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Girth in Graphs

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It is shown that a graph of large girth and minimum degree at least 3 share many properties with a graph of large minimum degree. For example, it has a contraction containing a large complete graph, it contains a subgraph of large cyclic vertex-connectivity (a property which guarantees, e.g., that many prescribed independent edges are in a common cycle), it contains cycles of all even lengths modulo a prescribed natural number, and it contains many disjoint cycles of the same length. The analogous results for graphs of large minimum degree are due to Mader (*Math. Ann.* **194** (1971), 295–312; *Abh. Math. Sem. Univ. Hamburg* **37** (1972), 86–97), Woodall (*J. Combin. Theory Ser. B* **22** (1977), 274–278), Bollobás (*Bull. London Math. Soc.* **9** (1977), 97–98) and Häggkvist (Equicardinal disjoint cycles in sparse graphs, to appear). Also, a graph of large girth and minimum degree at least 3 has a cycle with many chords. An analogous result for graphs of chromatic number at least 4 has been announced by Voss (*J. Combin. Theory Ser. B* **32** (1982), 264–285).

1. Introduction

Several authors have established the existence of various configurations in graphs of sufficiently large connectivity (independent of the order of the graph) or, more generally, large minimum degree (see, e.g., [2, 9]). A basic result of this type due to Mader [7] asserts that a graph of large minimum degree contains a subdivision of a large complete graph. Such investigations can be applied, for example, to graphs of large chromatic number or d-polytopal graphs. Large girth is, in a sense, a "dual property" of large edge-connectivity since a cycle in a planar graph corresponds to a minimal edge-cut in its dual graph and vice versa. Therefore, it seems natural also to search for configurations in graphs of minimum degree at least 3 and large fixed girth (independent of the order of the graph). Also, in several extremal problems in graph theory the extremal graphs are complete k-partite graphs and some extremal results may therefore change drastically if we restrict our attention to graphs of girth at least 5.

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In this paper we first show that any graph of minimum degree at least 3 and girth at least $4k - 3$ can be contracted into a graph of minimum degree at least k. Combined with the above result of Mader [7], this shows that a graph of minimum degree at least 3 and large girth can be contracted into a large complete graph. On the other hand, we conjecture that, for each natural number k, there exists a natural number $f(k)$ such that each graph of minimum degree $f(k)$ contains a subgraph of minimum degree at least 3 and girth at least k.

Mader [8] showed that any graph of minimum degree at least $4k$ contains a k-connected subgraph. As a counterpart to this we show that any graph of minimum degree at least 3 and girth at least $4k - 6$ contains an induced subgraph which is a subdivision of a graph of minimum degree at least 3 and cyclic vertex-connectivity (a concept to be defined in Section 2) at least k and we show that in any graph of minimum degree at least 3 and cyclic vertex-connectivity at least 2^{k+1} any k independent edges are in a common cycle analogous to a result of Woodall [12] and Häggkvist and Thomassen [6]. We apply the result on subgraphs of large cyclic vertex-connectivity to show that, for each natural number k, any graph of minimum degree at least 3 and girth at least 2^{k+10} contains cycles of all even lengths modulo k. The analogous result involving large minimum degree was obtained by Bollobás [1]. Reference [10] contains a common generalization of the results of Mader [7] and Bollobás [1].

Finally, we illustrate in Section 6 (which is independent of the previous sections) the above remark on extremal problems for graphs of girth at least 5 by investigating collections of disjoint cycles in graphs. If C is a shortest cycle in a graph and x is a vertex not in C, then x is joined to at most one vertex of C if C has length at least 5 and to at most two vertices of C if C has length 4. From this observation it follows that any graph of minimum degree at least $3k - 1$ (resp. $2k$, resp. $k + 1$) and girth at least 3 (resp. 4, resp. 5) has k disjoint cycles. The complete graphs and the complete bipartite graphs show that the first two assertions are best possible. The third assertion, however, is far from best possible. It turns out, surprisingly, that girth 5 and minimum degree 4 guarantees k disjoint cycles (provided, of course, that the graph has sufficiently many vertices). More precisely, any graph of minimum degree at least 4 girth at least 5 and order n has at least $\sqrt{n}/100(\log n)^2$ pairwise disjoint cycles of the same length. We use this to show that any graph of minimum degree at least $3k + 1$ (and sufficiently large order) has k disjoint cycles of the same length. This extends a result of Häggkvist [5].

