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Dynamical interpretation of the wavefunction of the universe

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ABSTRACT

In this paper, we study the physical meaning of the wavefunction of the universe. With the continuity equation derived from the Wheeler–DeWitt (WDW) equation in the minisuperspace model, we show that the quantity $\rho(a) = |\psi(a)|^2$ for the universe is inversely proportional to the Hubble parameter of the universe. Thus, $\rho(a)$ represents the probability density of the universe staying in the state a during its evolution, which we call the dynamical interpretation of the wavefunction of the universe. We demonstrate that the dynamical interpretation can predict the evolution laws of the universe in the classical limit as those given by the Friedmann equation. Furthermore, we show that the value of the operator ordering factor p in the WDW equation can be determined to be $p = -2$.

1. Introduction

In quantum mechanics, the state of a particle is completely described by its wavefunction $\Psi(x,t)$. At the time t, the probability of finding the particle in an interval Δx about the position x is proportional to $|\Psi(x,t)|^2\Delta x$, and thus $\rho(x,t) = |\Psi(x,t)|^2$ is interpreted as the probability density and $\Psi(x,t)$ is called the probability amplitude for the particle. This is the statistical interpretation or the ensemble interpretation of the wavefunction determined by the Schrödinger equation in the standard quantum mechanics. In quantum cosmology theory, the universe is described by a wavefunction $\psi(h_{ij},\phi)$ determined by the quantum gravity equation, called the Wheeler–DeWitt (WDW) equation [1], $\mathcal{H}\psi(h_{ij},\phi) = 0$, where h_{ij} is the 3d metric and ϕ is a scalar field, rather than the classical spacetime. In principle, the wavefunction should contain all information about the universe [2], although it is hard to extract all the information from it [3]. At first glance, the wavefunction of the universe should satisfy the statistical interpretation [4]. But, if we adopt the statistical interpretation for the wavefunction of the universe directly, it would be strange that the quantity $|\psi(h_{ij},\phi)|^2$ denotes the probability density of finding a universe somewhere.

The observed universe is unique. If we want to study the physical meaning of the wavefunction of the universe with only one universe, the statistical or ensemble interpretation should be abandoned. In this paper, we study the properties of the wavefunction of the universe in the minisuperspace model, in which the wavefunction of the universe can be determined by the unique parameter, the scale factor a to describe the time-dependent evolution of the universe. First of all, we find that the operator ordering factor p representing the ambiguity in the ordering of noncommuting operators in the WDW equation should take value $p = -2$ due to the requirement of finiteness of the wavefunction of the universe. Next, we show that the quantity $\rho(a) = |\psi(a)|^2$ for the universe is inversely proportional to the Hubble parameter of the universe, and represents the probability density of the universe staying in the state a during its evolution. We further show that the dynamical interpretation of the wavefunction of the universe can give inflation solutions of the small universe and the correct evolution laws of the universe in the classical limit, as required by the correspondence principle. This paper is organized as follows. In Section 2, the WDW equation is applied to the minisuperspace model. Then, the formula of $\rho(a)$ for the universe is obtained with a determined p, in Section 3. The dynamical interpretation of the wavefunction of the universe is given in Section 4. The correct evolution laws of the universe in the classical limit with dynamical interpretation is discussed in Section 5. In Section 6, the formula of $\rho(a)$ of the universe within the scalar field model is obtained. Finally, we discuss and conclude in Section 7.

2. WDW equation in the minisuperspace model

Assumed to be homogeneous and isotropic, the universe can be described by a minisuperspace model [5–7] with one single parameter, the scale factor a. The Einstein–Hilbert action for the model can be written as

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$$S = \int \left(\frac{Rc^3}{16\pi G} - \frac{9\varepsilon_n}{16c} \right) \sqrt{-g} d^4x, \tag{1}$$

where the ε_n represents the energy density of the universe, and the constant before ε_n is chosen for later convenience. Since the universe is homogeneous and isotropic, the metric of the universe in the minisuperspace model is given by

$$ds^2 = \sigma^2 \left[-N^2(t)c^2dt^2 + a^2(t)d\Omega_3^2 \right]. \tag{2}$$

Here, $d\Omega_3^2 = dr^2/(1-kr^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ is the metric on a unit three-sphere, $N(t)$ is an arbitrary lapse function, and $\sigma^2 = 2/3\pi$ is a normalizing factor chosen for later convenience. Note that r is dimensionless and the scale factor $a(t)$ has length dimension [8]. From Eq. (2), one can get the scalar curvature

