l)x\right) \dot{-} \left(((2^k-1)(n-m)-l)x\right) \qquad \text{(by (23))} \\ &\approx (nx) \dot{-} (mx),\end{aligned}$$

which completes the proof. ∎

COROLLARY 8.9. *The variety* $\mathbf{ISPP_u}(\boldsymbol{\omega})$ *is the smallest nontrivial quasivariety of cancellative pocrims.*

Proof. Since (18) implies (22), the previous lemma shows that the algebra $\boldsymbol{\omega}$ must belong to every nontrivial quasivariety of cancellative pocrims. ∎

We shall show that, in contrast with Corollary 8.9, there is no largest variety of cancellative pocrims. In the process we use the varieties $\mathscr{I}_n \cap \mathscr{C}$, $n \in \omega$, where $\mathscr{I}_n$ is as defined in Example VI of Section 4.

For each $k \in \omega$, let $A_k = \{2n\colon n = 0, \ldots, k,\ n \in \omega\} \cup \{n \in \omega\colon 2k < n\}$. Then the commutative integral linearly ordered monoid $\langle A_k; +, 0; \leq \rangle$ (where $+$ is natural addition and $\leq$ is the natural linear order) is a submonoid of $\boldsymbol{\omega}$ satisfying the conditions of Lemma 2.2, and is therefore residuated. The residuation operation $\dot{-}_k$ may be defined as follows: we have $n \dot{-}_k m = 0$ if $n \leq m$, while for $n \geq m$:

$$n \dot{-}_k m = \begin{cases} n - m, & \text{if } n - m \text{ is even or } n - m \geq 2k, \\ n - m + 1, & \text{otherwise.} \end{cases}$$

By Lemma 2.2, $\mathbf{A}_k = \langle A_k; +, \dot{-}_k, 0 \rangle$ is a pocrim. Since $\mathbf{A}_k$ satisfies (21), it is a cancellative pocrim. We require the following technical lemma.

LEMMA 8.10. *Let $a, b \in A_k$ with $a > b$ and let $1 \leq n \in \omega$.*

(a) *Suppose $b \geq 2k$ with a even and b odd (or vice versa). If $j_{2n}^{\mathbf{A}_k}(b, a) < j_{2n-1}^{\mathbf{A}_k}(b, a)$ then $j_{2n}^{\mathbf{A}_k}(b, a) = b - 2n - 1$ and $a - b < 2k - 2n$.*
(b) *In all other cases, $j_0^{\mathbf{A}_k}(a, b) = j_0^{\mathbf{A}_k}(b, a)$.*
(c) *If $k \geq 5$, then $j_m^{\mathbf{A}_k}(2^k, 2^k - 1) = 2^k - m - 1$ and $j_m^{\mathbf{A}_k}(2^k - 1, 2^k) = 2^k - m - 2$, for $m = -1, 0, 1, \ldots, 2k - 1$.*

Proof. (a) and (c) are proved by induction on n and m, respectively, while (b) is easily verified. We omit the tedious details. ∎

COROLLARY 8.11. *$\mathbf{A}_k \in \mathscr{J}_{2k}$ for all $k \in \omega$, and $\mathbf{A}_k \notin \mathscr{J}_{2k-1}$ when $k \geq 5$. Thus the sequence $\mathscr{C} \cap \mathscr{J}_n$, $n \geq 9$, is a strictly ascending chain of varieties of cancellative pocrims.*

Proof. Suppose $\mathbf{A}_k$ does not satisfy (J_{2k}). By [BR95, Lemma 17(i)], it follows that there exist $a, b \in A_k$ such that $j_{2k}^{\mathbf{A}_k}(b, a) < j_{2k-1}^{\mathbf{A}_k}(b, a)$. Clearly $a \neq b$ and, by symmetry, we may assume that $a > b$. However, Lemma 8.10(a), (b) show that this implies $a - b < 0$, a contradiction. Thus $\mathbf{A}_k \in \mathscr{J}_{2k}$. If $k \geq 5$ then $2^k - (2k - 1) - 2 \geq 2n$ so Lemma 8.10(c) shows that $\mathbf{A}_k \notin \mathscr{J}_{2k-1}$. ∎