2. Cyclic Vertex-Connectivity

Our terminology is essentially that of Bollobás [2]. In addition, we say that a collection of edges are (pairwise) *independent* if no two of them have an end in common. We shall also find it convenient to work with cyclic vertex-connectivity rather than cyclic edge-connectivity (as defined in [2, p. 143]). We say that a 2-connected graph is *cyclically k-vertex-connected* ($k \geqslant 2$) if it is not the union of two subgraphs G_1 and G_2 such that each of G_1 and G_2 has a cycle and $|V(G_1) \cap V(G_2)| \leqslant k-1$. Note that our definition implies that a cyclically k-vertex-connected graph has girth at least k and is cyclically k-edge-connected as well. Conversely, it is not difficult to show that a cubic cyclically k-edge-connected graph (of large order) is also cyclically k-vertex-connected and so the two types of cyclic connectivity are closely related. However, unlike the cyclic edge-connectivity, the cyclic vertex-connectivity may increase if we subdivide some edges of the graph as demonstrated by the complete graph of order 4 less one edge. Thus the cyclic vertex-connectivity is not a good measure of how connected a graph is when vertices of degree 2 are allowed but it turns out to be so for graphs of minimum degree at least 3 as we demonstrate in this paper.

3. Contractions of Graphs of Large Girth

Theorem 3.1. *If G is a graph of minimum degree at least 3 and girth at least $2k-3$ (where k is a natural number $\geqslant 3$), then G can be contracted into a multigraph H of minimum degree at least k such that no two vertices of H are joined by more than two edges.*

Proof. If $k = 3$, we put $H = G$ so assume that $k \geqslant 4$. We can also assume without loss of generality that G is connected. Now consider a partition of the vertex set of G into sets $A_1, A_2, \ldots, A_m$ such that $|A_i| \geqslant k-2$ and $G(A_i)$ is connected for each $i = 1, 2, \ldots, m$. Clearly such a partition exists with $m = 1$. Among all such partitions we choose one such that m is maximum.

Let T_i be a spanning tree of $G(A_i)$ for $i = 1, 2, \ldots, m$. We first show that $G(A_i) = T_i$ for each $i = 1, 2, \ldots, m$. For if $G(A_i)$ has an edge e not in T_i, then $T_i \cup \{e\}$ has a unique cycle C_i which has length at least $2k-3$ and hence T_i has an edge e' such that $T_i - e'$ consists of two trees with vertex sets A_i', A_i'' respectively, each of cardinality at least $k-2$. But then the partition $A_1, \ldots, A_{i-1}, A_i', A_i'', A_{i+1}, \ldots, A_m$ contradicts the maximality of m.

We next show that no two trees T_i and T_j ($1 \leqslant i < j \leqslant m$) are joined by more than two edges. For if e_1, e_2, e_3 are three edges joining T_i and T_j, then $T_i \cup T_j \cup \{e_1, e_2, e_3\}$ contains three internally disjoint paths P_1, P_2, P_3. Since G has girth at least $2k-3$ we can assume that P_1 and P_2 both have length at least $k-1$. Now let A_i' (resp. A_i'') be $k-2$ consecutive internal vertices of P_1 (resp. P_2). Then each of the three sets A_i', A_i'' and $(A_i \cup A_j)\setminus(A_i' \cup A_i'')$ induces a connected subgraph of G of order at least $k-2$ and by considering these sets instead of A_i and A_j we obtain a contradiction to the maximality of m.

Now the graph H obtained by contracting each T_i $(i = 1, 2,..., m)$ into a vertex satisfies the conclusion of the theorem.

Mader [7] demonstrated the existence of a function $q(k)$ such that each graph of minimum degree at least $q(k)$ contains a subgraph which can be contracted into a complete graph of order k. With this notation Theorem 3.1 implies

COROLLARY 3.2. *If G has minimum degree at least* 3 *and girth at least* $4q(k) - 3$, *then G has a subgraph which can be contracted into a complete graph of order k.*

Corollary 3.2 shows that many types of graphs can be found in graphs of minimum degree at least 3 and large girth. For example, any graph of minimum degree at least 3 and girth at least $4q(3k) - 3$ has k disjoint cycles. This result will be strengthened considerably in Section 6. If true, the following conjecture shows that results on graphs of large girth and minimum degree at least 3 can be applied to graphs of large minimum degree.