$$\mathcal{R} = 6\frac{\ddot{a}}{\sigma^2c^2N^2a} + 6\frac{\dot{a}^2}{\sigma^2c^2N^2a^2} + \frac{6k}{\sigma^2a^2}, \tag{3}$$

where the dot denotes the derivative with respect to the time, t. Inserting Eqs. (2) and (3) into Eq. (1), we can get

$$\begin{aligned} S &= \int \frac{6\sigma^2Nc^4}{16\pi G} \left(\frac{a^2\ddot{a}}{N^2c^2} + \frac{a\dot{a}^2}{N^2c^2} + ka - \frac{G\varepsilon_n a^3}{c^4} \right) d^4x, \\ &= \frac{6\sigma^2Nc^4V}{16\pi G} \int \left(\frac{a^2\ddot{a}}{N^2c^2} + \frac{a\dot{a}^2}{N^2c^2} + ka - \frac{G\varepsilon_n a^3}{c^4} \right) dt, \\ &= \frac{Nc^4}{2G} \int \left(-\frac{a\dot{a}^2}{N^2c^2} + ka - \frac{G\varepsilon_n a^3}{c^4} \right) dt. \end{aligned}$$

The Lagrangian of the bubble can thus be written as

$$\mathcal{L} = \frac{Nc^4}{2G} \left(ka - \frac{a\dot{a}^2}{N^2c^2} - \frac{G\varepsilon_n a^3}{c^4} \right), \tag{4}$$

and the momentum p_a can be obtained as

$$p_a = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -\frac{c^2a\dot{a}}{NG}.$$

Taking $N = 1$, the Hamiltonian is found to be

$$\begin{aligned} \mathcal{H} &= p_a\dot{a} - \mathcal{L} \\ &= -\frac{1}{2}\left(\frac{Gp_a^2}{c^2a} + \frac{c^4ka}{G} - \varepsilon_n a^3 \right). \end{aligned}$$

In quantum cosmology theory, the evolution of the universe is completely determined by its quantum state that should satisfy the WDW equation. With $\mathcal{H}\Psi = 0$ and $p_a^2 = -a^{-p}\frac{\partial}{\partial a}(a^p\frac{\partial}{\partial a})$, we get the WDW equation [1,3,9],

$$\left(\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - ka^2 + \varepsilon_n a^4 \right)\psi(a) = 0. \tag{5}$$

Here, $k = 1, 0, -1$ are for spatially closed, flat and open universe, respectively. The factor p represents the uncertainty in the choice of operator ordering. For simplicity, we take $\hbar = G = c = 1$ in this paper.

3. The square modulus of the wavefunction for the universe

In order to study the physical meaning of the wavefunction of the universe, we should discuss the square modulus of the wavefunction for the universe first. Since there is only one parameter a in Eq. (5), the complex function $\psi(a)$ can be formally rewritten as

$$\psi(a) = R(a)\exp(iS(a)), \tag{6}$$

where R and S are real functions. Then the square modulus of the wavefunction is

$$\rho(a) \equiv |\psi(a)|^2 = R(a)^2.$$

It should be pointed out that, no matter what the physical meaning it is, the value of $\rho(a)$ should be finite.

From Eq. (5), it is easy to construct a conserved current j^a as [11,15]

$$j^a = \frac{i}{2}a^p(\psi^*\partial_a\psi - \psi\partial_a\psi^*), \tag{7}$$

$$\partial_a j^a = 0. \tag{8}$$

Inserting Eq. (6) into Eq. (7), we can get

$$j^a = -a^pR^2S',$$

where the prime denotes the derivative with respect to a. Integrating Eq. (8) gives that

$$a^pR^2S' = c_0, \tag{9}$$

where the c_0 is a dimensionless integral constant.

In the quantum Hamilton–Jacobi theory, the relation between the action and the momentum can be written as [5,12],

$$p_a = \frac{\partial S}{\partial a} = \frac{\partial \mathcal{L}}{\partial \dot{a}} = -a\dot{a}. \tag{10}$$

According to Eqs. (9) and (10), we can get $\rho(a)$ for the universe

$$\rho(a) = R(a)^2 = -\frac{c_0}{a^{p+1}\dot{a}}. \tag{11}$$

Using the definition of the Hubble parameter

$$H(a) = \frac{\dot{a}}{a}, \tag{12}$$

we can rewrite $\rho(a)$ as

$$\rho(a) = -\frac{c_0}{a^{p+2}H(a)}. \tag{13}$$

In principle, the value of $\rho(a)$ should be finite for any a, i.e., no matter whether the universe is very small or large enough. In both the inflation stage and the dark energy stage, the value of the Hubble parameter should be finite. From

$$\rho(a \to 0) = -\frac{c_0}{a^{p+2}H(a)},$$

we can get a boundary condition

$$p + 2 \le 0.$$

On the other hand, the finiteness of $\rho(a)$ with large a

$$\rho(a \to \infty) = -\frac{c_0}{a^{p+2}H(a)}$$

requires that

$$p + 2 \ge 0.$$

The above two boundary conditions determine the value of the ordering factor to be $p = -2$. Thus, Eq. (13) is reduced to

$$\rho(a) = -\frac{c_0}{H(a)}. \tag{14}$$

With the above formula at hand, we can discuss now the physical meaning of the wavefunction of the universe.

