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Mixed superposition rules and the Riccati hierarchy

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ABSTRACT

Mixed superposition rules, i.e., functions describing the general solution of a system of first-order differential equations in terms of a generic family of particular solutions of first-order systems and some constants, are studied. The main achievement is a generalization of the celebrated Lie–Scheffers Theorem, characterizing systems admitting a mixed superposition rule. This somehow unexpected result says that such systems are exactly Lie systems, i.e., they admit a standard superposition rule. This provides a new and powerful tool for finding Lie systems, which is applied here to studying the Riccati hierarchy and to retrieving some known results in a more efficient and simpler way.

1. Introduction

Lie systems are systems of first-order differential equations whose general solutions can be described in terms of generic families of particular solutions and some constants by a particular type of maps: the *superposition rules* [1–5]. The importance of these systems is due, for instance, to their appearance in relevant physical and mathematical problems (see [6] and references therein). This has motivated a number of works devoted to the analysis of their properties and applications [7–13].

The Lie–Scheffers Theorem [1,4], characterizing systems admitting a superposition rule, shows that this property has geometrical roots and is rather exceptional. Although the superposition rules cannot be explicitly derived or effectively applied in many cases [14], their importance has motivated the search for generalizations [14–22]. For example, Vessiot pioneered the study of superposition rules for second-order differential equations [19], which was followed by Winternitz [21] and by Cariñena

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and coworkers [10,11,14,15]. Inselberg, in turn, proposed a concept of superposition rules for operators [22], while Winternitz and Schnider suggested a notion of superposition rules for equations on supermanifolds [20].

Our main aim in this note is the study of *mixed superposition rules*, whose original idea was briefly mentioned in [4,6] and implicitly used in [6,10,23–26] in analyzing a number of important differential equations that appear in the physics and mathematical literature. On the other hand, geometric properties of mixed superposition rules have not been thoroughly analyzed yet and we hope that this work will fill in this gap.

As a main result, we characterize systems possessing a mixed superposition rule. More specifically, we prove that a system of first-order differential equations admits a mixed superposition rule if and only if it is a Lie system. This somehow unexpected theorem, called hereafter the *extended Lie–Scheffers Theorem*, is a generalization of the Lie–Scheffers Theorem. In addition, it provides also a generalization of the *Lie's condition* [6].

The extended Lie–Scheffers Theorem furnishes a new method to determine whether a system is a Lie system: the search for a mixed superposition rule. As the latter is frequently easier than finding a standard superposition rule, we easily recover several recent achievements about Riccati equations [2,6], Milne–Pinney equations [10,21], second-order Riccati equations [14], and second-order Kummer–Schwarz equations [15] in a unified and simple way. Moreover, our approach can be applied in other cases, e.g. the linearization of certain differential equations [23], where Lie systems could also be employed.

Although we prove that the mixed superposition rules, like the standard superposition rules, can only appear in the context of Lie systems, the mixed superposition rules are much more versatile than the standard ones and allow us to express the general solution of a Lie system in a much broader variety of forms. Further, we show that mixed superposition rules can be used to investigate simultaneously different Lie systems.

Subsequently, in an analysis of mixed superposition rules, we demonstrate that there are some relations between properties of the involved Lie systems, which provides us with tools for the investigation of main features of one Lie system in terms of the characteristics of the others.

Our theoretical achievements provide a better understanding of geometrical properties of various particular equations of interest and furnish a framework for the study of questions of physical and mathematical relevance. Special attention we paid to the analysis of the *Riccati hierarchy* [23,13], which is described through an infinite family of Lie systems admitting mixed superposition rules. This permits us to combine the general theory of Lie systems with our new methods in studying the whole Riccati hierarchy and other related problems [2,6,10,13,14,23–26].

The structure of the paper goes as follows. Section 2 concerns the description of the basic notions to be used throughout the paper. In Section 3 we present the motivation for working with mixed superposition rules and in Section 4 we prove that mixed superposition rules can be understood as a certain type of flat connections that completes in depth an idea pointed out in [4]. Next, we provide in Section 5 a characterization of systems admitting a mixed superposition rule and related results. In Section 6 we investigate a relevant type of mixed superposition rules appearing in the literature. In Section 7 we use our theoretical results to study the members of the Riccati hierarchy and other equations of physical interest. Finally, our main results and perspectives of a future work are summarized in Section 8.

2. Preliminaries

For simplicity, we hereafter restrict ourselves to systems of differential equations on vector spaces and assume geometric objects and functions to be real, smooth, and globally defined. This allows us to highlight the main aspects of our work without discussing minor technical details.

Every system of first-order differential equations on $\mathbb{R}^{n_0}$,

$$\frac{dx^i}{dt} = X^i(t,x), \quad i = 1,\ldots,n_0, \tag{1}$$

is determined by the unique t-dependent vector field on $\mathbb{R}^{n_0}$,

$$X(t,x)=\sum_{i=1}^{n_0} X^i(t,x)\frac{\partial}{\partial x^i}, \tag{2}$$

whose integral curves (see [15] for details on this notion) are, up to a reparametrization, of the form $\gamma(t)=(t,x(t))$, with $x(t)$ being a solution of (1). Conversely, the t-dependent vector field X determines its so-called *associated system*, i.e. the system (1) whose solutions describe its integral curves of the form $\gamma: t\in\mathbb{R}\mapsto(t,x(t))\in\mathbb{R}\times\mathbb{R}^{n_0}$. This justifies the use of the symbol X for both, a t-dependent vector field and its associated system. Additionally, it can be proved that every t-dependent vector field X is equivalent to a t-parametrized family $\{X^t\}_{t\in\mathbb{R}}$ of vector fields $X^t:\mathbb{R}^{n_0}\ni x\mapsto X(t,x)\in \mathrm{T}\mathbb{R}^{n_0}$ [15].

Definition 1. Given a (finite or infinite) family $\mathcal{A}$ of vector fields on $\mathbb{R}^{n_0}$, we denote with $\mathrm{Lie}(\mathcal{A})$ the smallest Lie algebra V of vector fields on $\mathbb{R}^{n_0}$ containing $\mathcal{A}$. The *minimal Lie algebra* V^X of a t-dependent vector field X is $V^X=\mathrm{Lie}(\{X^t\}_{t\in\mathbb{R}})$.

Our work is mainly aimed to analyze mixed superposition rules. This concept represents a generalization of the notion of a superposition rule.

Definition 2. A *superposition rule* for a system X is a function $\Phi:\mathbb{R}^{mn_0}\times\mathbb{R}^{n_0}\to\mathbb{R}^{n_0}$ of the form

$$x=\Phi(x_{(1)},\dots,x_{(m)};k_1,\dots,k_{n_0}) \tag{3}$$

that allows us to write its general solution $x(t)$ as

$$x(t)=\Phi\big(x_{(1)}(t),\dots,x_{(m)}(t);k_1,\dots,k_{n_0}\big), \tag{4}$$

with $x_{(1)}(t),\dots,x_{(m)}(t)$ being a generic family of particular solutions and $k_1,\dots,k_{n_0}$ being constants related to the initial conditions for $x(t)$.

The characterization of systems admitting a superposition rule is given by the celebrated *Lie–Scheffers Theorem* [1,6].

Theorem 3. *A system X admits a superposition rule if and only if it can be written in the form*

$$X^t=\sum_{\alpha=1}^{r} b_\alpha(t)Y_\alpha, \tag{5}$$

for a family $Y_1,\dots,Y_r$ of vector fields closing on an r-dimensional real Lie algebra and t-dependent functions $b_1(t),\dots,b_r(t)$. In other words, X admits a superposition rule if and only if V^X is finite-dimensional.

The finite-dimensional real Lie algebras of vector fields which are related to Lie systems are usually called *Vessiot–Guldberg Lie algebras* in the literature [6]. Apart from the Lie–Scheffers Theorem, Lie also proved that if X admits a superposition rule depending on m particular solutions, then X possesses a Vessiot–Guldberg Lie algebra V of dimension at most $m\cdot n_0$. This is referred to as *Lie's condition* [6].

The Lie–Scheffers Theorem plays a central rôle in our work. Its geometric proof makes use of *diagonal prolongations* [4,6].

Definition 4. Given a t-dependent vector field X, the t-dependent vector field $\widetilde{X}$ on $\mathbb{R}^{n_0(m+1)}$ of the form

$$\widetilde{X} = \sum_{a=0}^{m} \sum_{i=1}^{n_0} X^i(t, x_{(a)}) \frac{\partial}{\partial x^i_{(a)}} \tag{6}$$

is called the *diagonal prolongation* of X to $\mathbb{R}^{n_0(m+1)}$.

It is important to note that, given two t-independent vector fields X_1, X_2, we have $\widetilde{[X_1, X_2]} = [\widetilde{X}_1, \widetilde{X}_2]$. Hence, if V is a Lie algebra of vector fields on $\mathbb{R}^{n_0}$, the diagonal prolongations of its elements to $\mathbb{R}^{n_0 m}$ span a Lie algebra of vector fields isomorphic to V. Given a Lie algebra of vector fields V on $\mathbb{R}^{n_0}$, we will denote with V_m the Lie algebra spanned by the diagonal prolongations of the elements of V to $\mathbb{R}^{n_0 m}$.

3. On the definition of mixed superposition rules

It is natural to generalize the notion of a superposition rule and to consider the mixed superposition rules that describe the general solution of a system in terms of generic families of particular solutions of (maybe different) first-order systems and a set of constants. To motivate this concept and to illustrate its usefulness, we will now provide a series of examples.