Conjecture 3.3. For any natural numbers k, g there exists a natural number $h(k, g)$ such that each $h(k, g)$-connected graph contains a k-connected subgraph of girth at least g.

Conjecture 3.3 combined with the aforementioned result of Mader [8] on k-connected subgraphs in graphs of minimum degree $4k$ implies that any graph of minimum degree at least $4h(k, g)$ contains a subgraph of minimum degree at least k and girth at least g. Note that this subgraph (and also the subgraph in Conjecture 3.3) cannot always be chosen to be a spanning subgraph as shown by the complete bipartite graphs $K_{a,b}$. For if $b > \frac{1}{2}a(a-1)$, then any spanning subgraph of $K_{a,b}$ of minimum degree at least 2 contains a cycle of length 4.

Erdös [4] made a conjecture analogous to Conjecture 3.3 with chromatic number instead of connectivity. An equivalent version of that conjecture is that every graph of infinite chromatic number has, for each natural number g, a subgraph of infinite chromatic number and girth greater than g. The analogous conjecture with "infinite chromatic number" replaced by "infinite connectivity" was also made by Erdös [4] and solved by the author [11] but that has no relation to Conjecture 3.3.

4. GIRTH, CYCLIC VERTEX-CONNECTIVITY, AND CYCLES THROUGH PRESCRIBED EDGES

The following result is analogous to Mader's result [8] that any graph of minimum degree $4k$ contains a k-connected subgraph.

THEOREM 4.1. *If G is a graph of minimum degree at least 3 and girth at least $4k-6$ (where k is a natural number $\geqslant 2$), then G contains an induced subgraph H which can be described as a graph obtained from a cyclically k-vertex-connected graph H^* of minimum degree at least 3 by inserting at most $2k-3$ vertices of degree 2 on distinct edges.*

Proof. For $k=2$ we just let H be an endblock of G so we assume that $k \geqslant 3$. Among all induced subgraphs of G that contain a cycle and have at most $2k-3$ vertices of degree less than 3 we select one, say H, of smallest order. Then H is 2-connected. For if H is the union of two proper subgraphs H_1 and H_2 which have precisely one vertex in common, then either H_1 or H_2 contradicts the minimality of H. Also, H does not contain two adjacent vertices x, y of degree 2 for otherwise $H-\{x, y\}$ contradicts the minimality of H. Since H has no cycle of length 3 or 4, H is a subdivision of a (unique) graph H^* of minimum degree at least 3.

It only remains to prove that H^* is cyclically k-vertex-connected. Suppose (reductio ad absurdum) that H^* is the union of two induced subgraphs H_1^* and H_2^* none of which is a forest and such that $|V(H_1^*) \cap V(H_2^*)| \leqslant k-1$. Then H_1^* and H_2^* correspond to two induced subgraphs H_1 and H_2, respectively, of H which are subdivisions of H_1^* and H_2^*, respectively. Let A be the set of those vertices of $V(H_1) \cap V(H_2)$ that have degree 2 in H. Let $H_1' = H_1 - A$. If H_1' has a cycle, then either H_1' or H_2 contradicts the minimality of H. So we can assume that H_1' is a forest. This forest has at most $k-1$ vertices of degree less than 2. But then it has at most $k-3$ vertics of degree greater than 2 and, since H has at most $2k-3$ vertics of degree 2, it follows that H_1 has at most $(k-1)+(k-3)+(2k-3)$ vertices. Since H_1 has a cycle, G has girth at most $4k-7$. This contradiction proves the theorem.

The next result (which will be used in Theorem 4.3 below) is of Menger type.

PROPOSITION 4.2. *If G has minimum degree at least 3 and is cyclically k-vertex-connected, $k \geqslant 3$, and A, B are disjoint vertex sets of G each of cardinality at least $2k-3$, then G contains a collection of k disjoint paths from A to B.*

Proof. Suppose (reductio ad absurdum) that the proposition is false. By Menger's theorem, the vertex set of G has a decomposition $V(G) = A' \cup$

$B' \cup S$ such that $|S| \leqslant k-1$, $A \subseteq A' \cup S$, $B \subseteq B' \cup S$, and G' has no edge from A' to B'. Since G is cyclically k-vertex-connected we can assume that $A' \cup S$ induces a forest F. Since $|A| \geqslant 2k-3$ we have $|A'| \geqslant k-2$ and hence F has at least $k-2$ vertices of degree greater than 2 and at most $k-1$ vertices of degree less than 3. But it is easy to see that no such forest exists. This contradiction proves the proposition.