Table 1
The classical evolution laws of the universe given by the Friedmann equation. The universe is dominated by different kinds of energies $\varepsilon_n = \lambda_n/a^n$ at different stages.

Dominator	Density	Evolution
radiation	$n = 4$	$a(t) \propto (t + t_0)^{1/2}$
matter	$n = 3$	$a(t) \propto (t + t_0)^{2/3}$
dark energy	$n = 0$	$a(t) \propto e^t$

4. Dynamical interpretation of the wavefunction of the universe

In quantum mechanics, the wavefunction $\Psi(x, t)$ of a particle is interpreted as probability amplitude and $|\Psi(x, t)|^2 \Delta x$ is the probability of finding the particle at time t in the interval Δx. This is the statistical interpretation or the ensemble interpretation of wavefunction. However, such an interpretation of wavefunction in quantum mechanics cannot be applied to the wavefunction of the universe in quantum cosmology. For an observer inside the universe, it is should be very strange if the wavefunction is related to the probability of finding a universe. So, the physical meaning of the wavefunction for the universe should be reinterpreted.

From Eq. (14), we can see that the value of $\rho(a)$ for the universe only depends on the Hubble parameter $H(a)$. $\rho(a)$ is large when the universe expands slowly, and it is small when the universe expands quickly. So, it is obvious that $\rho(a)$ for the universe represents the evolution velocity of the universe and thus relates to the dynamics of the universe. In this case, $\rho(a)$ can be treated as the dynamical interpretation. Similar to the statistical interpretation of the wavefunction in quantum mechanics, the dynamical interpretation for the wavefunction of the universe can be explicitly described as:

> $\int_{a_1}^{a_2} \psi^*(a)\psi(a)da$ is proportional to the time spent when the universe involves from the state a_1 to a_2.

In this way, we have showed that the physical meaning of $\rho(a)$ for the universe can be interpreted as the probability density of the universe staying in the state a during its evolution.

Generally speaking, an interpretation for the physical meaning of wavefunction should satisfy the correspondence principle, i.e., the quantum cosmology can reduce to the classical cosmology in the classical limit within the dynamical interpretation. In fact, the dynamical interpretation completely depends on the evolution equation Eq. (11), which we call the dynamical equation for the universe. As will be shown below, solutions of the WDW equation in the classical limit ($a \gg 1$) together with the dynamical equation can predict the same evolution laws of the universe as those given by the Friedmann equation. The exponential expansion solutions of the early universe ($a \ll 1$) can also be obtained from the WDW equation together with the dynamical equation (11).

5. The evolution of the universe with the dynamical equation

If we want to probe the rationality of the dynamical interpretation of the wavefunction of the universe, we should verify whether the dynamical interpretation can give the correct evolution laws of the universe in the classical limit or not [13]. On the other hand, the quantum behaviors of the small enough universe should be reserved with the dynamical interpretation. The classical evolution laws of the universe at different stages dominated by radiation, matter and dark matter, respectively, can be obtained by solving the Friedmann equation. The evolution equations of the classical universe at different stages are shown in Table 1.

Let us study the evolution laws of the universe with the dynamical interpretation of the universe. For simplicity, we only consider the case of the flat universe $k = 0$. The WDW equation for the flat universe with energy density ε_n can be written as

$$\left(\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} + \varepsilon_n a^4\right)\psi(a) = 0. \tag{15}$$

Here $\varepsilon_n = \lambda_n/a^n$, for $n = 4, 3, 0$ representing the universe dominated by radiation, matter and dark energy, respectively [14].

In principle, the universe contains all kinds of energies at the same time, so the energy density should take the form of $\varepsilon = \varepsilon_0 + \varepsilon_3 + \varepsilon_4$. In practice, the universe is dominated by one kind of energy ε_n at a specific stage. During the evolution of the universe, n changes slowly from $n = 4$ to $n = 0$. For an arbitrary n, we can obtain the analytical solutions of Eq. (15),

$$\psi_n(a) = a^{\frac{1-p}{2}}\left[ic_1 J_\nu\left(\frac{\sqrt{\lambda_n}a^{3-n/2}}{3-n/2}\right) + c_2 Y_\nu\left(\frac{\sqrt{\lambda_n}a^{3-n/2}}{3-n/2}\right)\right]. \tag{16}$$

Here, $J_\alpha(x)'s$ are Bessel functions of the first kind, $Y_\alpha(x)'s$ are Bessel function of the second kind, and $\nu = |(1-p)/(n-6)|$.

