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Inequalities of Hadamard's Type for Lipschitzian Mappings

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In this paper, we establish several inequalities of Hadamard's type for Lipschitzian mappings. © 2001 Academic Press

1. INTRODUCTION

The inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which hold for all convex mappings $f\colon [a,b] \to R$, are known in the literature as Hadamard's inequalities [5]. In [4], Fejér gave a weighted generalization of Hadamard's inequalities: If $f\colon [a,b] \to R$ is convex, and $g\colon [a,b] \to R$ is nonnegative, integrable, and symmetric to $x = \frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{1.2}$$

For various results which generalize, improve, and extend the inequalities (1.2), see [1, 7, 9]. In [9, Theorem 2, 5, 6], we proved the following

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two theorems:

THEOREM A. *Let* $f\colon [a, b] \to R$ *be a convex function*, $0 < \alpha < 1$, $0 < \beta < 1$, $A = \alpha a + (1-\alpha)b$, $u_0 = (b-a)\min\{\frac{\alpha}{1-\beta}, \frac{1-\alpha}{\beta}\}$, *and let* $g\colon [a, b] \to R$ *be nonnegative and integrable and*

$$g(A - \beta u) = g(A + (1-\beta)u), \qquad u \in [0, u_0]. \tag{1.3}$$

Then

$$\begin{aligned} f[\alpha a + (1-\alpha)b] &\int_{A-\beta u}^{A+(1-\beta)u} g(x)dx \\ &\le \frac{1-\beta}{\beta} \int_{A-\beta u}^{A} f(x)g(x)dx + \frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(x)g(x)dx \\ &\le [\alpha f(a) + (1-\alpha)f(b)] \int_{A-\beta u}^{A+(1-\beta)u} g(x)dx. \end{aligned} \tag{1.4}$$

THEOREM B. *Let f, A, g, and* u_0 *be defined as in Theorem* A, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta \le 1$, *and let P, Q be defined on* $[0, 1]$ *by*

$$\begin{aligned} P(t) = \int_0^u &\{(1-\beta)f(A-\beta tx)g(A-\beta x) \\ &+ \beta f[A+(1-\beta)tx]g[A+(1-\beta)x]\}dx \end{aligned} \tag{1.5}$$

$$\begin{aligned} Q(t) = \int_0^u &\{(1-\beta)f[A-\beta u+\beta x(1-t)]g[A-\beta(u-x)]+\beta f[A+(1-\beta)u \\ &-(1-\beta)(1-t)x]g[A+(1-\beta)(u-x)]\}dx \end{aligned} \tag{1.6}$$

for some $u \in [0, u_0]$. *Then P and Q are convex and monotonically increasing on* $[0, 1]$ *and*

$$\begin{aligned} f[\alpha a+(1-\alpha)b]&\int_{A-\beta u}^{A+(1-\beta)u} g(x)dx = P(0) \le P(t) \\ &\le P(1) = \frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(x)g(x)dx + \frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(x)g(x)dx = Q(0) \\ &\le Q(t) \le Q(1) = [(1-\beta)f(A-\beta u)+\beta f(A+(1-\beta)u)]\int_{A-\beta u}^{A+(1-\beta)u} g(x)dx \\ &\le [\alpha f(a)+(1-\alpha)f(b)]\int_{A-\beta u}^{A+(1-\beta)u} g(x)dx. \end{aligned} \tag{1.7}$$

We note that if $\alpha = \beta = \frac{1}{2}$ and $u = u_0 = b - a$, then

$$P(t) = \int_a^b f\left[tx + (1-t)\frac{a+b}{2}\right]g(x)dx, \tag{1.8}$$

$$Q(t) = \frac{1}{2}\int_a^b \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right)g\left(\frac{x+a}{2}\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right)g\left(\frac{x+b}{2}\right)\right]dx, \tag{1.9}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx &= P(0) \le P(t) \le P(1) = \int_a^b f(x)g(x)dx \\ &= Q(0) \le Q(t) \le Q(1) \\ &= \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \end{aligned} \tag{1.10}$$

Also, if $\alpha = \beta = \frac{1}{2}$, $u = u_0 = b - a$, and $g(x) \equiv 1$, then

$$P(t) = \int_a^b f\left[tx + (1-t)\frac{a+b}{2}\right]dx \tag{1.11}$$

and

$$Q(t) = \frac{1}{2}\int_a^b \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right)\right]dx, \tag{1.12}$$

where (1.11) and (1.12) were established by Dragomir [2] and Yang and Hong [8], respectively.

