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An Algorithm for Computing the Integral Closure

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In this article we describe an algorithm for computing the integral closure of a reduced Noetherian ring, provided this integral closure is finitely generated as a module.

1. Introduction

In this paper we describe an algorithm for computing the normalization for certain rings. This quite general algorithm is essentially due to Grauert and Remmert (1971, 1984). They proved a normality criterion in order to give a simple proof of a theorem of Oka, which says that the non-normal points of an analytic space is an analytic space itself. We reformulate their results to better suit our purpose.

Since writing a first version of this article, I have seen a few papers which treat special cases of our criterion, especially Gianni and Trager (1996) and Vasconcelos (1997, chapter 6). However, the criterion in this article is the most general and flexible. As described in Eisenbud *et al.* (1992), one obtains as a by-product an algorithm for finding the irreducible components of a reduced affine variety. For earlier algorithms for computing the integral closure, we refer to Seidenberg (1975) and Stolzenberg (1968).

An algorithm for computing the integral closure, based on the results in this article, has been implemented in MACAULAY, and in SINGULAR (Decker *et al.*, 1997). (For SINGULAR, see Greuel *et al.* (1997).) It took 4 seconds on a Pentium-Pro2000 in SINGULAR to compute the normalization of the following example of Huneke (see example 10.6.11 in Vasconcelos (1994)).

$$
\begin{aligned}
&5abcde - a^5 - b^5 - c^5 - d^5 - e^5,\\
&ab^3c + bc^3d + a^3be + cd^3e + ade^3,\\
&a^2bc^2 + b^2cd^2 + a^2d^2e + ab^2e^2 + c^2de^2,\\
&abc^5 - b^4c^2d - 2a^2b^2cde + ac^3d^2e - a^4de^2 + bcd^2e^3 + abe^5,\\
&ab^2c^4 - b^5cd - a^2b^3de + 2abc^2d^2e + ad^4e^2 - a^2bce^3 - cde^5,\\
&a^3b^2cd - bc^2d^4 + ab^2c^3e - b^5de - d^6e + 3abcd^2e^2 - a^2be^4 - de^6,\\
&a^4b^2c - abc^2d^3 - ab^5e - b^3c^2de - ad^5e + 2a^2bcde^2 + cd^2e^4,\\
&b^6c + bc^6 + a^2b^4e - 3ab^2c^2de + c^4d^2e - a^3cde^2 - abd^3e^2 + bce^5.
\end{aligned}
$$

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2. The Algorithm

Let R be a commutative, Noetherian and reduced ring with 1, $\widetilde{R}$ the integral closure (also called normalization) of R. We consider the set:

$$NNL := \{\boldsymbol{p} \in \operatorname{Spec}(R) : R_{\boldsymbol{p}} \text{ is not normal}\}.$$

Here NNL stands for Non-Normal Locus. Let I be an ideal of R containing a non-zero divisor. We have canonical inclusions:

$$R \subset \operatorname{Hom}_R(I, I) \subset \widetilde{R}.$$

The first inclusion is the map which sends an element of R to multiplication with this element. The second inclusion is sending $\phi \in \operatorname{Hom}_R(I, I)$ to $\frac{\phi(f)}{f}$ for any element $f \in I$ which is a non-zero divisor of R. It is easily checked that the map is independent of the choice of f. That we in fact land in $\widetilde{R}$ is a consequence of the Cayley–Hamilton theorem, and a proof can be found in any textbook which includes integral closure as a topic, see for example Eisenbud (1995, Theorem 4.3).

Theorem 2.1. *(Grauert and Remmert, 1971, pp. 220–221, 1984, pp. 125–127). Assume that the ideal I contains a non-zero divisor, and has the following property:*

$$NNL \subset V(I),$$

where $V(I) = \{\boldsymbol{p} \in \operatorname{Spec}(R) : I \subset \boldsymbol{p}\}$ denotes, as usual, the zero set of I. Suppose moreover that I has the property

$$\operatorname{Hom}_R(I, I) = \operatorname{Hom}_R(I, R) \cap \widetilde{\widetilde{R}}. \qquad (*)$$

