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Strongly indexable graphs

B.D. Acharya

Department of Science & Technology, Government of India, Technology Bhavan, New Mehrauli Road, New Dehli-110016, India

S.M. Hegde

Department of Mathematics, University of Delhi, Delhi-110007, India

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Abstract

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A (p, q)-graph $G = (V, E)$ is said to be *strongly k-indexable* if it admits a *strong k-indexer* viz., an injective function $f: V \to \{0, 1, 2, \ldots, p-1\}$ such that

$$f(x) + f(y) = f^+(xy) \in f^+(E) = \{k, k+1, k+2, \ldots, k+q-1\}.$$

In the terms defined here, k will be omitted if it happens to be unity. We find that a strongly indexable graph has exactly one nontrivial component which is either a star or has a traingle. In any strongly k-indexable graph the minimum point degree is at most 3. Using this fact we show that there are exactly three strongly indexable regular graphs, viz. K_2, K_3 and $K_2 \times K_3$. If an eulerian (p, q)-graph is strongly indexable then $q \equiv 0, 3 \pmod 4$.

Unless mentioned otherwise, we consider only finite simple graphs as treated in [9].

Labeling of the points of a given graph $G = (V, E)$ by integers is merely an assignment of certain distinct nonnegative integers to the points of G. In other words, it is simply an injective function $f: V \to A$, from the point set $V = V(G)$ of G into a subset A of the set $\mathbb{N}$ of nonnegative integers.

Several practical problems in real-life situations have motivated the study of labelings of graphs which are required to obey a variety of conditions depending on the structure of graphs. There is an enormous amount of literature built up on several kinds of labelings over the past two decades or so. It would be too unwieldy to cite all references, but a few pioneering papers like [1–3, 5–6, 10–11] are worth comprehension.

In this paper, we are interested to investigate the properties of (p, q)-graphs $G = (V, E)$ which admit a labeling $f: V \to \{0, 1, 2, \ldots, p-1\}$ such that the *values* $f^+(e)$ of the lines $e = uv$ of G, defined as $f(u) + f(v)$, are all distinct—that is, such that the so induced function $f^+: E(G) \to \mathbb{N}$ is injective too—such a labeling f of G being called an *indexer* of G. We shall call a graph *indexable* if it admits an indexer and by an *indexed graph* we shall mean an indexable graph together with an indexer.

For any labeling f of a given graph G, we shall write

$$f(G) = \{f(u): u \in V(G)\}, \qquad f^+(G) = \{f^+(e): e \in E(G)\},$$

and $f_{\max}(G)$ and $f^+_{\max}(G)$ will denote the maximum values in $f(G)$ and $f^+(G)$ respectively.

Observation 1. *For any indexer f of a given (p, q)-graph G, since $f(G) = \{0, 1, 2, \ldots, p-1\}$ we have $f^+(G) \subseteq \{1, 2, \ldots, 2p-3\}$.*

Hence, for any indexable (p, q)-graph G we must have

$$q \leqslant 2p - 3. \tag{1}$$

The bound in (1) is sharp. For example, for any integer $p \geqslant 3$, consider the *join* $G = K_2 + \bar{K}_{p-2}$. Let the points of K_2 be labeled u_1 and u_p, and those of $\bar{K}_{p-2}$ be labeled $u_2, u_3, \ldots, u_{p-1}$. Then the map $f: V(G) \to \{0, 1, 2, \ldots, p-1\}$ defined by

$$f(u_i) = i - 1, \quad 1 \leqslant i \leqslant p \tag{2}$$

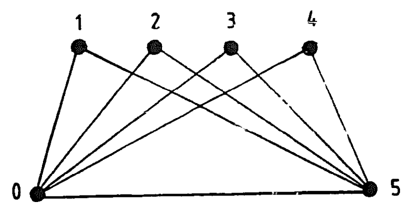
can be easily verified to be an indexer of G.

This motivates the following new notion. A (p, q)-graph $G = (V, E)$ is said to be *strongly indexable* if it admits a *strong indexer*, viz, an indexer f such that $f^+(G) = \{1, 2, \ldots, q\}$. In general, G is said to be *strongly k-indexable* if it admits a *strong k-indexer*, that is, an indexer f so that $f^+(G) = \{k, k+1, k+2, \ldots, k+q-1\}$ (see Fig. 1).