Consider a linear differential equation of the form

$$\frac{dx}{dt} = a(t)x + b(t), \tag{7}$$

with $a(t)$ and $b(t)$ being any pair of t-dependent functions. It is well known that its general solution can be written as

$$x(t) = x_{(1)}(t) + k x_{(2)}(t), \quad k \in \mathbb{R}, \tag{8}$$

where $x_{(1)}(t)$ and $x_{(2)}(t)$ are generic particular solutions of (7) and the homogeneous system

$$\frac{dx}{dt} = a(t)x, \tag{9}$$

respectively. In other words, linear systems admit their general solutions to be described in terms of particular solutions of two different systems, the original one and the homogeneous one, and a constant.

Let us now turn to Bernoulli equations, i.e. first-order differential equations

$$\frac{dx}{dt} = a(t)x + b(t)x^n, \quad n \neq 1, \tag{10}$$

with $a(t)$ and $b(t)$ being two arbitrary t-dependent functions. These equations form another class of first-order differential equations whose general solutions can be obtained from particular solutions of two different systems and a constant. Indeed, the change of variables $z = x^{1-n}$ transforms (10) into

$$\frac{dz}{dt} = (1-n)\big(a(t)z + b(t)\big),$$

whose general solution is $z(t) = z_p(t) + kz_h(t)$, where k is a real constant, $z_p(t)$ is a particular solution of the above equation, and $z_h(t)$ is a particular solution of

$$\frac{dz}{dt} = (1-n)a(t)z.$$

Undoing the previous change of variables, the solution of a Bernoulli equation can be cast in the form

$$x(t) = \left(x_{(1)}^{1-n}(t) + kx_{(2)}^{1-n}(t)\right)^{\frac{1}{1-n}}, \tag{11}$$

where $x_{(1)}(t)$ and $x_{(2)}(t)$ are particular solutions of (10) and (9), respectively.

Let us now discuss a family of systems admitting a feature analogue to the above ones. Let X be a Lie system, so that we can write

$$X^t = \sum_{\alpha=1}^{r} b_\alpha(t)Y_\alpha, \tag{12}$$

for certain t-dependent functions $b_1(t), \ldots, b_r(t)$ and vector fields $Y_1, \ldots, Y_r$ spanning an r-dimensional real Lie algebra V of vector fields. It is known that there always exists a (local) Lie group action $\Phi : G \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ whose fundamental vector fields coincide with V [27, Theorem XI]. It can also be proved that the general solution $x(t)$ of X can be brought into the form

$$x(t) = \Phi\left(g_{(1)}(t); k\right), \tag{13}$$

where $k \in \mathbb{R}^{n_0}$ and $g_{(1)}(t)$ is a particular solution of the Lie system

$$\frac{dg}{dt} = -\sum_{\alpha=1}^{r} b_\alpha(t)Y_\alpha^R(g), \quad g \in G, \tag{14}$$

where each Y_α^R is the unique right-invariant vector field on G corresponding to the fundamental vector field $Y_\alpha \in V$ (see [3,5,6] for details). This shows that the general solution of a Lie system can always be described by a particular solution of a system of the form (14). It is worth noting that this interesting result presents an important drawback: frequently the explicit expression of Φ cannot be determined.

Winternitz–Smorodinsky oscillators [28–31] and Milne–Pinney equations [24,32–35] can easily be described by means of systems of first-order differential equations

$$\begin{cases} \dfrac{dx}{dt} = p_x, \\ \dfrac{dp_x}{dt} = -\omega^2(t)x + \dfrac{c}{x^3}, \end{cases} \tag{15}$$

with c being a real constant and $x > 0$. The general solution $(x(t), p_x(t))$ of any such system can be determined through two particular solutions $(x_{(1)}(t), p_{(1)}(t))$, $(x_{(2)}(t), p_{(2)}(t))$ of the linear first-order system

$$\begin{cases} \dfrac{dx}{dt} = p, \\ \dfrac{dp}{dt} = -\omega^2(t)x, \end{cases} \tag{16}$$

obtained by adding a new variable $p \equiv dx/dt$ to the equation of a t-dependent frequency harmonic oscillator, through the expressions

$$x(t) = \frac{\sqrt{2}}{|W|}\big[k_1x_{(1)}^2(t) + k_2x_{(2)}^2(t) + \sqrt{4k_1k_2 - cW^2}x_{(1)}(t)x_{(2)}(t)\big]^{1/2},$$

$$p(t) = \frac{\sqrt{2}[k_1x_{(1)}(t)p_{(1)}(t) + k_2x_{(2)}(t)p_{(2)}(t) + \sqrt{k_1k_2 - c(W/2)^2}(p_{(1)}(t)x_{(2)}(t) + x_{(1)}(t)p_{(2)}(t))]}{|W|[k_1x_{(1)}^2(t) + k_2x_{(2)}^2(t) + \sqrt{4k_1k_2 - cW^2}x_{(1)}(t)x_{(2)}(t)]^{1/2}}, \tag{17}$$

where $W = x_{(1)}(t)p_{(2)}(t) - p_{(1)}(t)x_{(2)}(t)$ is a constant of motion of system (16) and k_1, k_2 are two real constants (see [10,35,36]). It is to be remarked that these expressions are interesting because they allow us to study systems appearing in the calculation of invariants for non-quadratic Hamiltonian systems, cosmology, quantum mechanics, Bose–Einstein condensates, etc. [33–35].

Finally, recall that we aim to introduce a geometric notion covering, as particular cases, expressions of the type (8), (11), (13), (17), and standard superposition rules. This leads us to the following.

Definition 5 *(Mixed superposition rule).* A *mixed superposition rule* for a system X is an $(m+1)$-tuple $(\Phi, X_{(1)}, \ldots, X_{(m)})$ consisting of a function $\Phi : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ and a series of systems $X_{(a)}$ on $\mathbb{R}^{n_a}$, with $a = 1, \ldots, m$, such that the general solution $x(t)$ of X can be cast in the form

$$x(t) = \Phi\big(x_{(1)}(t), \ldots, x_{(m)}(t); k_1, \ldots, k_{n_0}\big), \tag{18}$$

where $x_{(1)}(t), \ldots, x_{(m)}(t)$ is a generic family of particular solutions of $X_{(1)}, \ldots, X_{(m)}$, respectively, and $k_1, \ldots, k_{n_0}$ are constants related to initial conditions.

Note 6. Let us stress that the map Φ, describing a mixed superposition rule, does not depend on t.

In view of the above definition, a superposition rule for a system X on $\mathbb{R}^{n_0}$ can be viewed naturally as a mixed superposition rule of the form

$$\big(\Phi : \mathbb{R}^{n_0 m} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}, \overbrace{X, \ldots, X}^{m \text{ times}}\big). \tag{19}$$

In this language, we can say that linear system (7) admits the mixed superposition rule given by $(\Phi_1, X_{(1)}, X_{(2)})$, where $\Phi_1 : (x_{(1)}, x_{(2)}; k) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto x = x_{(1)} + kx_{(2)} \in \mathbb{R}$, $X_{(1)} = (a(t)x_{(1)} + b(t))\partial/\partial x_{(1)}$, and $X_{(2)} = a(t)x_{(2)}\partial/\partial x_{(2)}$.

The Bernoulli equations (10) possess the mixed superposition rule $(\Phi_2 : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, X_{(1)}, X_{(2)})$, where $\Phi_2(x_{(1)}, x_{(2)}; k) = (x_{(1)}^{1-n} + kx_{(2)}^{1-n})^{\frac{1}{1-n}}$, $X_{(1)} = (a(t)x_{(1)} + b(t)x_{(1)}^n)\partial/\partial x_{(1)}$ and $X_{(2)} = a(t)x_{(2)}\partial/\partial x_{(2)}$. Meanwhile, every Lie system (12) admits a mixed superposition rule $(\Phi; X_{(1)})$, where $X_{(1)} = -\sum_{\alpha=1}^r b_\alpha(t)Y_\alpha^R$ and Φ is given by (13).

Finally, Winternitz–Smorodinsky oscillators (15) or Milne–Pinney equations, written as a first-order system, admit the mixed superposition rule

$$\Phi_3 : (\xi_{(1)}, \xi_{(2)}; k_1, k_2) \in \mathrm{T}^*\mathbb{R}_+ \times \mathrm{T}^*\mathbb{R}_+ \times \mathbb{R}^2 \mapsto (x, p) \in \mathrm{T}^*\mathbb{R}_+,$$

with $\xi_{(1)} = (x_{(1)}, p_{(1)})$, $\xi_{(2)} = (x_{(2)}, p_{(2)})$, and

$$x = \frac{\sqrt{2}}{|x_{(1)}p_{(2)} - p_{(1)}x_{(2)}|}\big[k_1x_{(1)}^2 + k_2x_{(2)}^2 + \sqrt{4k_1k_2 - c(x_{(1)}p_{(2)} - p_{(1)}x_{(2)})^2}x_{(1)}x_{(2)}\big]^{1/2},$$

$$p = \frac{\sqrt{2}[k_1 x_{(1)} p_{(1)} + k_2 x_{(2)} p_{(2)} + \sqrt{k_1 k_2 - c(p_{(1)} x_{(2)} + p_{(2)} x_{(1)})^2/4}(p_{(1)} x_{(2)} + p_{(2)} x_{(1)})]}{|x_{(1)} p_{(2)} - p_{(1)} x_{(2)}|[k_1 x_{(1)}^2 + k_2 x_{(2)}^2 + \sqrt{4k_1 k_2 - c(x_{(1)} p_{(2)} - p_{(1)} x_{(2)})^2} x_{(1)} x_{(2)}]^{1/2}}.$$

Apart from Milne–Pinney equations, many other systems posses a similar property, i.e., their general solutions can be obtained through solutions of a t-dependent linear homogeneous system of first-order differential equation. For instance, Riccati equations, certain second-order Riccati equations, second-order Kummer–Schwarz equations, and other systems appearing in the linearization of differential equations admit a similar feature [2,23,26,37,35].