First, consider the wavefunction of the WDW equation in the classical limit ($a \gg 1$). For $x \gg |\nu^2 - 1/4|$, Bessel functions take the following asymptotic forms,

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x - \nu\pi/2 - \pi/4),$$
$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\sin(x - \nu\pi/2 - \pi/4).$$

If the free parameters c_1 and c_2 in Eq. (16) take the values of $c_1 = c_2 = c_-\sqrt{\pi/2}$, the wavefunction can be rewritten as

$$\psi_n(a) = c_- a^{\frac{n-2p-4}{4}}\exp\left[\frac{-i\sqrt{\lambda_n}a^{3-n/2}}{3-n/2} + i\theta\right], \tag{17}$$

where $\theta = (3n - 2p - 16)\pi/(4n - 24)$, and $S' < 0$. From Eq. (10), we know that the wavefunction in Eq. (17) describes an expanding universe as suggested by Vilenkin [11]. With the wavefunction above, we can get

$$\rho(a) \equiv |\psi(a)|^2 = c_-^2 a^{-p-2+n/2}.$$

Together with the dynamical equation (11), the above equation gives

$$\dot{a} = \frac{-c_0}{c_-^2 a^{-1+n/2}}, \tag{18}$$

where $c_0 < 0$. Transforming the formula in Eq. (18) into

$$a^{-1+n/2}da = \frac{-c_0}{c_-^2}dt,$$

and then integrating both sides of the equation, we have

$$a(t) \propto \begin{cases} (t + t_0)^{2/n}, & n \neq 0, \\ e^{t+t_0}, & n = 0. \end{cases}$$

The evolution laws of the universe from the WDW equation in the classical limit ($a \gg 1$) are completely consistent with the solutions of the Friedmann equation as shown in Table 1. It is interesting that the evolution equation of the universe derived from the WDW equation is independent of the operator ordering factor p, which definitely means that p only represents the quantum effects of the universe [15].

Next, consider the evolution of the early universe ($a \ll 1$) within the dynamical interpretation. For $x \ll |\nu^2 - 1/4|$, Bessel functions take the following asymptotic forms,

$$J_\nu(x) \sim \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)},$$
$$Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{x}{2}\right)^{-\nu}.$$

So the wavefunction Eq. (16) for the small scale factor ($a \ll 1$) can be rewritten as

$$\psi_n(a) = c_-\sqrt{\frac{\pi}{2}}\left(\frac{i\lambda_n^{\nu/2}a^{1-p}}{\Gamma(\nu+1)(6-n)^\nu} - \frac{\Gamma(\nu)(6-n)^\nu}{\pi\lambda_n^{\nu/2}}\right).$$

Here, we have assumed that $p < 1$, otherwise the wavefunction is divergent. c_1 and c_2 take the same value $c_-\sqrt{\pi/2}$ as that determined by the solutions in the classical limit. In this case, the probability density of the early universe is

$$\rho(a \ll 1) \equiv |\psi_n(a \ll 1)|^2 = c_-^2\frac{\Gamma^2(\nu)(6-n)^{2\nu}}{2\pi\lambda_n^\nu}.$$

It is obvious that, when $a \ll 1$, $\rho(a)$ approximates a constant denoted as $\rho_0(n)$. When $a \ll 1$, the dynamical equation (11) can be rewritten as

$$a^{p+1}\dot{a} = -\frac{c_0}{\rho_0(n)},$$

which is solved by

$$a(t) = \begin{cases} [-(p+2)c_0(t+t_0)/\rho_0(n)]^{\frac{1}{p+2}}, & p \neq -2, \\ e^{H(t+t_0)}, & p = -2, \end{cases}$$

where $H = -c_0/\rho_0(n)$.

When the universe is very small, its behaviors are dominated by quantum effects and the evolution of the universe depends on the operator ordering factor p: different p gives different evolution equations of the scale factor $a(t)$. In fact, the ordering factor p has been determined by the boundary conditions of finite density matrix of the universe as $p = -2$. This specific value $p = -2$ gives exponential expansion solutions for the early universe, which is consistent with the result from quantum trajectory theory [15,16]. Thus, we conclude that the WDW equation together with the dynamical interpretation can give exponential expansion solutions of the early universe and the correct evolution laws of the universe in the classical limit.