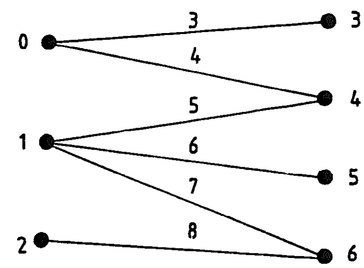
Before we proceed with our investigations on these new notions, a few words about their relation to certain other known graph labeling problems are worth mentioning here.

Grace [7–8] calls a (p, q)-graph $G = (V, E)$ *sequential* if it admits a *sequential labeling of characteristic* k, viz., a labeling $f: V \to \{0, 1, 2, \ldots, M\}$ such that $f^+(G) = \{k, k+1, k+2, \ldots, k+q-1\}$ where $M = q$ if G is a tree and $M = q - 1$ if G is not a tree. Clearly, by this definition it follows that the notions of sequential labelings of characteristic k and strong k-indexers agree on the classes of trees and unicyclic graphs.

Chang et al. [4] define a (p, q)-graph $G = (V, E)$ to be *strongly k-elegant* if it admits a *strong k-elegant* labeling which is a labeling $f: V \to \{0, 1, 2, \ldots, q\}$ of G such that $f^+(G) = \{k, k+1, k+2, \ldots, k+q-1\}$. Here too, we observe that the notions of strong k-elegant labelings and strong k-indexers agree on the class of trees.



A STRONGLY INDEXED GRAPH



A STRONGLY 3 - INDEXED GRAPH

Fig. 1.

Chang et al. [4] also define a (p, q)-graph $G=(V, E)$ to be *strongly k-harmonious* if it admits a *strong k-harmonious* labeling which is a labeling $f: V \to \{0, 1, 2, \ldots, q-1\}$ such that $f^+(G)=\{k, k+1, k+2, \ldots, k+q-1\}$. It is easily seen that the notions of strongly k-harmonious graphs and strongly k-indexable graphs agree on the class of unicyclic graphs.

Some examples of non-indexable graphs are provided by the following result.

Theorem 1. *The complete graph K_n is indexable if and only if $n \leq 3$.*

The following result sheds some light on the structure of strongly indexable graphs.

Theorem 2. *Every strongly indexable graph has exactly one nontrivial component which is either a star or has a triangle.*

Proof. It is enough to show that a strongly indexable triangle-free graph (i.e., a graph without triangles) is a forest having exactly one nontrivial component which is a star. Towards this end, let $G=(V, E)$ be a triangle-free (p, q)-graph having a strong indexer f. Let $u \in V$ be such that $f(u)=0$. Since f is a strong indexer, $1 \in f^+(G)$ so that there exists a point v_1 in the *neighbourhood* $N(u)=\{v \in V: uv \in E\}$ *of* u with $f(v_1)=1$. Similarly, $2 \in f^+(G)$ and since f is injective one can easily check that there exists a point $v_2 \in N(u)$ such that $f(v_2)=2$. Let t be the largest integer in $f^+(G)$ such that $1,2,3,\ldots,t-1 \in f^+(E_u(G))$, where $E_u(G)$ denotes the set of lines of G that are incident at u, and let $f^+(xy)=t$. Since $t \notin f^+(E_u(G))$ we must have $0<f(x)<t$, $0<f(y)<t$ and $t=f^+(xy)=f(x)+f(y)$ so that $x, y \in N(u)$. But then it follows that u, x, y is a triangle in G, a contradiction to our supposition that G is triangle-free. □

Thus, the class of connected triangle-free graphs which are not stars is an infinite class of graphs which are not strongly indexable. This class contains, for example, all bipartitie graphs other than stars, all cycles C_n, $n \geq 4$, etc.