We now construct a cancellative pocrim $\mathbf{A}$, into which each $\mathbf{A}_k$ is embeddable, such that a subalgebra $\mathbf{B}$ of $\mathbf{A}$ has a homomorphic image which is not a pocrim. To do this, consider chains E_i, $i \in \omega$, defined as follows:

$$E_0 = \{0_0, 2_0, 4_0, \ldots\},$$

$$E_1 = \{\ldots, -4_1, -2_1, 0_1, 1_1, 2_1, \ldots\},$$

and

$$E_i = \{\ldots, -2_i, -1_i, 0_i, 1_i, 2_i, \ldots\}, \qquad i > 1,$$

where, for each i, $n_i < m_i$ in E_i iff $n < m$. Let $A = \bigcup_{i=0}^{\infty} E_i$ and define a linear order $\leq$ on A by specifying that $\leq$ restricts to the existing order on each E_i, while for distinct $i, j \in \omega$, $n_i < m_j$ iff $i < j$. (In other words, as a poset, $\langle A; \leq \rangle$ is the ordinal sum of the $\langle E_i; \leq \rangle$, $i \in \omega$.) Define a binary operation $\oplus$ on A by

$$n_i \oplus m_j = (n+m)_{i+j} = m_j \oplus n_i.$$

The commutative integral pomonoid $\langle A; \oplus, 0, \leq \rangle$ is residuated as follows: for integers n, m and $i, j \in \omega$, we have $n_i \dot{-} m_j = 0_0$ whenever $n_i \leq m_j$, while for $n_i > m_j$, we have

$$n_i \dot{-} m_j = \begin{cases} (n-m)_{i-j}, & \text{if } n-m \text{ is even, or } i-j>1, \\ & \text{or } i-j=1 \text{ and } n-m \geq 0, \\ (n-m+1)_{i-j}, & \text{otherwise.} \end{cases}$$

Thus, $\mathbf{A} = \langle A; \oplus, \dot{-}, 0 \rangle$ is a pocrim. Notice that $\mathbf{A}$ is cancellative.

Fact 8.12. $\mathbf{A}$ is isomorphic to a subalgebra of an ultraproduct of the pocrims $\mathbf{A}_k$, $k \in \omega$.

The lengthy but straightforward proof of this fact is given in [BR93], for a sequence of algebras very similar to our $\mathbf{A}_k$, $k \in \omega$, and will therefore be omitted here. As a matter of interest, it can be checked quite easily that $\mathbf{A}$ has exactly one nontrivial congruence, viz., the congruence φ_I, where I is the (unique nontrivial) ideal E_0 of $\mathbf{A}$. Thus $\mathbf{H}(\mathbf{A}) \subseteq \mathscr{C}$, and by Proposition 8.5, $\mathbf{H}(\mathbf{A}^-) \subseteq \mathscr{BCK}$.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{A}$ with universe $B = \bigcup_{i=0}^{\infty} E_{2i}$. Write E_{2i}^o and E_{2i}^e, respectively, for the sets $\{n_{2i}\colon n \text{ odd}\}$ and $\{n_{2i}\colon n \text{ even}\}$. Consider the relation η on $\mathbf{B}$ defined by: $a\,\eta\,b$ iff $a, b \in E_0$, or $a, b \in E_{2i}^o$ for some i, or $a, b \in E_{2i}^e$ for some i. It is readily verified that η is a congruence of $\mathbf{B}$ (hence also of $\mathbf{B}^-$) with $0/\eta = E_0$. Note that η is not a $\mathscr{BCK}$-congruence of $\mathbf{B}^-$, since $1_2/\eta \dot{-} 0_2/\eta = 2_0/\eta = 0_0/\eta = 0_2/\eta \dot{-} 1_2/\eta$ but $1_2/\eta \neq 0_2/\eta$. Thus, $\mathbf{HS}(\mathbf{A}) \not\subseteq \mathscr{M}$ and $\mathbf{HS}(\mathbf{A}^-) \not\subseteq \mathscr{BCK}$. Each of the pocrims $\mathbf{A}_k$, however, belongs to one of the varieties $\mathscr{J}_n \cap \mathscr{C}$ of cancellative pocrims. Since $\mathbf{B} \in \mathbf{SP_u}(\{\mathbf{A}_k\colon k \in \omega\})$ has a homomorphic image $\mathbf{B}/\eta$ which is not a pocrim, $\mathbf{B}$ cannot belong to any variety of (cancellative) pocrims, whence we conclude:

THEOREM 8.13. *There is no largest variety of cancellative pocrims.*

Notice that the above argument also gives an alternative proof of Proposition 8.3. In [WK84] and [BR93], it was shown that there is no largest variety of *BCK*-algebras and no largest variety of pocrims. The method of proof in [BR93] is similar to the argument above but used noncancellative (in fact, finite) algebras in place of our $\mathbf{A}_k$. The essential difference between the two arguments lies in our (seemingly unavoidable) use of the varieties $\mathscr{J}_n \cap \mathscr{C}$ here.