4. Mixed superposition rules as flat connections

Similarly to standard superposition rules, mixed superposition rules can be associated with special flat connections. This idea, originally proposed in [4], is now developed in depth.

Consider a mixed superposition rule $(\Phi, X_{(1)}, \dots, X_{(m)})$, with $\Phi : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$, for a system $X_{(0)}$ on $\mathbb{R}^{n_0}$ (we shall see briefly why this notation is appropriate for our purposes). The Implicit Function Theorem shows that, fixing a point $p = (x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$, the map $\Phi|_p : k \in \mathbb{R}^{n_0} \mapsto x_{(0)} = \Phi(p; k) \in \mathbb{R}^{n_0}$ can locally be inverted to define a mapping $\Psi : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}$ such that

$$\Psi(x_{(0)}, \dots, x_{(m)}) = k,$$

where $k = (k_1, \dots, k_{n_0})$ is the only point of $\mathbb{R}^{n_0}$ satisfying

$$x_{(0)} = \Phi(x_{(1)}, \dots, x_{(m)}; k).$$

Hence, the map Ψ determines an n_0-codimensional (generally local) foliation of $\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$. As Φ is a mixed superposition rule, for a generic family of particular solutions $x_{(0)}(t), \dots, x_{(m)}(t)$ of $X_{(0)}, \dots, X_{(m)}$, we have

$$x_{(0)}(t) = \Phi\big(x_{(1)}(t), \dots, x_{(m)}(t); k\big) \quad \Longleftrightarrow \quad \Psi\big(x_{(0)}(t), \dots, x_{(m)}(t)\big) = k.$$

Differentiating the latter expression with respect to t, we get

$$\sum_{a=0}^{m} \sum_{i=1}^{n_a} X_{(a)}^i\big(t, x_{(a)}(t)\big) \frac{\partial \Psi^j}{\partial x_{(a)}^i} = 0, \quad j = 1, \dots, n_0,$$

with $\Psi = (\Psi^1, \dots, \Psi^{n_0})$. Hence,

$$\sum_{a=0}^{m} X_{(a)}\big(t, x_{(a)}(t)\big) \Psi^j\big(x_{(0)}(t), \dots, x_{(m)}(t)\big) = Z^t \Psi^j\big(x_{(0)}(t), \dots, x_{(m)}(t)\big) = 0, \tag{20}$$

for $j = 1, \dots, n_0$, where we define Z to be the t-dependent vector field on $\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$ of the form

$$Z = \sum_{a=0}^{m} \sum_{i=1}^{n_a} X_{(a)}^i(t, x_{(a)}) \frac{\partial}{\partial x_{(a)}^i}. \tag{21}$$

This will be called the *direct product*, $Z = X_{(0)} \times \cdots \times X_{(m)}$, of the t-dependent vector fields $X_{(0)}, \dots, X_{(m)}$. More precisely, we have the following definition.

Definition 7. Given a set $X_{(0)},\ldots,X_{(m)}$ of t-dependent vector fields defined, respectively, on $\mathbb{R}^{n_0},\ldots,\mathbb{R}^{n_m}$, their *direct product* (or *direct prolongation*) $Z = X_{(0)} \times \cdots \times X_{(m)}$ is the unique t-dependent vector field Z on $\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$ such that $\mathrm{pr}_{a*} Z^t = (X_{(a)})^t$, with $\mathrm{pr}_a : (x_{(0)},\ldots,x_{(m)}) \in \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m} \mapsto x_{(a)} \in \mathbb{R}^{n_a}$ for $a = 0,\ldots,m$ and $t \in \mathbb{R}$.

We say that a t-dependent vector field Z on $\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$ is a *direct prolongation* if Z can be written as the direct product of a family of t-dependent vector fields on the spaces $\mathbb{R}^{n_0},\ldots,\mathbb{R}^{n_m}$. In other words, Z is a direct prolongation if the vector fields $\{Z^t\}_{t\in\mathbb{R}}$ are projectable onto the spaces $\mathbb{R}^{n_a}$, with $a = 0,\ldots,m$.

As equalities (20) hold for a generic family of particular solutions $x_{(0)}(t),\ldots,x_{(m)}(t)$, it turns out that a mixed superposition rule $(\Phi, X_{(1)},\ldots,X_{(m)})$ for a system $X_{(0)}$ implies the existence of n_0 common first-integrals, namely $\Psi^1,\ldots,\Psi^{n_0}$, of the vector fields $\{Z^t\}_{t\in\mathbb{R}}$ for the direct product Z. These functions give rise to an n_0-codimensional foliation $\mathfrak{F}$ such that the vector fields Z^t are tangent to its leaves, $\mathfrak{F}_k$ with $k \in \mathbb{R}^{n_0}$. Hence, vector fields from V^Z span an integrable distribution over a dense and open subset of $\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$.

The foliation $\mathfrak{F}$ has another important property. Given a level set $\mathfrak{F}_k$ corresponding to $k = (k_1,\ldots,k_{n_0})$ and $(x_{(1)},\ldots,x_{(m)}) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$, there is a unique point $x_{(0)} \in \mathbb{R}^{n_0}$ such that $(x_{(0)}, x_{(1)},\ldots,x_{(m)}) \in \mathfrak{F}_k$. Then, the projection

$$\mathrm{pr} : (x_{(0)},\ldots,x_{(m)}) \in \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m} \mapsto (x_{(1)},\ldots,x_{(m)}) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \tag{22}$$

induces local diffeomorphisms among the leaves $\mathfrak{F}_k$ of $\mathfrak{F}$ and $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$. Such foliation we will call *horizontal* foliation on the bundle (22).

This property shows that $\mathfrak{F}$ corresponds to a zero curvature connection ∇ in the bundle $\mathrm{pr} : \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m} \to \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$. Indeed, the restriction of pr to each leaf gives a (local) one-to-one map. In this way, there exists a linear map among vector fields on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ and horizontal vector fields tangent to a leaf. This connection (foliation), along with the vector fields $X_{(1)},\ldots,X_{(m)}$, provides us with a *mixed superposition rule* for $X_{(0)}$ without referring to the map Ψ. Indeed, if we take a point $x_{(0)}$ and m particular solutions $x_{(1)}(t),\ldots,x_{(m)}(t)$ of the systems $X_{(1)},\ldots,X_{(m)}$, respectively, then $x_{(0)}(t)$ is the unique curve in $\mathbb{R}^{n_0}$ such that

$$\big(x_{(0)}(t), x_{(1)}(t),\ldots,x_{(m)}(t)\big) \in \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$$

and $(x_{(0)}(0), x_{(1)}(0),\ldots,x_{(m)}(0))$ belong to the same leaf. Thus, it is the foliation $\mathfrak{F}$ and the systems $X_{(1)},\ldots,X_{(m)}$, what really matters if the *mixed superposition rule* for a system $X_{(0)}$ is concerned.

Conversely, assume that we are given a system $X_{(0)}$, whose general solution we want to analyze. Let $X_{(1)},\ldots,X_{(m)}$ be the family of systems whose particular solutions will be used to analyze $X_{(0)}$ and ∇ be a flat connection on the bundle $\mathrm{pr} : \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m} \to \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$, which can be integrated to an n_0-codimensional foliation $\mathfrak{F}$ on $\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$ such that the vector fields Z^t, for $Z = X_{(0)} \times X_{(1)} \times \cdots \times X_{(m)}$, are tangent to the leaves of $\mathfrak{F}$. Then, the above procedure provides us with a mixed superposition rule for the system $X_{(0)}$ in terms of solutions of $X_{(1)},\ldots,X_{(m)}$.

Indeed, let $k \in \mathbb{R}^{n_0}$ enumerate smoothly the leaves $\mathfrak{F}_k$ of $\mathfrak{F}$, i.e. there exists a smooth map $\iota : \mathbb{R}^{n_0} \to \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$ such that $\iota(\mathbb{R}^{n_0})$ intersects every $\mathfrak{F}_k$ at a unique point. Then, if $x_{(0)} \in \mathbb{R}^{n_0}$ is the unique point such that

$$(x_{(0)}, x_{(1)},\ldots,x_{(m)}) \in \mathfrak{F}_k,$$

we can define a *mixed superposition rule* for $X_{(0)}$ of the form

$$x_{(0)} = \Phi(x_{(1)},\ldots,x_{(m)}; k),$$

in terms of solutions of the systems $X_{(1)},\ldots,X_{(m)}$.