Proposition 4.2 is best possible in the sense that it becomes false if $2k-3$ is replaced by $2k-4$. To see this we consider a cubic cyclically $(2k-2)$-vertex-connected graph $(k \geqslant 4)$. We consider a shortest cycle and select two disjoint paths P_1 and P_2 of length $k-4$ on this cycle such that no vertex of the graph is joined to both P_1 and P_2. Then we let A (resp. B) consists of all vertices of P_1 (resp. P_2) and all their neighbours. Then $|A| = |B| = 2k-4$ and clearly there are no k disjoint paths from A to B.

THEOREM 4.3. *If A is a set of k independent edges in a cyclically 2^{k+1}-vertex-connected graph G of minimum degree at least* 3, *then G has a cycle through A.*

Proof (by induction on k). For $k = 1, 2$ there is nothing to prove so we proceed to the induction step and assume that $k \geqslant 3$. We first consider the case where G has a path of length at most $2k$ connecting two ends of A. Let $P_0\colon x_0 x_1 \cdots x_m$ be a shortest such path and let e_1 and e_2 be the edges of A incident with x_1 and x_m respectively. Then we let G' denote the graph obtained from $G - V(P_0)$ by adding the edge e_0' between the ends of e_1 and e_2 not in P_0 and we put $A' = (A \backslash \{e_1, e_2\}) \cup \{e_0'\}$. Any vertex which in G' has degree 2 is adjacent to P_0 and hence G' has no two adjacent vertices of degree 2 (for otherwise G would have a cycle of length at most $m+3$). Since G' has no 3-cycle or 4-cycle it follows that G' is a subdivision of a graph G'' of minimum degree at least 3. To each edge of G' we associate in the obvious way an edge of G'' and the edge set of G'' corresponding to A' is denoted A''.

No three edges of A'' are incident with the same vertex of G''. For if this were the case, then two of those edges would correspond to a subdivided edge and hence two vertices of distance 2 in G' would be joined to P_0 and consequently G would have a cycle of length at most $m+4$, a contradiction. By a similar argument, no three edges of A'' form a path or cycle. So A'' consists of disjoint paths of length 1 or 2.

If A'' consists of independent edges we put $G''' = G''$ and $A''' = A''$. Otherwise, define G''' and A''' as follows: If A'' contains two edges incident with the same vertex, then the two corresponding edges of A' have distance 1 from each other, i.e., $m = 1$. We then consider a maximal collection $P_0, P_1, P_2, \ldots, P_q$ of pairwise disjoint paths of length 3 in G such that each P_i $(1 \leqslant i \leqslant q)$ contains two edges of A and has an intermediate vertex joined to an intermediate vertex of one of P_0, $P_1,\dots,P_{i-1}$. In particular, $V(P_0)\cup V(P_1)\cup\cdots\cup V(P_q)$ induces a connected subgraph T of G. Note that T has order at most $2k$ because its vertices are all endvertices of edges of A and hence T is a tree. For each P_i $(0\leqslant i\leqslant q)$ we delete the two intermediate vertices and add the edge e_i' between the endvertices. In the resulting graph each endvertex of T has unchanged degree and no two vertices of degree 2 are adjacent because each vertex of degree 2 is adjacent in G to T and G has no cycle of length at most $2k+2$. Also, the resulting graph has no 3-cycle or 4-cycle so it is a subdivision of a (unique) graph G''' of minimum degree at least 3. We let A''' consist of those edges of G''' which correspond to edges of A (including the edges e_0', $e_1',\dots,e_q'$) and we shall now apply the induction hypothesis to the pair G''', A'''. The maximality of q easily implies that the edges of A''' are pairwise independent. Note that $|A'''|\leqslant k-1$ and that the set S of intermediate vertices of the paths $P_0,\dots,P_q$ has cardinality at most k.