Recently Dragomir *et al.* [3] and Matic and Pečarić [6] proved some results for Lipschitzian functions related to (1.11) and (1.12), respectively. In this paper, we will prove some inequalities for Lipschitzian functions related to the functions P, Q.

2. MAIN RESULTS

For the function P, defined by (1.5), we have the following theorem:

THEOREM 1. *Let $f\colon [a,b] \to R$ be an M-Lipschitzian function. Then*

$$\left|\left[\frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(x)g(x)dx + \frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(x)g(x)dx\right] - P(t)\right| \le \left[2\beta(1-\beta)M\int_0^u xg(A-\beta x)dx\right](1-t); \tag{2.1}$$

$$\left|P(t) - f(\alpha a + (1-\alpha)b)\int_{A-\beta u}^{A+(1-\beta)u} g(x)dx\right| \le \left[2\beta(1-\beta)M\int_0^u xg(A-\beta x)dx\right]t \tag{2.2}$$

and

$$\left|P(t) - t\left[\frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(x)g(x)dx + \frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(x)g(x)dx\right]\right.$$
$$\left. - (1-t)f(\alpha a + (1-\alpha)b)\int_{A-\beta u}^{A+(1-\beta)u} g(x)dx\right|$$
$$\leq \left[4\beta(1-\beta)M\int_0^u xg(A-\beta x)dx\right]t(1-t) \tag{2.3}$$

for all $t \in [0, 1]$.

Proof. For $t_1, t_2, \in [0, 1]$, we have

$$\left|P(t_2) - P(t_1)\right|$$
$$= \left|\int_0^u \{(1-\beta)[f(A-\beta t_2 x) - f(A-\beta t_1 x)]g(A-\beta x)\right.$$
$$\left. + \beta[f(A+(1-\beta)t_2 x) - f(A+(1-\beta)t_1 x)]g(A+(1-\beta)x)\}dx\right|$$
$$\leq \int_0^u \{(1-\beta)\left|f(A-\beta t_2 x) - f(A-\beta t_1 x)\right|g(A-\beta x)$$
$$+ \beta\left|f(A+(1-\beta)t_2 x) - f(A+(1-\beta)t_1 x)\right|g(A+(1-\beta)x)\}dx$$
$$\leq \beta(1-\beta)M\left|t_2 - t_1\right|\int_0^u [xg(A-\beta x) + xg(A+(1-\beta)x)]dx.$$

Thus, from the identity

$$g(A-\beta x) = g(A+(1-\beta)x), \qquad x \in [0, u],$$

we obtain

$$\left|P(t_2) - P(t_1)\right| \leq \left[2\beta(1-\beta)M\int_0^u xg(A-\beta x)dx\right]|t_2 - t_1|. \tag{2.4}$$

Now, the inequalities (2.1) and (2.2) follow from (2.4) by setting $t_1 = t$, $t_2 = 1$ and $t_1 = 0$, $t_2 = t$, respectively. The inequality (2.3) is obtained multiplying (2.1) by t and multiplying (2.2) by $(1-t)$ and taking their sum. This completes the proof.

Similarly, for the function Q, defined by (1.6), we have the following theorem:

THEOREM 2. *Let f be defined as in Theorem* 1. *Then*

$$\left|\frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(x)g(x)dx + \frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(x)g(x)dx - Q(t)\right|$$
$$\leq [2\beta(1-\beta)M\int_0^u xg(A-\beta(u-x))dx]t; \tag{2.5}$$

$$\left|Q(t) - [(1-\beta)f(A-\beta u) + \beta f(A+(1-\beta)u)]\int_{A-\beta u}^{A+(1-\beta)u} g(x)dx\right|$$
$$\leq [2\beta(1-\beta)M\int_0^u xg(A-\beta(u-x))dx](1-t) \tag{2.6}$$

and

$$\left|Q(t) - (1-t)\left[\frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(x)g(x)dx + \frac{\beta}{1-\beta}\int_A^{A+(1-\beta)u} f(x)g(x)dx\right] - t[(1-\beta)f(A-\beta u) + \beta f(A+(1-\beta)u)]\int_{A-\beta u}^{A+(1-\beta)u} g(x)dx\right|$$
$$\leq \left[4\beta(1-\beta)M\int_0^u xg(A-\beta(u-x))dx\right]t(1-t) \tag{2.7}$$

for all $t \in [0, 1]$.