Let us analyze the above claim in detail. The Implicit Function Theorem shows that there exists a function $\Psi:\mathbb{R}^{n_0}\times\cdots\times\mathbb{R}^{n_m}\to\mathbb{R}^{n_0}$ such that

$$\Psi(x_{(0)},\ldots,x_{(m)})=k$$

is equivalent to $(x_{(0)},\ldots,x_{(m)})\in\mathfrak{F}_k$. If we fix k and take solutions $x_{(1)}(t),\ldots,x_{(m)}(t)$ of $X_{(1)},\ldots,X_{(m)}$, then $x_{(0)}(t)$, defined by $\Psi(x_{(0)}(t),\ldots,x_{(m)}(t))=k$, is an integral curve of $X_{(0)}$. Indeed, since the vector fields Z^t are tangent to $\mathfrak{F}$, if $x'_{(0)}(t)$ is a solution of $X_{(0)}$ with the initial value $x'_{(0)}(0)=x_{(0)}$, then the curve

$$t\mapsto\big(x_{(0)}(t),x_{(1)}(t),\ldots,x_{(m)}(t)\big)$$

lies entirely in a leaf of $\mathfrak{F}$, so in $\mathfrak{F}_k$. But the point of one leaf is entirely determined by its projection pr, so $x'_{(0)}(t)=x_{(0)}(t)$ and $x_{(0)}(t)$ is a solution. Summarizing, we have proved the following proposition.

Proposition 8. *A mixed superposition rule* $(\Phi,X_{(1)},\ldots,X_{(m)})$, *with* $\Phi:\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}\times\mathbb{R}^{n_0}\to\mathbb{R}^{n_0}$, *for a system* X *on* $\mathbb{R}^{n_0}$ *amounts to a flat connection* (*equivalently, horizontal foliation* $\mathfrak{F}$) *on the bundle* $\mathrm{pr}:\mathbb{R}^{n_0}\times\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}\to\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}$ *such that the vector fields* $\{Z^t\}_{t\in\mathbb{R}}$, *associated with the direct product* $Z=X\times X_{(1)}\times\cdots\times X_{(m)}$, *are horizontal vector fields with respect to the connection* (*resp., are tangent to the leaves of* $\mathfrak{F}$).

According to the above proposition, the horizontal foliation $\mathfrak{F}$ contains the generalized foliation $\mathfrak{F}^0$ associated with the generalized distribution $\mathcal{D}$ generated by the Lie algebra $V^Z=\mathrm{Lie}(\{Z^t\}_{t\in\mathbb{R}})$. One can regard $\mathfrak{F}^0$ as a regular foliation on an open and dense subset of $\mathbb{R}^{n_0}\times\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}$. If only the projection

$$\mathrm{pr}_{\hat d}:\mathbb{R}^{n_0}\times\cdots\times\mathbb{R}^{n_m}\to\mathbb{R}^{n_0}\times\cdots\times\widehat{\mathbb{R}^{n_d}}\times\cdots\times\mathbb{R}^{n_m},\tag{23}$$

where $\widehat{\mathbb{R}^{n_d}}$ indicates that this space is not included in the direct product, induces diffeomorphisms on the leaves of $\mathfrak{F}^0$, i.e., it is an injective map on the fibers of $\mathcal{D}$, then $\mathfrak{F}^0$ can be extended to an n_d-codimensional foliation $\mathfrak{F}^{(d)}$, horizontal on the bundle (23), which will define a mixed superposition rule for the system $X_{(d)}$. In this way, we get the following.

Proposition 9. *Consider* t*-dependent systems* $X_{(a)}$ *on* $\mathbb{R}^{n_a}$, $a=0,\ldots,m$, *their direct product* $Z=X_{(0)}\times X_{(1)}\times\cdots\times X_{(m)}$, *and the corresponding generalized distribution* $\mathcal{D}$ *generated by the Lie algebra* $V^Z=\mathrm{Lie}(\{Z^t\}_{t\in\mathbb{R}})$. *Then,* $X_{(d)}$ *admits a mixed superposition rule*

$$\big(\Phi',X_{(0)},X_{(1)},\ldots,X_{(d-1)},X_{(d+1)},\ldots,X_{(m)}\big)$$

if and only if the projection (23) *induces injective maps on the fibers of* $\mathcal{D}$.

5. Characterization of systems possessing a mixed superposition rule

In this section, we first analyze the properties of direct products of t-dependent vector fields and other related notions we define. Subsequently, our results are used to characterize and to analyze systems admitting a mixed superposition rule.

Let us recall that a t-dependent vector field Z on $\mathbb{R}^{n_0}\times\cdots\times\mathbb{R}^{n_m}$ is called a *direct prolongation* if Z can be written as the direct product of a family of t-dependent vector fields on the spaces $\mathbb{R}^{n_0},\ldots,\mathbb{R}^{n_m}$, i.e., Z can be then brought into the form

$$Z=\sum_{a=0}^{m}\sum_{i=1}^{n_a}Z_a^i(t,x_{(a)})\frac{\partial}{\partial x^i_{(a)}},\tag{24}$$

for certain functions $Z_a^i:\mathbb{R}\times\mathbb{R}^{n_a}\to\mathbb{R}$, with $a=0,\ldots,m$ and $i=1,\ldots,n_a$.

Note 10. The term *direct prolongation* is coined so as to highlight that this notion is a generalization of the concept of *diagonal prolongations* which appears in the theory of standard superposition rules [4].

In view of (24), the following is obvious.

Lemma 11. *The Lie bracket of t-independent direct prolongations is a t-independent direct prolongation.*

The following lemma provides us with the key property of direct prolongations to characterize systems admitting mixed superposition rules.

Lemma 12. *Consider a family $Z_1, \dots, Z_r$ of t-*independent *direct prolongations on $\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$ such that $\mathrm{pr}_* Z_\alpha$, for $\alpha = 1, \dots, r$, are linearly independent at a generic point. If a vector field $\sum_{\alpha=1}^r f_\alpha Z_\alpha$, with $f_1, \dots, f_r \in C^\infty(\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m})$, is a direct prolongation, then $f_1, \dots, f_r$ depend only on $x_{(1)}, \dots, x_{(m)}$.*

Proof. As $Z_1, \dots, Z_r$ are t-independent direct prolongations, we can write

$$Z_\alpha = \sum_{a=0}^{m} \sum_{i=1}^{n_a} Z^i_{a\alpha}(x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}, \quad \alpha = 1, \dots, r,$$

for certain functions $Z^i_{a\alpha} : \mathbb{R}^{n_a} \to \mathbb{R}$. Likewise, if $\sum_{\alpha=1}^r f_\alpha Z_\alpha$ is a t-independent direct prolongation, there exist functions $B^i_a : \mathbb{R}^{n_a} \to \mathbb{R}$, with $a = 0, \dots, m$ and $i = 1, \dots, n_a$ such that

$$\sum_{\alpha=1}^{r} f_\alpha Z_\alpha = \sum_{a=0}^{m} \sum_{i=1}^{n_a} B^i_a(x_{(a)}) \frac{\partial}{\partial x^i_{(a)}}.$$

Hence,

$$\sum_{\alpha=1}^{r} f_\alpha Z^i_{a\alpha}(x_{(a)}) = B^i_a(x_{(a)}), \quad a = 0, \dots, m, \ i = 1, \dots, n_a.$$

In particular, we have the subset of equations

$$\sum_{\alpha=1}^{r} f_\alpha Z^i_{a\alpha}(x_{(a)}) = B^i_a(x_{(a)}), \quad a = 1, \dots, m, \ i = 1, \dots, n_a.$$

Since the projections $\mathrm{pr}_* Z_\alpha$ are linearly independent at a generic point, the above system has a unique solution $f_1, \dots, f_r$ whose value is determined by the functions B^i_a, $Z^i_{a\alpha}$, for $a = 1, \dots, m$ and $i = 1, \dots, n_a$, which depend on $x_{(1)}, \dots, x_{(m)}$ only. Hence, $f_1, \dots, f_r$ depend exclusively on these variables. □

Let us now prove the central result of our paper.

Theorem 13 *(The extended Lie–Scheffers Theorem). A system X admits a mixed superposition rule if and only if it is a Lie system.*

Proof. Let $(\Phi : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}, X_{(1)}, \dots, X_{(m)})$ be a mixed superposition rule for X and let Z be the direct product of $X \times X_{(1)} \times \cdots \times X_{(m)}$. According to Proposition 8, the elements of $\mathrm{Lie}(\{Z^t\}_{t\in\mathbb{R}})$ span a generalized distribution $\mathcal{D}$ over $\mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_m}$ which is regular and integrable on an open and dense subset of this space. Further, we can always choose a basis $Z_1, \dots, Z_r$ of $\mathcal{D}$

whose elements belong to $\mathrm{Lie}(\{Z^t\}_{t\in\mathbb{R}})$, so, due to Lemma 11, are t-independent direct prolongations. As $\mathcal{D}$ describes a mixed superposition rule, the elements of such a basis project, via pr_*, onto $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}$, giving rise to a family of linearly independent vector fields at a generic point of this space.