Suppose G''' is not cyclically 2^k-vertex-connected. Then it is the union of two induced subgraphs H_1, H_2 none of which is a forest and such that $|V(H_1)\cap V(H_2)|<2^k$. But then the induced subgraphs of G with vertex sets $V(H_1)\cup S\cup W$ and $V(H_2)\cup S\cup W$, where W is the set of vertices of G corresponding to subdivided edges in $H_1\cap H_2$ show that G is not cyclically 2^{k+1}-vertex-connected, a contradiction. (Note that $|W|\leqslant\frac{1}{2}|V(H_1)\cap V(H_2)|$ since otherwise two of the edges corresponding to W would be incident with the same vertex and then G would have a cycle of length at most $2k+3$). This proves that G''' has a cycle containing A''' and this clearly corresponds to a cycle of G through A.

So Theorem 4.3 is proved if the path P_0 exists. We can therefore assume that any path connecting the ends of two edges of A has length greater than $2k$. Consider an edge xy of A. Now we delete all those vertices from G which in $G-y$ have distance less than k from x and we add all edges from x to the set D of those vertices which have distance (in $G-y$) k from x. We do this for each endvertex of each edge of A and obtain thereby a graph H. Since G has girth greater than $2k+1$, there are at least 2^k vertices of distance k from x in $G-y$ and x and y have no common neighbours in H. Consider another edge $x'y'$ of A. The set D' of vertices which in $G-y'$ have distance k from x' is disjoint from D because the distance from x to x' is greater than $2k$. By Proposition 4.2, G has a collection of $k+1$ disjoint paths from D to D' and thus H has $k+1$ internally disjoint paths from x to x'. Now it follows from the result of [6] that H has a cycle through A and hence also G has a cycle through A. This completes the proof.*

* *Note added in proof*: The above proof is inaccurate, since the $k+1$ paths from D to D' are only edge-disjoint in H. Therefore we define H slightly differently: Instead of identifying all vertices of distance (in $G-y$) less than k from x we contract an appropriate tree with $2k-1$ vertices including x, and we use the fact that contracting an edge reduces the cyclic vertex-connectivity by at most one.

5. Cycles Modulo k and Cycles with Many Chords

If we exclude the existence of small cycles in a graph of minimum degree at least 3 (i.e., we require that the graph has large girth) then it turns out that certain types of large cycles are present. In this section we apply Theorem 4.1 to find a configuration in a graph of large girth and minimum degree at least 3 which implies that such a graph contains cycles of all even lengths modulo a fixed natural number k. The analogous result for graphs of large minimum degree is due to Bollobás [1]. We also use the above configuration to find cycles with many chords.

In Section 4 we derived some connectivity properties of graphs of large cyclic vertex-connectivity analogous to properties of graphs of large connectivity. If we delete a set of m vertices of a $(k+m)$-connected graph, then the resulting graph is k-connected. The analogous statement for graphs of given cyclic vertex-connectivity is not true since we may even be able to make the graph disconnected by deleting few vertices. However, we can prove the following (which we shall use in our main result of this section).

Proposition 5.1. *If A is a set of m vertices in a cyclically $(3k+m-1)$-vertex-connected graph G $(k \geqslant 2)$ of minimum degree at least 3 such that A induces a connected subgraph of G, then $G-A$ is cyclically $(3k-1)$-vertex-connected. Furthermore, any two vertices of degree 2 in $G-A$ have distance at least 3 in $G-A$ and the unique graph G^* of minimum-degree at least 3 of which $G-A$ is a subdivision is cyclically $2k$-vertex-connected.*

Proof. If $G-A$ is the union of two subgraphs H_1 and H_2 each of which contains a cycle and such that $|V(H_1) \cap V(H_2)| < 3k-1$, then G is the union of the subgraphs induced by $V(H_1) \cup A$ and $V(H_2) \cup A$, a contradiction. Moreover, $G-A$ is 2-connected. For if this were not the case, then $G-A$ would have a vertex of degree 1 or 0. This vertex would be joined to at least two vertices of A and, since $G(A)$ is connected, G would have a cycle of length at most $m+1$, a contradiction.