6. The dynamical interpretation for the minisuperspace model with a scalar field

In quantum cosmology, the most widely used model is the minisuperspace model with a scalar field. For a FRW universe filled with a scalar field ϕ, the WDW equation can be written as [10,17]

$$\left[\frac{1}{a^p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - \frac{1}{a^2}\frac{\partial^2}{\partial\phi^2} - U(a,\phi)\right]\psi(a,\phi) = 0,$$

where $U(a,\phi) = a^2(k - a^2V(\phi))$. We can rewrite the wavefunction $\psi(a,\phi)$ as $\psi(a,\phi) = R(a,\phi)e^{iS(a,\phi)}$, where both $R(a,\phi)$ and $S(a,\phi)$ are real functions. Since the wavefunction $\psi(a,\phi)$ is a function of a and ϕ, the currents for WDW equation can be obtained as [10]

$$j^a = \frac{i}{2}a^p(\psi^*\partial_a\psi - \psi\partial_a\psi^*),$$
$$= -a^pR^2(a,\phi)\partial_aS(a,\phi), \tag{19}$$
$$j^\phi = \frac{-i}{2}a^{p-2}(\psi^*\partial_\phi\psi - \psi\partial_\phi\psi^*),$$
$$= a^{p-2}R^2(a,\phi)\partial_\phi S(a,\phi). \tag{20}$$

The quantum Hamilton–Jacobi theory gives the guidance relations [6]

$$\partial_aS(a,\phi) = -a\dot{a},$$
$$\partial_\phi S(a,\phi) = a^3\dot{\phi}.$$

The currents j^a and j^ϕ satisfy the continuity equation,

$$\partial_aj^a + \partial_\phi j^\phi = 0. \tag{21}$$

Inserting Eqs. (19) and (20) into Eq. (21), we can get

$$\partial_a(a^pR^2\partial_aS) - \partial_\phi(a^{p-2}R^2\partial_\phi S) = 0.$$

Applying the guidance relation to the equation above, we can obtain

$$\partial_a(a^pR^2\partial_aS) + \partial_a(a^{p-2}R^2\partial_\phi S)a^2\dot{\phi}/\dot{a} = 0. \tag{22}$$

Integrating Eq. (22) over a, we get

$$a^{p+1}R^2\dot{a} - a^{p+3}R^2\dot{\phi}^2/\dot{a} + \int a^{p+1}R^2\dot{\phi}d(a^2\dot{\phi}/\dot{a}) = -c_0.$$

Let $A(a,\phi) = R^{-2}\int R^2a^{p+2}\dot{\phi}(2\phi_a + a\phi_{aa})da$, which gives

$$\rho(a,\phi) = R^2(a,\phi) = \frac{-c_0}{a^{p+1}\dot{a}(1-a^2\phi_a^2) + A(a,\phi)}, \tag{23}$$

where $\phi_a = d\phi/da = \dot{\phi}/\dot{a}$, and $\phi_{aa} = d\phi_a/da$. We can see when $\dot{\phi} \to 0$, i.e., $A(a,\phi) \to 0$ and $a^2\phi_a^2 \to 0$, $\rho(\phi,a)$ in Eq. (23) will return to $\rho(a)$ in Eq. (11), which indicates that the dynamical interpretation still holds for the wavefunction of the early universe with a slowly-rolling scalar field.

7. Discussion and conclusion

In summary, we have found the mathematical relation between the quantity $\rho(a)$ for the universe and the Hubble parameter that $\rho(a)$ is inversely proportional to the Hubble parameter $H(a)$. We argue that $\rho(a)$ is not the probability density of finding a universe somewhere, but represents the probability density of the universe staying in the state a. This presents a dynamical interpretation of the wavefunction of the universe. We have demonstrated that the dynamical interpretation can give the same evolution laws of the universe as those given by the Friedmann equation, which satisfies the requirement of the correspondence principle that the quantum cosmology should reduce to the classical cosmology in the classical limit. In this way, we have presented an investigation of the physical meaning of the wavefunction of the universe.

Another result is that the value of the operator ordering factor p that represents the ambiguity in the ordering of noncommuting operators has also been determined. With the requirement of the finiteness of the wavefunction of the universe, the ordering factor should take value of $p = -2$. This specific value of p can predict exponential expansion solutions of the small universe. In fact, when the universe becomes large enough, the evolution of the universe is independent of the value of the ordering factor p, which implies that p represents the rules of the quantization of the early universe and only dominates quantum behaviors of the universe.

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