Since each $[Z_\alpha, Z_\beta]$ is a direct prolongation that belongs to $\mathcal{D}$ and $Z_1,\ldots,Z_r$ project to a family of linearly independent vector fields at a generic point of $\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}$, Lemma 12 ensures that

$$[Z_\alpha, Z_\beta]=\sum_{\gamma=1}^{r} f_{\alpha\beta\gamma}Z_\gamma,\quad \alpha,\beta=1,\ldots,r,$$

for certain r^3 functions $f_{\alpha\beta\gamma}$ depending on the variables $x_{(1)},\ldots,x_{(m)}$. Further, each $[Z_\alpha, Z_\beta]$ is projectable onto $\mathbb{R}^{n_0}$ under the projection $\mathrm{pr}_0:(x_{(0)},\ldots,x_{(m)})\in\mathbb{R}^{n_0}\times\cdots\times\mathbb{R}^{n_m}\mapsto x_{(0)}\in\mathbb{R}^{n_0}$. Hence,

$$\mathrm{pr}_{0*}[Z_\alpha, Z_\beta]=[\mathrm{pr}_{0*}Z_\alpha,\mathrm{pr}_{0*}Z_\beta]=[Y_\alpha,Y_\beta]=\sum_{\gamma=1}^{r} f_{\alpha\beta\gamma}(x_{(1)},\ldots,x_{(m)})Y_\gamma,\quad \alpha,\beta=1,\ldots,r,$$

where $Y_\alpha\equiv\mathrm{pr}_{0*}Z_\alpha$, with $\alpha=1,\ldots,r$, are vector fields on $\mathbb{R}^{n_0}$. Note that the above equality implies that, for every fixed $(x_{(1)},\ldots,x_{(m)})$, the vector field $[Y_\alpha,Y_\beta]$ on $\mathbb{R}^{n_0}$ is a linear combination of the vector fields $Y_1,\ldots,Y_r$ on $\mathbb{R}^{n_0}$, that is, $Y_1,\ldots,Y_r$ span a finite-dimensional real Lie algebra V of vector fields. Note that this result is due to the fact that all functions $f_{\alpha\beta\gamma}$ just depend on $x_{(1)},\ldots,x_{(m)}$ only. Now, the direct product $Z=X\times X_{(1)}\times\cdots\times X_{(m)}$ can be cast in the form

$$Z=\sum_{\alpha=1}^{r} b_\alpha Z_\alpha,$$

where $b_\alpha\in C^\infty(\mathbb{R}\times\mathbb{R}^{n_0}\times\cdots\times\mathbb{R}^{n_m})$ for $\alpha=1,\ldots,r$. Using again the fact that $\mathrm{pr}_*Z_1,\ldots,\mathrm{pr}_*Z_r$ are properly defined and linearly independent at a generic point, we get out of Lemma 12 that $b_1,\ldots,b_r$ depend only on t and $x_{(1)},\ldots,x_{(m)}$. Proceeding as above, we get that $\mathrm{pr}_{0*}Z^t=X^t$ takes values in V, i.e., X is a Lie system.

Conversely, if X is a Lie system, the Lie–Scheffers Theorem guarantees that it admits a superposition rule. As superposition rules form a particular class of mixed superposition rules, the theorem follows. □

The extended Lie–Scheffers Theorem not only characterizes systems admitting a mixed superposition rule but also provides a new tool to ensure that a system is a Lie system: it is enough to find a mixed superposition rule. For instance, observe that the extended Lie–Scheffers Theorem easily shows that the systems (15) related to Milne–Pinney equations and Winternitz–Smorodinsky oscillators, linear systems of differential equations (7), and Bernoulli equations are Lie systems. This easily retrieves as particular cases many of the results that were obtained in several recent works [10,11,13,38] through the standard Lie–Scheffers Theorem. Additionally, we prove for the first time that all Bernoulli equations are Lie systems, which completes a result merely pointed out in [6]. In Section 7, we will describe some new results in this direction.

Corollary 14 *(Extended Lie's condition). If X is a system admitting a mixed superposition rule $(\Phi:\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}\times\mathbb{R}^{n_0}\to\mathbb{R}^{n_0}, X_{(1)},\ldots,X_{(m)})$, then X admits a Vessiot–Guldberg Lie algebra V such that* $\dim V\leqslant\sum_{a=1}^m n_a$.

Proof. Following the proof of the extended Lie–Scheffers Theorem, we see that the mixed superposition rule for X induces a family of t-independent direct prolongations $Z_1,\ldots,Z_r$ tangent to the leaves of its associated foliation and satisfying that pr_*Z_α, with $\alpha=1,\ldots,r$, must be linearly independent at a generic point. Therefore, $r\leqslant\dim(\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m})=\sum_{a=1}^m n_a$. In addition, since the vector fields

$\mathrm{pr}_{0*} Z_\alpha$, with $\alpha = 1, \ldots, r$, span a finite-dimensional real Lie algebra V containing the vector fields $\{X^t\}_{t\in\mathbb{R}}$, our corollary easily follows. □

Note 15. The above corollary obviously includes the Lie's condition as a particular case. Indeed, it shows that if X admits a superposition rule depending on m particular solutions, then X admits a Vessiot–Guldberg Lie algebra V satisfying that $\dim V \leqslant n_0 \cdot m$.

Let us now discuss our previous results and their relations to the usual notions and properties of superposition rules.

Observe first that the direct product of m copies of a t-dependent vector field $X = \sum_{i=1}^{n_0} X^i(t,x)\partial/\partial x^i$ on $\mathbb{R}^{n_0}$ is a t-dependent vector field on $\mathbb{R}^{n_0 m}$ given by (6). This is exactly the expression for the diagonal prolongation to $\mathbb{R}^{n_0 m}$ of X. In other words, diagonal prolongations are actually direct prolongations involving direct products of several copies of the same t-dependent vector field. On the other hand, some properties of diagonal prolongations are stronger than those for direct prolongations. In particular, the functions $f_1, \ldots, f_r$ that appear in Lemma 12 turn out to be just constant for diagonal prolongations (see [4, Lemma 1]). This simplifies the proof of the Lie–Scheffers Theorem in comparison with our proof of its extended version.

As mentioned in the introduction, mixed superposition rules provide a new method to study solutions of systems of differential equations. Although the extended Lie–Scheffers Theorem shows that mixed superposition rules, like the standard ones, can only be used to study Lie systems, the area of possible applications is much broader as mixed superposition rules are much more versatile.

First of all, mixed superposition rules can be constructed in various ways and sometimes much easily than the strict superposition rules. In particular, Lie systems may admit mixed superposition rules in terms of non-Lie systems. Consider the following trivial example. The equation

$$\frac{dx}{dt} = 0$$

is associated with a Lie system (as every autonomous system [17,18,39]) $X = 0$. Its general solution reads $x(t) = k$, with $k \in \mathbb{R}$. Therefore, the map $\Phi : (x_{(1)}; k) \in \mathbb{R} \times \mathbb{R} \mapsto k \in \mathbb{R}$ gives rise to a mixed superposition rule $(\Phi, X_{(1)})$ for X in terms of any system $X_{(1)}$, e.g. a non-Lie one.

Second, as Proposition 9 shows, mixed superposition rules may enable us to study general solutions of different Lie systems at the same time.

6. Mixed superposition rules in terms of Lie systems

In spite of the fact that non-Lie systems can appear in mixed superposition rules, most of mixed superposition rules appearing in the literature involve exclusively Lie systems [6,10,23]. This motivates the study of this special case. Let us start with some necessary definitions and auxiliary facts to prove the main results of this section.

Definition 16. Given a vector space V of vector fields on $\mathbb{R}^{n_0}$, we say that V admits a *modular basis* if V possesses a basis of vector fields linearly independent at a generic point of $\mathbb{R}^{n_0}$.

Note 17. Note that the term modular basis refers to the fact that the basis of the real Lie algebra V consists of elements which are linearly independent over $C^\infty(\mathbb{R}^{n_0})$ (thus over $\mathbb{R}$). In particular, the space V is finite-dimensional.

Example 1. The Lie algebra $V = \langle x\partial/\partial x, y\partial/\partial y\rangle$ of vector fields on $\mathbb{R}^2$ obviously possesses a modular basis. On the other hand, the Lie algebra $V = \langle \partial/\partial x, x\partial/\partial x\rangle$ of vector fields on $\mathbb{R}$ does not, as there exist no two vector fields in V linearly independent at a generic (actually any) point of $\mathbb{R}$.

Lemma 18. *If V is a Lie algebra of vector fields on $\mathbb{R}^{n_0}$ admitting a modular basis, then every basis of V is modular. In particular, if $Y_1, \ldots, Y_s \in V$ are vector fields linearly independent over $\mathbb{R}$ and $X \in V$ is of the form*

$$X = \sum_{j=1}^{s} b_j Y_j, \tag{25}$$

where $b_1, \ldots, b_s \in C^\infty(\mathbb{R}^{n_0})$, then $b_1, \ldots, b_s$ must be constant.

Proof. Let $\mathcal{D}$ be the generalized distribution spanned by V and let $r = \dim V$. Since V admits a modular basis, the dimension of $\mathcal{D}_p$ is r for a generic point p. Suppose that $X_1, \ldots, X_r$ is a basis of V over $\mathbb{R}$. As the vector fields $X_1, \ldots, X_r$ span $\mathcal{D}_p$ for each p, the vectors $X_1(p), \ldots, X_r(p)$ span an r-dimensional space, thus are linearly independent, for a generic p. Hence, $X_1, \ldots, X_r$ is a modular basis of V. To prove the last statement, let us observe that we can enlarge the family $\{Y_1, \ldots, Y_s\}$ to a basis $Y_1, \ldots, Y_r$ of V which is necessarily a modular basis. We can then write X also in the form

$$X = \sum_{j=1}^{r} c_j Y_j, \tag{26}$$

for some constants c_j, $j = 1, \ldots, r$. As (25) and (26) imply

$$\sum_{j=1}^{s} (c_j - b_j) Y_j + \sum_{l=s+1}^{r} c_l Y_l = 0$$

and $Y_1, \ldots, Y_r$ are linearly independent over $C^\infty(\mathbb{R}^{n_0})$, it must be $b_j = c_j$, so b_j are constants for all $j = 1, \ldots, s$. $\square$

Theorem 19. *If a system X on $\mathbb{R}^{n_0}$ admits a mixed superposition rule in terms of m copies of a Lie system $X_{(1)}$ and $V_m^{X_{(1)}}$ admits a modular basis, then the direct product Z of X and m times $X_{(1)}$ is a Lie system, $V^Z \simeq V^{X_{(1)}}$ and the projection*

$$\mathrm{pr}_0 : \mathbb{R}^{n_0} \times \mathbb{R}^{m \cdot n_1} \to \mathbb{R}^{n_0}$$

induces a Lie algebra homomorphism $\mathrm{pr}_{0*} : V^Z \to V^X$.