It follows that $G-A$ is cyclically $(3k-1)$-vertex-connected. Since G has no cycle of length at most $m+3$, there are no two vertices of degree 2 in $G-A$ that have distance 1 or 2 in $G-A$. We prove by contradiction that G^* is cyclically $2k$-vertex-connected. For if G^* is the union of two subgraphs H_1^* and H_2^* such that $|V(H_1^*) \cap V(H_2^*)| \leqslant 2k-1$, and such that H_1^* and H_2^* both contain cycles, then the corresponding subgraphs of $G-A$ have at most $3k-2$ vertices in common which contradicts that $G-A$ is cyclically $(3k-1)$-vertex-connected.

Theorem 5.2. *Let G be a graph of minimum degree at least 3 and girth at least $2k(3 \cdot 2^k + k^2 - 1)$ where k is an integer greater than one. Let d be* *any natural number $\leqslant k$. Then G contains a path P with k vertices, a cycle C disjoint from P and k pairwise disjoint paths $P_1, P_2, \ldots, P_k$ of length d from P to C each of which has only its ends in common with $P \cup C$.*

Proof. By Theorem 4.1, G contains a subgraph which is a subdivision of graph H of minimum degree at least 3 and cyclic vertex-connectivity at least $3 \cdot 2^k + k^2 - 1$ such that less than $2(3 \cdot 2^k + k^2 - 1)$ edges of H are subdivided in G. Consider a shortest cycle C of H. For each vertex x of C we choose a path P_x of length $d-1$ starting with x and an edge not in C. Then P_x has only x in common with C and the paths P_x, $x \in V(C)$, are pairwise disjoint. Now C corresponds to a cycle of G of length at least $2k(3 \cdot 2^k + k^2 - 1)$ and hence C contains $2(3 \cdot 2^k + k^2 - 1)$ disjoint paths which in G correspond to paths of length $k-1$. Since H has fewer than $2(3 \cdot 2^k + k^2 - 1)$ subdivided edges at least one of these paths on C, say P, together with the paths P_x, $x \in V(P)$, contain no subdivided edge. Now we delete from H the path P and the paths P_x, where $x \in V(P)$ and by Proposition 5.1, the resulting graph is the subdivision of a cyclically 2^{k+1}-vertex-connected graph H' of minimum degree at least 3. For each vertex x of P we select in H' an edge e_x such that e_x is incident with a vertex of H' (or corresponds to a vertex of degree 2 in the graph obtained from H by deleting P and the paths P_x, $x \in V(P)$) which is adjacent in H to the endvertex of P_x distinct from x. Then the edges e_x, $x \in V(P)$, are independent in H' and, by Theorem 4.3, H' has a cycle containing all edges e_x, $x \in V(P)$. This cycle corresponds to a cycle C of G and the proof is complete.

THEOREM 5.3. *If G is a graph of minimum degree at least 3 and girth at least $2(k^2+1)(3 \cdot 2^{k^2+1} + (k^2+1)^2 - 1)$, then G contains cycles of all even lengths modulo k.*

Proof. Let d be any natural number $\leqslant k$. We apply Theorem 5.2 with k^2+1 instead of k. Then we consider the $k+1$ vertices of P such that the distance on P between any two consecutive ones is k and we consider the corresponding paths from P to C. As noted by Bollobás [1] the endvertices on C of two of these paths, say P_1 and P_2, are connected on C by a path whose length is divisible by k. Now $P \cup P_1 \cup P_2 \cup C$ contains a cycle of length $2d$ modulo k.

By a more careful reasoning in Theorems 5.2 and 5.3 we can prove Theorem 5.3 under the weaker condition that G has girth 2^{k+10}. However, this is probably still far from best possible. In [12] Voss announced the result that any graph of chromatic number at least 4 and girth at least $g \geqslant 4$ contains an even (resp. odd) cycle with at least $2^{(g/14)-1}$ chords. Theorem 5.2 implies similar results with the condition on the chromatic number replaced by the weaker condition that each vertex has degree at least 3. For example, we get immediately

THEOREM 5.4. *For each natural number k, there exists a natural number* $q(k)$ *such any graph G of minimum degree at least* 3 *and girth at least* $q(k)$ *contains a cycle C with k independent chords* $e_1, e_2, \ldots, e_k$ *such that either any two of these chords cross one other or else, for each* $i = 2, 3, \ldots, k-1$, e_i *partitions C into two segments* P_1^i *and* P_2^i *such that* P_1^i *contains all ends of* $e_1, e_2, \ldots, e_{i-1}$ *and no end of any of* $e_{i+1}, e_{i+2}, \ldots, e_k$.