Then one has the following normality criterion:

$$R = \operatorname{Hom}_R(I, I) \iff R \text{ is normal}.$$

Proof. The implication $\Longleftarrow$ is trivial. To prove the converse, let $h = \frac{f}{g} \in \widetilde{R}$. Consider the following ideal in R:

$$\{\phi \in R : h\phi \in R\} = \operatorname{Ann}(hR/(hR \cap R)).$$

We call its zero set the “pole set” of h:

$$P(h) := \{\boldsymbol{p} \in \operatorname{Spec}(R) : h \notin R_{\boldsymbol{p}}\}.$$

It is immediate that $P(h) \subset NNL$. Let J be the ideal of $P(h)$. There exists a $c > 0$ such that $hJ^c \subset R$, by the Nullstellensatz. By the Nullstellensatz again $\sqrt{I} \subset J$. Therefore there exists a $d > 0$ such that $hI^d \subset R$. Let $d \geq 0$ be minimal with this property. The theorem follows, if we can prove that $d = 0$. Suppose the converse, that is, $d > 0$. Then there exists an $a \in I^{d-1}$ with $ha \notin R$. Furthermore, $ha \in \widetilde{R}$ and $(ha)I \subset R$. It follows from $(*)$ that $ha \in \operatorname{Hom}_R(I, I)$. From the assumption $R = \operatorname{Hom}_R(I, I)$ it follows that $ha \in R$, in contradiction with the choice of a.

We have to find an ideal which satisfies condition $(*)$. This is provided by:

Theorem 2.2. *Every radical ideal I containing a non-zero divisor satisfies condition $(*)$.*

PROOF. The proof is in Grauert and Remmert (1971, 1984), but because it is so simple we give it here. Let $h \in \widetilde{R}$, so we have an equation:

$$h^n = a_0 + a_1 h + \cdots + a_{n-1} h^{n-1}; \qquad a_i \in R.$$

If $hI \subset R$, then for each $f \in I$:

$$(hf)^n = a_0 f^n + a_1 (hf) f^{n-1} + \cdots + a_{n-1} (hf)^{n-1} f \in I.$$

As I is supposed to be radical it follows that $hf \in I$, and that is what we had to prove.

These results give rise to the following algorithm.

ALGORITHM

INPUT: A reduced Noetherian ring R.
OUTPUT: The normalization $\widetilde{R}$ of R.

STEP 1: Determine a non-zero ideal I with $NNL \subset V(I)$.
STEP 2: Take a non-zero element $f \in I$, and compute $J := \mathrm{Ann}(f)$. If $J = 0$, GOTO STEP 4.
STEP 3: Put $R := R/\mathrm{Ann}(J) \oplus R/J$ and GOTO STEP 1.
STEP 4: Compute the radical $\sqrt{I}$ of I. Put $I := \sqrt{I}$.
STEP 5: Compute $\mathrm{Hom}_R(I, I)$. If $R = \mathrm{Hom}_R(I, I)$ then put $\widetilde{R} := R$ and STOP.
STEP 6: Set $R := \mathrm{Hom}_R(I, I)$ and GOTO STEP 1.

This algorithm terminates exactly when the normalization $\widetilde{R}$ is finitely generated as an R-module. This happens for example for affine rings, due to a classical result of E. Noether.

Some remarks are in order.

1. To determine an ideal I with $NNL \subset V(I)$ one can take any I which contains the non-regular locus of R. This is what one probably always has to do in the first step. Having such an I, which is radical, one can make a new one by taking $J := \mathrm{Ann}(\mathrm{Hom}_R(I, I)/R)$. Indeed the space defined by J is exactly the non-normal locus of R. This is what Grauert and Remmert used to prove that the non-normal locus of an analytic space is analytic.
2. Algorithms for computing the radical of an ideal are described in Eisenbud *et al.* (1992) and probably will use most of the time in the algorithm. If possible, one should avoid this computation by taking an ideal for which one knows in advance that it is radical. This idea can be applied in the example of Huneke mentioned in the introduction. One computes that the projective variety defined by those equations is smooth. Therefore, for I one can take the irrelevant maximal ideal, and one can avoid computation of the radical in this case.
3. Is it possible to find other ideals satisfying property $(*)$, which are easier to compute than the radical? This would speed up the computation. Buchweitz remarked that the proof of Theorem 2 shows that an integrally closed ideal (which contains a non-zero divisor) in fact satisfies property $(*)$.