Proof. In view of Proposition 9, the vector fields from $\mathrm{Lie}(\{Z^t\}_{t\in\mathbb{R}})$ span a generalized distribution $\mathcal{D}$ over $\mathbb{R}^{n_0} \times \mathbb{R}^{n_1 m}$ which is regular over an open and dense subset of this space, and the projection

$$\mathrm{pr} : \mathbb{R}^{n_0} \times \mathbb{R}^{m \cdot n_1} \to \mathbb{R}^{m \cdot n_1}$$

induces injective maps on fibers of $\mathcal{D}$.

We can choose among the elements of $V^Z = \mathrm{Lie}(\{Z^t\}_{t\in\mathbb{R}})$ a basis $Z_1, \ldots, Z_r$ of $\mathcal{D}$. It follows that such $Z_1, \ldots, Z_r$ are projectable, *via* pr_*, onto $\mathbb{R}^{n_1 m}$ and their projections $\mathrm{pr}_* Z_1, \ldots, \mathrm{pr}_* Z_r$ are linearly independent at a generic point. Note also that such projections are diagonal prolongations, i.e. $\mathrm{pr}_* Z_\alpha = \widetilde{Y}_\alpha$ for certain vector fields $Y_\alpha \in V^{X_{(1)}}$, with $\alpha = 1, \ldots, r$. Moreover, we can prove that $Y_1, \ldots, Y_r$ form a basis of $V^{X_{(1)}}$. Let us start by showing that they span $V^{X_{(1)}}$. Indeed, as $\mathrm{pr}_{1*}(V^Z) = V^{X_{(1)}}$, for any element $\bar{Y} \in V^{X_{(1)}}$ there exists a t-independent direct prolongation $\bar{Z} \in V^Z$ such that $\mathrm{pr}_{1*} \bar{Z} = \bar{Y}$. Then, $\bar{Z} = \sum_{\alpha=1}^{r} f_\alpha Z_\alpha$ and $\mathrm{pr}_* \bar{Z} = \widetilde{Y} = \sum_{\alpha=1}^{r} f_\alpha \widetilde{Y}_\alpha$, where $f_1, \ldots, f_r$ are certain functions depending only on $x_{(1)}, \ldots, x_{(m)}$. Since $V_m^{X_{(1)}}$ admits a modular basis, Lemma 18 ensures that $f_\alpha = c_\alpha$ for certain constants $c_1, \ldots, c_r$, $\alpha = 1, \ldots, r$. Therefore, $\bar{Y}$ is a linear combination of

$Y_1, \dots, Y_r$. Additionally, $Y_1, \dots, Y_r$ must be linearly independent over $\mathbb{R}$ because otherwise $\widetilde{Y}_1, \dots, \widetilde{Y}_r$ would not be linearly independent at a generic point. Hence, $Y_1, \dots, Y_r$ form a basis for $V^{X_{(1)}}$.

Lemma 12 implies now that

$$[Z_\alpha, Z_\beta] = \sum_{\gamma=1}^{r} f_{\alpha\beta\gamma} Z_\gamma, \quad \alpha, \beta = 1, \dots, r,$$

for a unique family of r^3 functions $f_{\alpha\beta\gamma}$ depending on the variables $x_{(1)}, \dots, x_{(m)}$. Hence,

$$\mathrm{pr}_*[Z_\alpha, Z_\beta] = [\widetilde{Y}_\alpha, \widetilde{Y}_\beta] = \sum_{\gamma=1}^{r} f_{\alpha\beta\gamma} \widetilde{Y}_\gamma, \quad \alpha, \beta = 1, \dots, r.$$

As $[\widetilde{Y}_\alpha, \widetilde{Y}_\beta]$ belongs to $V_m^{X_{(1)}}$ and $\widetilde{Y}_1, \dots, \widetilde{Y}_r$ form a modular basis of this Lie algebra, Lemma 18 again yields $f_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}$ for certain r^3 constants $c_{\alpha\beta\gamma}$. Hence,

$$[Z_\alpha, Z_\beta] = \sum_{\gamma=1}^{r} c_{\alpha\beta\gamma} Z_\gamma, \quad \alpha, \beta = 1, \dots, r,$$

and $Z_1, \dots, Z_r$ span an r-dimensional Lie algebra V. Furthermore, as $Y_1, \dots, Y_r$ share the same structure constants as $\widetilde{Y}_1, \dots, \widetilde{Y}_r$ and $Z_1, \dots, Z_r$, then the latter family of vector fields spans a Lie algebra isomorphic to $V^{X_{(1)}}$, i.e. $V^Z \simeq V^{X_{(1)}}$. Note that the vector fields $\{\mathrm{pr}_* Z^t\}_{t\in\mathbb{R}}$ are diagonal prolongations of elements $V^{X_{(1)}}$ and they are therefore spanned by linear combinations (with constant coefficients) of the diagonal prolongations $\mathrm{pr}_* Z_\alpha = \widetilde{Y}_\alpha$. In other words, there exist t-dependent functions $b_1(t), \dots, b_r(t)$ such that $\mathrm{pr}_* Z^t = \sum_{\alpha=1}^{r} b_\alpha(t)\widetilde{Y}_\alpha$, with $\alpha = 1, \dots, r$. Hence, as Z^t can be described as a unique linear combination of $Z_1, \dots, Z_r$, it follows that

$$Z^t = \sum_{\alpha=1}^{r} b_\alpha(t) Z_\alpha.$$

In other words, Z is a Lie system related to a Vessiot–Guldberg Lie algebra V^Z. Obviously $V^Z \simeq V^{X_{(1)}}$ and, as $\mathrm{pr}_{0*} \mathrm{Lie}(\{Z^t\}_{t\in\mathbb{R}}) = \mathrm{Lie}(\{\mathrm{pr}_{0*} Z^t\}_{t\in\mathbb{R}}) = \mathrm{Lie}(\{X^t\}_{t\in\mathbb{R}}) = V^X$, we have $V^X = \mathrm{pr}_{0*} V^Z$. $\square$

Let us give a second result which allows us to link, under certain conditions, the properties of a Lie system to those of the Lie systems involved in its mixed superposition rules.

Theorem 20. *If a system X on $\mathbb{R}^{n_0}$ admits a mixed superposition rule*

$$\big(\Phi : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}, X_{(1)}, \dots, X_{(m)}\big)$$

in terms of m Lie systems $X_{(1)}, \dots, X_{(m)}$, and, for a certain $k \in \mathbb{R}^{n_0}$, the mapping

$$\Phi_k : (x_{(1)}, \dots, x_{(m)}) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m} \mapsto \Phi(x_{(1)}, \dots, x_{(m)}; k) \in \mathbb{R}^{n_0}, \quad k \in \mathbb{R}^{n_0},$$

has open and dense image in $\mathbb{R}^{n_0}$, then its tangent map induces a Lie algebra epimorphism $\Phi_{k*} : V^{\widehat{X}} \to V^X$, with $\widehat{X} = X_{(1)} \times \cdots \times X_{(m)}$.

Proof. Note that, under the above assumptions,

$$x(t) = \Phi\big(x_{(1)}(t), \ldots, x_{(m)}(t); k\big) = \Phi_k\big(x_{(1)}(t), \ldots, x_{(m)}(t)\big)$$

is a solution of X for every generic family of particular solutions $x_{(1)}(t), \ldots, x_{(m)}(t)$ of the systems $X_{(1)}, \ldots, X_{(m)}$, respectively. Differentiating in terms of t and using that $\operatorname{Im}\Phi_k$ is dense and open in $\mathbb{R}^{n_0}$, we obtain

$$X^t = \Phi_{k*}(X_{(1)} \times \cdots \times X_{(m)})_t = \Phi_{k*}\widehat{X}^t, \quad \forall t \in \mathbb{R}.$$

As $X_{(1)}, \ldots, X_{(m)}$ are Lie systems, the vector fields corresponding to

$$\begin{array}{ccccccccc} X_{(1)} & \times & 0 & \times & \cdots & \times & 0 & \times & 0, \\ 0 & \times & X_{(2)} & \times & \cdots & \times & 0 & \times & 0, \\ & & & & \cdots & & & & \\ 0 & \times & \cdots & \times & \cdots & \times & 0 & \times & X_{(m)} \end{array}$$

span a finite-dimensional Lie algebra of vector fields containing $V^{\widehat{X}}$, which shows that $\widehat{X}$ is a Lie system. From here, it can easily be proved that Φ_{k*} is well defined over $V^{\widehat{X}}$, i.e. its elements are projectable under Φ_{k*}, and $V^X = \Phi_{k*}V^{\widehat{X}}$. □

Corollary 21. *If a non-zero Lie system X admits a mixed superposition rule $(\Phi: \mathbb{R}^{n_1 m} \times \mathbb{R}^{n_0} \to \mathbb{R}^{n_0}, X_{(1)}, \ldots, X_{(1)})$ in terms of m particular solutions of a Lie system $X_{(1)}$ such that $V^{X_{(1)}}$ is a simple Lie algebra and at least one of the following conditions holds*:

1. *$V_m^{X_{(1)}}$ admits a modular basis,*
2. *there exists a $k \in \mathbb{R}^{n_0}$ such that $\operatorname{Im}\Phi_k$ is open and dense in $\mathbb{R}^{n_0}$,*

then $V^X \simeq V^{X_{(1)}}$.