Corollary 2.1. *For the star $K_{1,p-1}$ there is a unique strong indexer, viz., the one which assigns* 0 *to the central point and the numbers* $1, 2, \ldots, p-1$ *to the points of unit degree.*

Corollary 2.2. *If G is a strongly indexable graph with a triangle then any strong indexer of G must assign* 0 *to a point of a triangle in G.*

Remark 1. Hence, if a unicyclic graph G is strongly indexable then its unique cycle must be a triangle. Let U_3 denote the class of unicyclic graphs in each of which the unique cycle is a triangle. For any integer $d \geq 1$, we have a construction of a strongly indexable graph of diameter d in U_3. However, to determine precisely which graphs in U_3 are strongly indexable is an open problem.

Conjecture 1. All unicyclic graphs are indexable.

In support of this conjecture we have been able to establish that all cycles are indexable.

For any (p, q)-graph $G=(V, E)$ and for any function $f: V \to \mathbb{N}$, the identity

$$\sum_{u \in V(G)} d(u)f(u) = \sum_{e \in E(G)} f^+(e) \tag{3}$$

can be proved by easy counting arguments.

Given a labeling $f: V(G) \to \{0, 1, 2, \ldots, p-1\}$ of a (p, q)-graph $G=(V, E)$, we may have a *canonical* labeling of the points of G so that a point u with $f(u)=i$ is labeled u_i, $0 \leq i \leq p-1$. If G has a strong k-indexer f then for a canonical labeling of G, with respect to f, we may derive from (3) the identity

$$\sum_{i=0}^{p-1} i \cdot d(u_i) = qk + \binom{q}{2} \tag{4}$$

and hence if f is a strong indexer of G then (4) gives us

$$\sum_{i=1}^{p-1} i \cdot d(u_i) = q(q+1)/2. \tag{5}$$

Identity (4) will be used in the proof of the following result.

Theorem 3. *The minimum point degree δ of any strongly k-indexable graph with at least two points is at most* 3.

Proof. Let $G=(V, E)$ be a (p, q)-graph, $p \geqslant 2$, having a strong k-indexer f. Then using the canonical labeling of the points of G with respect to f and (4) we find that

$$(q^2+(2k-1)q)/2=\sum_{i=0}^{p-1} i \cdot d(u_i) \geqslant \delta \sum_{i=0}^{p-1} i = \delta p(p-1)/2;$$

that is,

$$q^2+(2k-1)q \geqslant \delta p(p-1). \tag{6}$$

By Observation 1 we have $q+k-1 \leqslant 2p-3$ which amounts to saying

$$q \leqslant 2(p-1)-k \tag{7}$$

for any strongly k-indexable graph. Hence, using (7) in (6) we find

$$[2(p-1)-k]^2+(2k-1)[2(p-1)-k) \geqslant \delta p(p-1)$$

which, on simplifying, yields

$$4p^2-10p-k^2+k+6 \geqslant \delta p(p-1). \tag{8}$$

Now, if $\delta > 3$, since δ is an integer we have $\delta \geqslant 4$ whence (8) yields

$$-6(p-1)-k(k-1) \geqslant 0.$$

This inequality is never possible for $p \geqslant 2$ and $k \geqslant 1$. □

Corollary 3.1. *For any strongly k-indexable graph G, the connectivity $k(G)$, the line connectivity $\lambda(G)$ and $\delta(G)$ stand in the relation $k(G) \leqslant \lambda(G) \leqslant \delta(G) \leqslant 3$.*

The following result determines the regular strongly indexable graphs.

Corollary 3.2. *There are exactly three nontrivial regular graphs which are strongly indexable, viz., K_2, K_3 and $K_2 \times K_3$.*

The following result gives a special necessary condition for an eulerian graph to be strongly indexable.

Theorem 4. *If an eulerian (p, q)-graph is strongly indexable then $q \equiv 0, 3 \pmod 4$.*

Proof. Since the degree of every point of an eulerian graph is even this result follows from (5). □

For example, $K_2 + \bar{K}_{p-2}$ for odd $p \geq 3$ is a strongly indexable eulerian graph. The cycles C_n, $3 < n \equiv 0, 3 \pmod 4$ are counterexamples to the converse of Theorem 4.

The following is a general method of constructing new strongly indexable graphs from an arbitrarily given strongly indexable graph.