The following corollaries are simple consequences of Theorem 1 and Theorem 2.

COROLLARY 1. *Under the assumptions of Theorem* 1 *and Theorem* 2, *let* $\beta = 1 - \alpha$ *and* $u = u_0 = b - a$. *Then, for* $t \in [0, 1]$, *we have*

$$P(t) = \int_0^{b-a}\{\alpha f(A-(1-\alpha)tx)g(A-(1-\alpha)x) + (1-\alpha)f(A+\alpha tx)g(A+\alpha x)\}dx; \tag{2.8}$$

$$\left|\left[\frac{\alpha}{1-\alpha}\int_a^A f(x)g(x)dx + \frac{1-\alpha}{\alpha}\int_A^b f(x)g(x)dx\right] - P(t)\right|$$
$$\leq \left[2\alpha(1-\alpha)M\int_0^{b-a} xg(A-(1-\alpha)x)dx\right](1-t); \tag{2.9}$$

$$\left|P(t) - f(\alpha a + (1-\alpha)b)\int_a^b g(x)dx\right|$$
$$\leq \left[2\alpha(1-\alpha)M\int_0^{b-a} xg(A-(1-\alpha)x)dx\right]t; \tag{2.10}$$

$$\left|P(t) - t\left[\frac{\alpha}{1-\alpha}\int_a^A f(x)g(x)dx + \frac{1-\alpha}{\alpha}\int_A^b f(x)g(x)dx\right] - (1-t)f(\alpha a + (1-\alpha)b)\int_a^b g(x)dx\right|$$
$$\leq \left[4\alpha(1-\alpha)M\int_0^{b-a} xg(A-(1-\alpha)x)dx\right]t(1-t) \tag{2.11}$$

and

$$Q(t) = \int_0^{b-a} \{\alpha f[a + (1-\alpha)x(1-t)]g[a + (1-\alpha)x] + (1-\alpha)f[b - \alpha x(1-t)]g[b - \alpha x]dx\}; \tag{2.12}$$

$$\left|\left[\frac{\alpha}{1-\alpha}\int_a^A f(x)g(x)dx + \frac{1-\alpha}{\alpha}\int_A^b f(x)g(x)dx\right] - Q(t)\right| \leq \left[2\alpha(1-\alpha)M\int_0^{b-a} xg(a+(1-\alpha)x)dx\right]t; \tag{2.13}$$

$$\left|Q(t) - [\alpha f(a) + (1-\alpha)f(b)]\int_a^b g(x)dx\right| \leq \left[2\alpha(1-\alpha)M\int_0^{b-a} xg(a+(1-\alpha)x)dx\right](1-t); \tag{2.14}$$

$$\left|Q(t) - (1-t)\left[\frac{\alpha}{1-\alpha}\int_a^A f(x)g(x)dx + \frac{1-\alpha}{\alpha}\int_A^b f(x)g(x)dx\right] - t[\alpha f(a) + (1-\alpha)f(b)]\int_a^b g(x)dx\right| \leq \left[4\alpha(1-\alpha)M\int_0^{b-a} xg(a-(1-\alpha)x)dx\right]t(1-t). \tag{2.15}$$

COROLLARY 2. *In Corollary* 1, *if we let* $t = 0$, *then* (2.9) *and* (2.14) *reduce to*

$$\left|f(A)\int_a^b g(x)dx - \left[\frac{\alpha}{1-\alpha}\int_a^A f(x)g(x)dx + \frac{1-\alpha}{\alpha}\int_A^b f(x)g(x)dx\right]\right| \leq 2\alpha(1-\alpha)M\int_0^{b-a} xg(A-(1-\alpha)x)dx \tag{2.16}$$

and

$$\left|[\alpha f(a) + (1-\alpha)f(b)]\int_a^b g(x)dx - \left[\frac{\alpha}{1-\alpha}\int_a^A f(x)g(x)dx + \frac{1-\alpha}{\alpha}\int_A^b f(x)g(x)dx\right]\right| \leq 2\alpha(1-\alpha)M\int_0^{b-a} xg(a+(1-\alpha)x)dx, \tag{2.17}$$

respectively.