The above construction shows that given a (cancellative) pocrim $\mathbf{A}$, the condition $\mathbf{H}(\mathbf{A}) \subseteq \mathscr{C}$ does not imply the condition $\mathbf{HS}(\mathbf{A}) \subseteq \mathscr{C}$. It is also the case that $\mathbf{HS}(\mathbf{A}) \subseteq \mathscr{C}$ does not imply $\mathbf{HSP}(\mathbf{A}) \subseteq \mathscr{C}$. Consider the same sequence of cancellative pocrims $\mathbf{A}_k$, $k \in \omega$. By the remarks preceding Theorem 8.13, no variety of (cancellative) pocrims contains all of the $\mathbf{A}_k$. Let $\mathbf{D}$ be the "direct sum" of these algebras, i.e., the subalgebra of $\prod_k \mathbf{A}_k$ whose elements are just those of $\prod_k A_k$ which are zero in all but finitely many co-ordinates k. Since each A_k is in $\mathbf{IS}(\mathbf{D})$, we clearly have $\mathbf{HSP}(\mathbf{D}) \not\subseteq \mathscr{C}$. To conclude that $\mathbf{HS}(\mathbf{D}) \subseteq \mathscr{C}$, we need only check that homomorphic images of 2-generated subalgebras of $\mathbf{D}$ satisfy the quasi-identity (5). This is indeed so: every finitely generated subalgebra of $\mathbf{D}$ is a subalgebra of the direct product $\mathbf{E}$ of some finite subfamily of $\{\mathbf{A}_k\colon k \in \omega\}$ and by Corollary 8.11, any such finite product $\mathbf{E}$ lies in the variety $\mathscr{C} \cap \mathscr{J}_n$ for some $n \in \omega$, so that $\mathbf{HS}(\mathbf{E}) \subseteq \mathscr{C} \cap \mathscr{J}_n$. In particular, any algebra in $\mathbf{HS}(\mathbf{E})$ satisfies (5), as required. These distinctions were already drawn for the quasivarieties $\mathscr{M}$ and $\mathscr{BCK}$ in [RS92] and [BR93].

9. PROBLEMS

Problem 1. Is the poset $\mathbf{P}^{v}(\mathscr{M})$ of all varieties of pocrims a lattice? In other words, is the union of any pair of pocrim varieties always contained in a pocrim variety?

Is the lattice $\mathbf{L}^{v}(\mathscr{C})$ of varieties of cancellative pocrims a chain? At this stage, our only knowledge of $\mathbf{L}^{v}(\mathscr{C})$ is that it contains the infinite chain

$$\mathscr{O} \prec \mathbf{ISPP_u}(\boldsymbol{\omega}) = \mathscr{J}_0 \cap \mathscr{C} \subseteq \mathscr{J}_1 \cap \mathscr{C} \subseteq \mathscr{J}_2 \cap \mathscr{C} \subseteq \cdots,$$

where $\mathscr{O}$ is the trivial variety. How many varieties of cancellative pocrims are there?

Problem 2. Are $\mathscr{G}$ and $\mathbf{ISPP_u}(\boldsymbol{\omega})$ the only atoms of the poset $\mathbf{P}^{v}(\mathscr{M})$?

Problem 3. Is every variety of pocrims congruence 3-permutable? If not, is there a bound on the degrees of permutability of pocrim varieties?

Problem 4. Let $\mathscr{V}$ be a variety of pocrims. Is $\mathbf{S}(\mathscr{V}^-)$ necessarily a variety? In other words, is $\mathbf{HS}(\mathscr{V}^-) \subseteq \mathscr{BCK}$? (If so, then the first question of Problem 3 has an affirmative answer, by Theorem 3.3.) In general, we do not even know whether $\mathbf{H}(\mathscr{V}^-) \subseteq \mathscr{BCK}$. If $\mathscr{V}$ is a variety of cancellative pocrims then $\mathbf{H}(\mathscr{V}^-) \subseteq \mathscr{BCK}$, by Proposition 8.5, but we do not know whether $\mathbf{HS}(\mathscr{V}^-) \subseteq \mathscr{BCK}$.

Problem 5. If $\mathscr{K}$ is a variety of *BCK*-algebras with an equational base Σ then the class of all pocrims which are models of Σ is a variety $\mathscr{V}$ with $\mathbf{S}(\mathscr{V}^-) \subseteq \mathscr{K}$ (this inclusion being sharp in some cases), so $\mathbf{S}(\mathscr{V}^-)$ is a variety of *BCK*-algebras.

Under what conditions does the equality $\mathbf{S}(\mathscr{V}^{-}) = \mathscr{K}$ obtain? In particular, for integers $n > 1$, is every BCK-algebra in $\mathscr{E}_n$ a residuation subreduct of an $(n+1)$-potent pocrim? The answer is affirmative if the BCK-algebra is in $\mathscr{E}_n^{s+}$, by Theorem 6.9.

Problem 6. Are there integers $n > 2$ for which the variety $\mathscr{E}_n^{s+}$ (or, equivalently, $\mathscr{E}_n^{s}$) is locally finite?

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