Let $G = (V, E)$ be any (p, q)-graph having a strong indexer f, and let $G' = (V', E')$ be a (p', q')-graph together with a strong indexer f' such that $p' \geq (2p - q)$. Define a function f'' on G' by

$$f''(u) =: f'(u) + p \text{ for every } u \in V'.$$

We then construct another graph $H = (V \cup V', E \cup E' \cup F)$ where F is a set of new lines defined by joining the point u of G with $f(u) = q - p + 1$ to the points v of G' with $f'(v) = p + i$, $0 \leq i \leq 2p - q - 1$. Since $2p - q - 1 = 2p - (q + 1) \leq 2p - 1$ we see that this construction is well-defined. Then one can easily see that the function $g = f \cup f''$ is a strong indexer of H. This construction is illustrated in Fig. 2(b) for the connected 'maximal' strongly indexed graph (i.e., with $q = 2p - 3$) of Fig. 2(a).

For any finite set X, a partition $\{X_1, X_2\}$ of X is said to be *central* if the cardinalities of X_1 and X_2 differ by at most unity. By a *partition of a graph G* we mean actually a partition of the point set $V(G)$ of G.

Theorem 5. *If a (p, q)-graph $G = (V, E)$ is strongly k-indexable then G has a central partition $\{V_0, V_e\}$ such that exactly $\lceil q/2 \rceil$ lines of G have their both ends not belonging to the same subset V_a, $a = 0, e$, where $\lceil\ \rceil$ denotes the least integer function.*

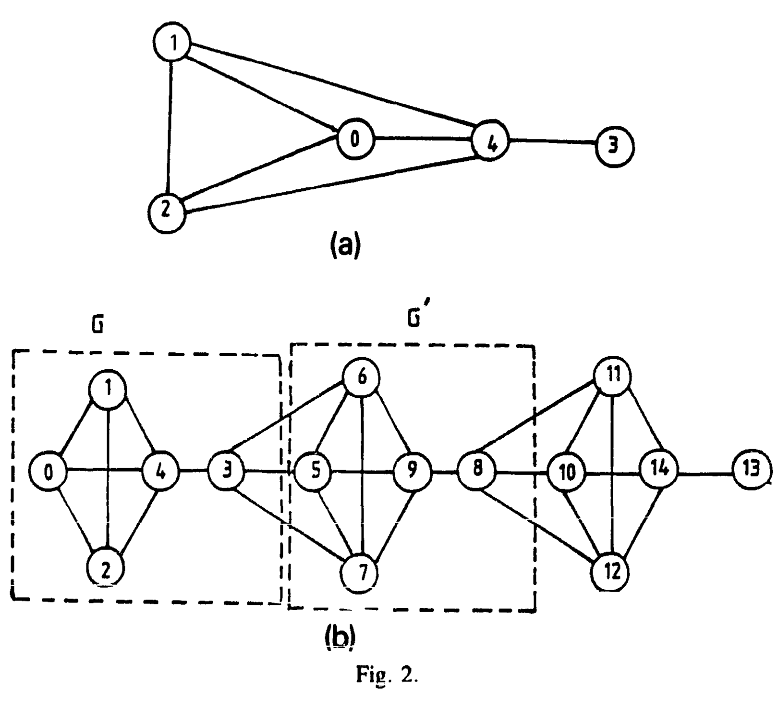


Fig. 2.

The following theorem brings forth an interesting and useful property of the off-diagonals of the standard adjacency matrix of a canonically labeled indexed graph. We were motivated in this direction by the matrix representation of sequential graphs propounded originally by Grace [7].

Theorem 6. *Let $G = (V, E)$ be a (p, q)-graph together with an indexer f. Let $A(G) = (a_{ij})$ denote the adjacency matrix of G with respect to the canonical labeling of the points of G obtained from f. Then every off-diagonal of $A(G)$ has either no nonzero entry or has exactly two 1's viz., those given by*

$$a_{ij} = a_{ji} = 1 \quad \textit{where } f^+(u_iu_j) = f^+(u_ju_i) = f(u_i) + f(u_j) = i + j.$$

Furthermore, if we label by $i + j$ the off-diagonal containing the entries a_{ij} and a_{ji}, then f is a strong k-indexer of G if and only if the only nonzero off-diagonals of $A(G)$ are labeled $k, k+1, k+2, \ldots, k+q-1$.

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