Remark 1. Let f be defined as in Corollary 2 and let g: $[a,b] \to R$ be nonnegative, integrable, and symmetric to $x = \frac{a+b}{2}$. Then we have the inequalities

$$\left| f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right| \leq \frac{M}{2} \int_0^{b-a} xg\left(\frac{a+b}{2} - \frac{x}{2}\right)dx \tag{2.18}$$

and

$$\left| \frac{f(a)+f(b)}{2} \int_a^b g(x)dx - \int_a^b f(x)g(x)dx \right| \leq \frac{M}{2} \int_0^{b-a} xg\left(a+\frac{x}{2}\right)dx. \tag{2.19}$$

Proof. The inequalities (2.18) and (2.19) follow immediately from (2.16) and (2.17) by letting $\alpha = \frac{1}{2}$ in Corollary 2.

Remark 2. If we set $g(x) \equiv 1$, $x \in [a,b]$ in Remark 1, then (2.18) and (2.19) reduce to

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{M}{4}(b-a) \tag{2.20}$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{M}{4}(b-a), \tag{2.21}$$

respectively. The inequality (2.20) was proved by Dragomir *et al*. [3], and the inequality (2.21) was proved by Matic and Pečarić [6].

Another result which is connected in a sense with the inequality (2.21) is also given in the following:

COROLLARY 3. *With the assumptions in Remark* 1, *we have the inequalities*

$$\left| \frac{f(ta+(1-t)((a+b)/2)) + f(tb+(1-t)((a+b)/2))}{2} \times \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} g(x)dx - \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(x)g(x)dx \right| \leq \frac{M}{2} \int_0^{t(b-a)} xg\left(a+\frac{x}{2}\right)dx \tag{2.22}$$

for all $t \in [0, 1]$, *and*

$$\left| \frac{f(ta + (1-t)((a+b)/2)) + f(tb + (1-t)((a+b)/2))}{2} \right.$$
$$\left. - \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx \right| \le \frac{Mt}{4}(b-a) \tag{2.23}$$

for all $t \in [0, 1]$.

Proof. If we let $u = ta + (1-t)\frac{a+b}{2}$ and $v = ta + (1-t)\frac{a+b}{2}$, then, using the inequality (2.19) applied for u and v, we have the inequality (2.22). The inequality (2.23) follows immediately from (2.22) by letting $g(x) = 1$ for all $x \in [a, b]$.

Remark 3. The inequality (2.23) is stronger than the inequality (3.5) in [3] which has $\frac{M}{3}$ in place of our $\frac{M}{4}$.

COROLLARY 4. *Under the assumptions of Theorem* 1 *and Theorem* 2, *let* $\beta = \alpha = \frac{1}{2}$, $A = \frac{a+b}{2}$, *and* $u = u_0 = b - a$. *Then* P *is defined by* (1.8),

$$\left| \int_a^b f(x)g(x)dx - P(t) \right| \le \frac{M(1-t)}{2} \int_0^{b-a} xg\left(\frac{a+b-x}{2}\right) dx; \tag{2.24}$$

$$\left| P(t) - f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \right| \le \frac{Mt}{2} \int_c^{b-a} xg\left(\frac{a+b-x}{2}\right) dx; \tag{2.25}$$

$$\left| P(t) - t\int_a^b f(x)g(x)dx - (1-t)f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \right|$$
$$\le Mt(1-t) \int_0^{b-a} xg\left(\frac{a+b-x}{2}\right) dx \tag{2.26}$$

and Q *is defined by* (1.9),

$$\left| \int_a^b f(x)g(x)dx - Q(t) \right| \le \frac{Mt}{2} \int_0^{b-a} xg\left(a + \frac{x}{2}\right) dx; \tag{2.27}$$

$$\left| Q(t) - \frac{f(a)+f(b)}{2} \int_a^b g(x)dx \right| \le \frac{M(1-t)}{2} \int_0^{b-a} xg\left(a + \frac{x}{2}\right) dx; \tag{2.28}$$

$$\left| Q(t) - (1-t)\int_a^b f(x)g(x)dx - t\left(\frac{f(a)+f(b)}{2}\right) \int_a^b g(x)dx \right|$$
$$\le Mt(1-t) \int_0^{b-a} xg\left(a + \frac{x}{2}\right) dx \tag{2.29}$$

for all $t \in [0, 1]$.

Remark 4. In Corollary 4, if we let $g(x) \equiv 1$, then (2.24)–(2.26) and (2.27)–(2.29) reduce to some results established by Dragomir *et al*. [3] and Matic and Pečarić [6], respectively.

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