Outline of proof. By Theorem 5.2 (with $d = 1$), G contains a cycle with many chords each having one end in the path P in Theorem 5.2. By Ramsey's theorem, either many of these cross each other or else many of these are pairwise noncrossing. Since they all have one end in P they satisfy the assertion of Theorem 5.4.

It is easy to see that the configuration in Therem 5.4 contains an even cycle with many chords (in fact, if m is any natural number and k is sufficiently large, then the configuration contains a cycle of length 2 modulo m and with at least m chords). Also Theorem 5.3 can be used to show that a 2-connected non-bipartite graph of large girth contains an odd cycle with at least k chords. We shall not go into details since it is probably possible to obtain much more general results. Using the results in [10] one can establish a number of results on cycles with many chords in graphs of large minimum degree.

6. DISJOINT CYCLES OF THE SAME LENGTH

In this section we refine methods of Häggkvist [5] to find disjoint cycles of the same length. It is well known and easy to prove that any graph of order n and minimum degree at least 3 contains a cycle of length less than $2 \log n$ (where log denotes the base 2 logarithm). We shall need a similar result where some vertices are allowed to have degree less than 3.

LEMMA 6.1. *If G is a graph of order n such that s vertices have degree* 0 *or* 1 *and t vertices have degree* 2 *and such that* $3s + 2t \leqslant n$, *then G has a cycle of length less than* $8 \log n$.

Proof (by induction on n). For $n = 1, 2, 3$, there is nothing to prove so we proceed to the induction step and assume that $n \geqslant 4$. If G has a vertex of degree zero we delete it and use induction. If G has a vertex x of degree 1 we consider the unique path $P: x x_1 x_2 \cdots x_{q+1}$ such that x_{q+1} has degree 1 or degree at least 3 in G and each $x_1, x_2, \ldots, x_q$ has degree 2 in G. If x_{q+1} has degree 1 in G we delete P and use induction. Otherwise, we delete $P - x_{q+1}$ and use induction.

So we can assume that G has minimum degree at least 2 and hence G is a subdivision of a multigraph H (possibly with loops) of minimum degree at least 3. If some edge or loop of H corresponds to a path or cycle of G of length at least 5 we delete the intermediate vertices of that path or cycle and use induction. So we can assume that each edge of H corresponds to a path of G of length at most 4. Since H has a cycle of length less than $2 \log n$, G has a cycle of length less than $8 \log n$.

THEOREM 6.2. *Let k be any natural number. If G is a graph of minimum degree at least 4, girth at least 5 and of order n such that $n/(\log n)^4 > 2^{13}(k-1)^2$, then G contains a collection of k pairwise disjoint cycles of the same length.*

Proof. We let C_1 be a shortest cycle of G, C_2 a shortest cycle of $G_1 = G - V(C_1)$, C_3 a shortest cycle of $G_2 = G_1 - V(C_2)$, etc., until we get a graph G_m with no cycle of length less than $8 \log n$. We claim that k of the cycles $C_1, C_2, \ldots, C_m$ have the same length. For suppose this were false and put $H_m = C_1 \cup C_2 \cup \cdots \cup C_m$. Then $m < 8(k-1) \log n$ and hence

$$|V(H_m)| < 8m \log n < 64(k-1)(\log n)^2.$$

First, we combine this with the inequality of the theorem to obtain an upper bound on the number of pairs of vertices of H_m:

$$\tfrac{1}{2} |V(H_m)|^2 < 2^{11}(k-1)^2(\log n)^4 < n/4.$$

Next, we note that G_m has order greater than $n - 64(k-1)(\log n)^2$ and, by Lemma 6.1, it has at least $(n/3) - 22(k-1)(\log n)^2$ vertices of degree less than 3. Each such vertex in G_m is joined to at least a pair of vertices in H_m, and the number of such vertices in G_m exceeds the number of pairs of H_m. Therefore, G_m has two vertices with two common neighbours in H_m, a contradiction to the assumption that G has girth at least 5.