Proof. If (1) holds, Theorem 19 shows that $V^Z \simeq V^{X_{(1)}}$, with $Z = X \times \overbrace{X_{(1)} \times \cdots \times X_{(1)}}^{m \text{ times}}$. So if $V^{X_{(1)}}$ is simple, then V^Z is simple and any non-trivial Lie algebra homomorphism from V^Z is an isomorphism. In particular, $V^X = \mathrm{pr}_*(V^Z) \simeq V^Z \simeq V^{X_{(1)}}$.

In a similar way, if (2) is satisfied, Theorem 20 states that there exists a Lie algebra epimorphism from $V^{\widehat{X}}$ to V^X, with $\widehat{X} = X_{(1)} \times \cdots \times X_{(1)}$ (m times). As $V^{X_{(1)}}$ is simple, $V^{\widehat{X}} \simeq V^{X_{(1)}}$ is also. Hence, as $V^X \neq \{0\}$, then $\Phi_{k*}: V^{\widehat{X}} \to V^X$ is an isomorphism and $V^X \simeq V^{\widehat{X}} \simeq V^{X_{(1)}}$. □

Observe that Theorems 19 and 20 as well as Corollary 21 provide information about the Vessiot–Guldberg Lie algebras associated to a system admitting a mixed superposition rule in terms of other Lie systems. This will be used in the following section to describe a new infinite family of Lie systems with interesting applications as well as to give simple proofs of some known results about Kummer–Schwarz, Ermakov, Riccati and second-order Riccati equations.

7. Mixed superposition rules and the Riccati hierarchy

Consider a linear homogeneous differential equation

$$\frac{d^s x}{dt^s} = -\sum_{l=0}^{s-1} b_l(t)\frac{d^l x}{dt^l}, \tag{27}$$

with $b_0(t), \ldots, b_{s-1}(t)$ being arbitrary functions of time, $d^0x/dt^0 \equiv x$ and $s \geqslant 2$. This equation is invariant under dilations. This symmetry induces a change of variables $y = x^{-1}\,dx/dt$, which transforms the above linear system into a differential equation

$$\frac{d^{s-1}y}{dt^{s-1}} = F_b\left(t, y, \frac{dy}{dt}, \ldots, \frac{d^{s-2}y}{dt^{s-2}}\right), \tag{28}$$

for a nonlinear function $F_b : \mathbb{R} \times \mathbb{R}^{s-1} \to \mathbb{R}$ whose form depends on $b = (b_0(t), \ldots, b_{s-1}(t))$.

Eqs. (28) are called *higher-order Riccati equations*. These equations describe almost the whole family of differential equations of the so-called *Riccati hierarchy* [26,40], which is of importance in the study of soliton solutions for PDEs and other relevant physical topics [14,26]. Its first members read [26]

$$\frac{dy}{dt} = -b_0(t) - b_1(t)y - y^2, \quad s = 2, \tag{29}$$

$$\frac{d^2y}{dt^2} = -3y\frac{dy}{dt} - y^3 - b_0(t) - b_1(t)y - b_2(t)\left(y^2 + \frac{dy}{dt}\right), \quad s = 3. \tag{30}$$

The first is a well-known *Riccati equation*, which is almost ubiquitous in physics and mathematics [6]. The second one can be understood as a generalization of the Painlevé–Ince equation [22,41] and it has been recently frequently studied [14,40].

The general solution $y(t)$ of (28) can be written as

$$\begin{aligned} y(t) &= \varphi\left(x_{(1)}(t), \ldots, x_{(s)}(t), \frac{dx_{(1)}}{dt}(t), \ldots, \frac{dx_{(s)}}{dt}(t), k\right) \\ &\equiv \left(\sum_{a=1}^{s} k_a \frac{dx_{(a)}}{dt}(t)\right)\left(\sum_{a=1}^{s} k_a x_{(a)}(t)\right)^{-1} \end{aligned} \tag{31}$$

in terms of a generic family $x_{(1)}(t), \ldots, x_{(s)}(t)$ of solutions of (27) and a generic $k = (k_1, \ldots, k_s) \in \mathbb{R}^s$. Note that the particular solution induced by k and λk is the same for every $\lambda \in \mathbb{R}/\{0\}$, i.e.

$$\varphi\left(x_{(1)}(t), \ldots, x_{(s)}(t), \frac{dx_{(1)}}{dt}(t), \ldots, \frac{dx_{(s)}}{dt}(t), k\right) = \varphi\left(x_{(1)}(t), \ldots, x_{(s)}(t), \frac{dx_{(1)}}{dt}(t), \ldots, \frac{dx_{(s)}}{dt}(t), \lambda k\right).$$

Hence, expression (31) enables us to describe a generic solution $y(t)$ of (28) in terms of $x_{(1)}(t), \ldots, x_{(m)}(t)$, arbitrary constants $k_1, \ldots, k_{s-1}$, and $k_s = 1$. This implies that the successive derivatives of $y(t)$ can be brought into the form

$$\frac{d^l y}{dt^l}(t) = \Phi^l\left(x_{(1)}(t), \ldots, x_{(s)}(t), \ldots, \frac{d^{s-1}x_{(1)}}{dt^{s-1}}(t), \ldots, \frac{d^{s-1}x_{(s)}}{dt^{s-1}}(t); k_1, \ldots, k_{s-1}\right), \tag{32}$$

for certain functions $\Phi^l : \mathbb{R}^{s^2} \times \mathbb{R}^{s-1} \to \mathbb{R}$, with $l = 1, \ldots, s-2$. Let us view these relations as a mixed superposition rule.

The higher-order differential equations (27) and (28) can be written as systems of first-order differential equations

$$\begin{cases} \dfrac{du^i}{dt} = u^{i+1}, \quad i = 0, \ldots, s-2, \\ \dfrac{du^{s-1}}{dt} = -\displaystyle\sum_{l=0}^{s-1} b_l(t)u^l, \end{cases} \tag{33}$$

and

$$\begin{cases} \dfrac{dv^i}{dt} = v^{i+1}, \quad i = 0, \ldots, s-3, \\ \dfrac{dv^{s-2}}{dt} = F_b\big(t, v^0, \ldots, v^{s-2}\big), \end{cases} \tag{34}$$

where $u^0 = x$ and $v^0 = y$. Let $X^b_{(1)}$ and X_{F_b} be the t-dependent vector fields associated with (33) and (34), respectively. Expressions (31) and (32) allow us to write the general solution $\mathbf{v}(t) = (v^0(t), v^1(t), \ldots, v^{s-2}(t))$ of X_{F_b} in terms of a generic family of particular solutions $\mathbf{u}_{(\mathbf{a})}(t) = (u^0_{(a)}(t), \ldots, u^{s-1}_{(a)}(t))$, with $a = 1, \ldots, s$, of the system $X^b_{(1)}$ and constants $k_1, \ldots, k_{s-1}$. More geometrically, if we define

$$\Phi^0(\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(s)}; k_1, \ldots, k_{s-1}) = \left(\sum_{a=1}^{s-1} k_a u^1_{(a)} + u^1_{(s)}\right)\left(\sum_{a=1}^{s-1} k_a u^0_{(a)} + u^0_{(s)}\right)^{-1},$$

we can construct a map

$$\Phi : (\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(s)}; k_1, \ldots, k_{s-1}) \in \mathbb{R}^{s^2} \times \mathbb{R}^{s-1} \mapsto \Phi(\mathbf{u}_{(1)}, \ldots, \mathbf{u}_{(s)}; k_1, \ldots, k_{s-1}) \in \mathbb{R}^{s-1},$$

with $\Phi = (\Phi^0, \ldots, \Phi^{s-2})$, which enables us to write the general solution $\mathbf{v}(t)$ of X^{F_b} in terms of s generic particular solutions of $X^b_{(1)}$ as

$$\mathbf{v}(t) = \Phi\big(\mathbf{u}_{(1)}(t), \ldots, \mathbf{u}_{(s)}(t); k_1, \ldots, k_{s-1}\big).$$

That is, we have defined a mixed superposition rule $(\Phi, \overbrace{X^b_{(1)}, \ldots, X^b_{(1)}}^{s \text{ times}})$ for X_{F_b}. Therefore, in view of the extended Lie Theorem, each X_{F_b} is a Lie system. For $s = 1$ and $s = 2$, this retrieves that Riccati equations and second-order Riccati equations of the form (30) are Lie systems [2,14] and shows that these cases are particular instances of a general property of the systems X_{F_b} related to the members of Riccati hierarchy (28).

Depending on the particular form of the functions $b_0(t), \ldots, b_{s-1}(t)$, the minimal Lie algebras for systems (34) range from a unidimensional one, when $b_0(t), \ldots, b_{s-1}(t)$ are constant, to the Lie algebra $V^{X_{F_b}}$ corresponding to the case $b = (b_0(t), \ldots, b_{s-1}(t))$ where the vectors $(b_0(t), \ldots, b_{s-1}(t)) \in \mathbb{R}^s$, with $t \in \mathbb{R}$, span the whole $\mathbb{R}^s$. We will hereafter focus on this latter case, as it is a generic case and whose $V^{X_{F_b}}$ describes a Vessiot–Guldberg Lie algebra for all the systems (34) with the same s. Note that, from the change of variables mapping (27) to (28), it easily follows that $\dim V^{X_{F_b}} > 1$ (cf. [26,40]).