4. Let $I, J \subset R$ be ideals such that both I and J contain a non-zero divisor. One has a canonical inclusion $\mathrm{Hom}_R(I, I) \subset \mathrm{Hom}_R(IJ, IJ)$. If the inclusion $R \subset \mathrm{Hom}_R(I, I)$ is strict, then also the inclusion $R \subset \mathrm{Hom}_R(IJ, IJ)$ is strict. Since $R \subset \mathrm{Hom}_R(I, I) \subset \mathrm{Hom}_R(IJ, IJ) \subset \widetilde{R}$, one can by replace I by IJ in STEP 5. One can take in particular $J = I^k$ for some $k \geq 1$. This might make the computation faster in some cases, but might slow down the calculation in other cases. A slower calculation will certainly result when the inclusion $\mathrm{Hom}_R(I, I) \subset \mathrm{Hom}_R(I^k, I^k)$ is not strict. For an example where the computation is faster, consider the hypersurface defined by the equation $zy^3 - zx^4 - x^8 = 0$. In this case, the reduced ideal of the singular locus is $I = (x, y)$. One can compute that $\mathrm{Hom}_R(I, I)$ is not normal, but $\mathrm{Hom}_R(I^2, I^2)$ computes the normalization of R.
5. In Gianni and Trager (1996), Propositions 4 and 5 it is shown that one can compute $\mathrm{Hom}_R(I, I) \subset Q(R)$ as $\frac{1}{f}(fI :_R I)$ for any non-zero divisor $f \in I$.
6. In STEP 4, having found a zero divisor f, the new ring defined is the ring which separates two parts of $\mathrm{Spec}(R)$, the one part consisting of the union of the components on which f vanishes, the other part (defined by J) consists of the union of the residual components. The extension $R \subset R/\mathrm{Ann(J)} \oplus R/J$ is indeed finite. It is isomorphic to $R[u]/K$, where the ideal K is generated by $u^2 - u$ and elements of type $uh_1, (u-1)h_2$ where h_1 runs over $\mathrm{Ann}(J)$ and h_2 runs over J.

3. The Ring Structure

In the algorithm, the computation of $\mathrm{Hom}_R(I, I)$ has two parts. In several computer algebra systems, one can compute $\mathrm{Hom}_R(I, I)$ as an R-module. The description of $\mathrm{Hom}_R(I, I)$ as a *ring* is essentially due to Catanese (1984). We will describe this now. Take generators $u_0 := 1, u_1, \ldots, u_t$ of $\mathrm{Hom}_R(I, I)$ as R-module. Consider the map:

$$R \cdot X_0 \oplus R \cdot X_1 \oplus \cdots \oplus R \cdot X_t \xrightarrow{\phi} \mathrm{Hom}_R(I, I), \qquad X_i \mapsto u_i.$$

Computing the kernel of the map ϕ gives "linear equations":

$$L_i = \sum_{j=0}^{t} \alpha_{ij} X_j = 0 \qquad \alpha_{ij} \in R;\ i = 1, \ldots, s.$$

Because $\mathrm{Hom}_R(I, I)$ is a ring, we have that $u_i u_j$ is in $\mathrm{Hom}_R(I, I)$ again (for all $1 \leq i \leq j \leq t$). Therefore, we can find elements $\beta_{ijk} \in R$ such that:

$$u_i u_j = \sum_{k=0}^{t} \beta_{ijk} u_k.$$

This gives $\frac{t(t+1)}{2}$ "quadratic equations":

$$Q_{ij} := X_i X_j - \sum_{k=0}^{t} \beta_{ijk} X_k.$$

For the easy proof of the following theorem, which might speed up the computation of the the ring $\mathrm{Hom}_R(I, I)$ as quotient of a polynomial ring, we refer to Catanese (1984).

Theorem 3.1. *Put $X_0 = 1$, and consider the ideal $J \subset R[X_1, \ldots, X_t]$ generated by the*

$L_i, i = 1, \ldots, s$ *and the* Q_{ij} *for* $1 \leq i \leq j \leq t$. *Then we have a ring isomorphism:*

$$\mathrm{Hom}_R(I, I) \cong R[X_1, \ldots, X_t]/J.$$

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