Theorem 6.2 is best possible in the sense that it becomes false if we allow G to contain 4-cycles (as demonstrated by the complete bipartite graphs $K_{3,m}$) and it becomes false if we allow vertices to have degree 3 as shown by a graph obtained from a cycle $x_0 x_1 x_2 \cdots x_{4m-1}$ by adding four vertices y_1, y_2, y_3, y_4 and joining y_i to all x_j for which $j \equiv i \bmod 4$. This graph has minimum degree 3 and girth 6 and it has no five disjoint cycles. However, we can prove

THEOREM 6.3. *For each natural number k there exists a natural number*

n_k such that each graph of minimum degree at least 3, girth at least 7 and order at least n_k contains k disjoint cycles of the same length.

The proof of Theorem 6.3 is similar to that of Theorem 6.2 except we use Lemma 6.4 instead of Lemma 6.1.

LEMMA 6.4. *Let G be a graph of order n having s vertices of degree 0 or 1 and at most t independent edges joining vertices of degree 2 in G. If*

$$9s + 7t \leqslant n,$$

then G has a cycle of length less than $12 \log n$.

The proof of Lemma 6.4 is the same as that of Lemma 6.1 except we show that each edge of H corresponds to a path of length at most 6 in G.

Using Theorem 6.2 we strengthen a result of Häggkvist [5].

THEOREM 6.5. *For each natural number k, there exists a natural number m_k such that any graph G of order $n \geqslant m_k$ and minimum degree at least $3k+1$ contains a collection of k pairwise disjoint cycles of the same length.*

Proof. Let $C_1, C_2, \ldots, C_s$ be a maximal collection of disjoint 3-cycles of G and $C_1', C_2', \ldots, C_t'$ a maximal collection of disjoint 4-cycles of $G' = G - (V(C_1) \cup V(C_2) \cup \cdots \cup V(C_s))$. Then $G'' = G' - (V(C_1') \cup \cdots \cup V(C_t'))$ has girth at least 5. We can assume that $s \leqslant k-1$ and $t \leqslant k-1$. Any vertex of G'' is joined to at most two vertices of each C_i' $(1 \leqslant i \leqslant t)$ because of the maximality of s. Also, there cannot be $k-t$ distinct vertices of G'' each of which is joined to two or three vertices of $k-t$ distinct 3-cycles among C_1, $C_2, \ldots, C_s$. Otherwise G would contain k disjoint 4-cycles. This means that all vertices of G'' (except possibly $k-t-1$) are joined to at most

$$2t + 3(k-1-t) + t = 3k-3$$

vertices in $V(C_1) \cup V(C_2) \cup \cdots \cup V(C_s) \cup V(C_1') \cup \cdots \cup V(C_t')$. Hence all vertices (except possibly $k-t-1$) have degree at least 4 in G''. Now the proof is completed by Theorem 6.2. In fact, for each $\varepsilon > 0$ and k sufficiently large, m_k can be chosen to be $k^{2+\varepsilon}$. For if k is large then, by Theorem 6.2, any graph of order at least $k^{2+\varepsilon}$ and minimum degree at least 4 has $7k$ disjoint cycles of the same length. Now the graph G'' may have $k-1$ vertices of degree less than 4. So in order to apply Theem 6.2, we consider five disjoint copies H_1, H_2, H_3, H_4, H_5 of G'', we form the union $G''' = H_1 \cup H_2 \cup H_3 \cup H_4 \cup H_5$ and delete successively the vertices of degree one or zero in G''' until we get a graph H of minimum degree at least 2. (Note that we delete fewer than $\frac{3}{2}k$ vertices in each of H_1, H_2, H_3, H_4, H_5.) We add edges joining vertices of degree 2 or 3 in H in such a way that the new edges form a cycle (of length >4 and such that the resulting graph H' has minimum degree at least 4. Then H' has order at least $k^{2+\varepsilon}$ (provided that k is large) and hence H has at least $7k$ disjoint cycles of the same length. At least k of these must be in the same H_i and hence in G.

Corradi and Hajnal [3] proved that any graph of minimum degree at least $2k$ and order at least $3k$ has k disjoint cycles. Perhaps even the following holds:

Conjecture 6.6. For each natural number k there exists a natural number p_k such that any graph of minimum degree at least $2k$ and order at least p_k contains k disjoint cycles of the same length.

The restriction of this conjecture to the case $k = 2$ was first made by Häggkvist [5].

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