As every $X^b_{(1)}$ is a linear system, it is a Lie system. Moreover, it is easy to prove that $V^{X^b_{(1)}}$ is spanned by the linear vector fields

$$X_{i,j} = u^j \frac{\partial}{\partial u^i}, \quad i, j = 0, \ldots, s-1. \tag{35}$$

Indeed, the linear spaces spanned by $\{(X^b_{(1)})^t\}_{t\in\mathbb{R}}$, on one hand, and the vector fields $X_{s-1,j}$, with $j = 0, \ldots, s-1$, and $\Delta = X_{0,1} + \cdots + X_{s-2,s-1} + X_{s-1,0}$, on the other, are the same. So, they are also the smallest Lie algebras containing their elements. Since

$$[X_{i,j}, \Delta] = X_{i-1,j} - X_{i,j+1}, \qquad [X_{i,s-1}, \Delta] = X_{i-1,s-1} - X_{i,0}, \quad i = 1, \ldots, s-1, \ j = 0, \ldots, s-2,$$

it inductively follows that the successive Lie brackets of elements of $\{(X^b_{(1)})^t\}_{t\in\mathbb{R}}$ span the vector fields $X_{i,j}$, which clearly generate $V^{X^b_{(1)}}$. The elements of this family form a basis of a real Lie algebra isomorphic to $\mathfrak{gl}(s,\mathbb{R})$. Moreover, it is straightforward to prove that their prolongations to $\mathbb{R}^{s^2}$ form a modular basis of a real Lie algebra isomorphic to $\mathfrak{gl}(s,\mathbb{R})$. Hence, Theorem 19 ensures that $V_s^{X^b_{(1)}} \simeq V^{X^b_{(1)}} \simeq \mathfrak{gl}(s,\mathbb{R})$ is a Lie extension of $V^{X_{F_b}}$. As $\mathfrak{gl}(s,\mathbb{R})$ admits a *Levi decomposition* $\mathfrak{sl}(s,\mathbb{R})\oplus\mathbb{R}$, the Lie algebra $V^{X^b_{(1)}}$ admits only proper ideals isomorphic to $\mathbb{R}$ and $\mathfrak{sl}(s,\mathbb{R})$, respectively. Then, as $\dim V^{X_{F_b}} > 1$, we obtain that $V^{X_{F_b}}$ must be isomorphic to $\mathfrak{sl}(s,\mathbb{R})$ or $\mathfrak{gl}(s,\mathbb{R})$.

There exist no real Lie algebras of vector fields isomorphic to $\mathfrak{gl}(2,\mathbb{R})$ and $\mathfrak{gl}(3,\mathbb{R})$ over $\mathbb{R}$ and $\mathbb{R}^2$, respectively (cf. [42,43]). Hence, the above result, for $s=1$ and $s=2$, shows that $V^{X_{F_b}}$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(3,\mathbb{R})$, correspondingly. Therefore, the Riccati equations and second-order Riccati equations (30) can be described as Lie systems related to Vessiot–Guldberg Lie algebras isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(3,\mathbb{R})$. This has been proved by a direct application of the Lie–Scheffers Theorem in [6,14] that had required the determination of the corresponding complicated Lie algebras of nonlinear vector fields. Here, this result appears as a geometric consequence of admitting a mixed superposition rule that requires no calculation. Actually, we can determine a Vessiot–Guldberg algebra for every X_{F_b} with a generic s as follows.

Proposition 22. *The system X_{F_b} associated with a Riccati equation of order s is a Lie system possessing a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(s,\mathbb{R})$.*

Proof. As X_{F_b} is a Lie system, $V^{X_{F_b}}$ is a Vessiot–Guldberg Lie algebra for X_{F_b}. Let us analyze its algebraic structure through Theorem 20.

It can readily be proved that the mixed superposition rule for X^{F_b} can be brought into the form

$$\Phi(p;k) = \big(\Phi^0(p;k), D_T\Phi^0(p;k), \dots, D_T^{s-2}\Phi^0(p;k)\big) \in \mathbb{R}^{s-1},$$

where $k\in\mathbb{R}^{s-1}$ and we write $p = (\mathbf{u}_{(1)},\dots,\mathbf{u}_{(s)}) \in \mathbb{R}^{s^2}$ and

$$D_T = \sum_{a=1}^{s}\sum_{i=0}^{s-2} u^{i+1}_{(a)}\frac{\partial}{\partial u^i_{(a)}}.$$

From the above expressions, it can easily be demonstrated that $\Phi_k(\cdot) = \Phi(\cdot;k)$ is surjective for a generic k. Therefore, Theorem 20 implies that Φ induces a Lie algebra epimorphism $\Phi_{k*}: V_s^{X^b_{(1)}} \to V^{X_{F_b}}$ for a generic $k\in\mathbb{R}^{s-1}$. Let us prove that this map has a non-trivial kernel.

The function $\Phi^0_k(\cdot) \equiv \Phi^0(\cdot;k)$ is homogeneous. In other words, $\Phi^0_k(\lambda\mathbf{u}_{(1)},\dots,\lambda\mathbf{u}_{(s)}) = \Phi^0_k(\mathbf{u}_{(1)},\dots,\mathbf{u}_{(s)})$ for all $\lambda\in\mathbb{R}/\{0\}$. From here, we obtain that the functions $D^j_T\Phi^0_k$, with $j = 1,\dots,s-2$, are also homogeneous. Consequently, Φ_k is homogeneous. Since the flow of the vector field

$$X_0 = \sum_{i=0}^{s-1}\sum_{a=1}^{s} u^i_{(a)}\partial/\partial u^i_{(a)} \in V_s^{X^b_{(1)}}$$

reads $g:(\lambda;p)\in\mathbb{R}\times\mathbb{R}^{s^2} \mapsto e^\lambda p \in \mathbb{R}^{s^2}$, then

$$(\Phi_{k*}X_0)(p) = \frac{d}{d\lambda}\bigg|_{\lambda=0}\Phi_k\circ g(\lambda;p) = \frac{d}{d\lambda}\bigg|_{\lambda=0}\Phi_k(p) = 0$$

and Φ_{k*} has a non-trivial kernel. Taking into account that $V_s^{X^b_{(1)}} \simeq \mathfrak{sl}(s,\mathbb{R})\oplus\mathbb{R}$ and $\dim(V^{X_{F_b}}) > 1$, we see that $V^{X_{F_b}} \simeq \mathfrak{sl}(s,\mathbb{R})$. $\square$

Note that most of our above procedure does not depend on the explicit form of the mixed superposition rule. Hence, it can be applied, under slight modifications, to analyze systems admitting a mixed superposition rule in terms of solutions of a linear system $X^b_{(1)}$. This occurs, for instance, in the study of second-order Kummer–Schwarz equations and Milne–Pinney equations, whose general solutions can be determined in terms of two particular solutions of a system $X^b_{(1)}$ and two constants [6,23,25,35,37]. In these cases, we have

$$X_F = v\frac{\partial}{\partial x} + F(t,x,v)\frac{\partial}{\partial v}, \qquad X^b_{(1)} = v\frac{\partial}{\partial x} - \omega^2(t)x\frac{\partial}{\partial v},$$

where F is a t-dependent function, related to one of the previous second-order differential equations, and $X^b_{(1)}$ is associated with a t-dependent frequency harmonic oscillator $d^2x/dt^2 = -\omega^2(t)x$. Thus, $b_0(t) = \omega^2(t)$, $b_1(t) = 0$, and $V_2^{X^b_{(1)}} \simeq \mathfrak{sl}(2,\mathbb{R})$ admits a modular basis as before. As $V^{X_F} \neq 0$ and $V^{X^b_{(1)}}$ is simple, Corollary 21 shows that the system X_F, corresponding to a second-order Kummer–Schwarz or Milne–Pinney equation, is a Lie system possessing a Vessiot–Guldberg Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. This was recently discovered through the Lie–Scheffers Theorem [2,10,15], but our method yields this result much simpler and in a unified form.

8. Conclusions and outlook

Following an idea briefly suggested in [4], we have proposed a definition of a mixed superposition rule that generalizes the concept of a superposition rule of Lie and Scheffers. We have characterized systems admitting a mixed superposition rule as certain flat connections and proved a result, called here the extended Lie–Scheffers Theorem, stating that only Lie systems admit mixed superposition rules. Additionally, we have shown that mixed superposition rules are more versatile than the standard ones and may be utilized to analyze simultaneously the general solutions of different Lie systems. The extended Lie–Scheffers Theorem provided us also with a new powerful tool for recognizing Lie systems which was then applied for retrieving, in a simple way, some results known from the recent literature.

Our methods have been illustrated by various examples of physical and mathematical interest, in particular, by relevant results about the hierarchy of Riccati equations. This gave rise to the description of a new interesting infinite family of Lie systems with relevant applications. Furthermore, our methods seem to apply to certain differential equations appearing in the linearization of higher-order differential equations [23,44].

Finally, it is natural to search further for a characterization of families of systems of first-order differential equations admitting a t-dependent common mixed superposition rule [18]. This could lead to the joint analysis of the whole Riccati hierarchy as well as other systems of differential equations possessing common t-dependent mixed superposition rules. We aim to study these questions and other possible applications in a future work.

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